Research Report

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Overview

- **Lecture on Numerical Simulation on Coupled Field Problems**
  - Fluid-Structure-Acoustics
  - Piezoelectricity
  - Magnetomechanics
  - MHD and Porous Media (Marco Discacciati)
- **Workshop on** (together with B. Kaltenbacher)
  Direct and Inverse Problems in Piezoelectricity
- **Computation of thin (flat) structures (D. Braess)**
  - Mechanical structures
  - Piezoelectric structures
- **Perfectly matched layers (PMLs) (J. Schöberl)**
  - Splitting approach
  - Analytic continuation approach
  - Time domain approach
- **Further cooperations**

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Computation of thin (flat) Structures (I)

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- **Reissner-Mindlin-formulation**

$$\Pi(\vartheta, \omega) = \frac{1}{2} \int_\Omega \varepsilon(\vartheta) \mathbf{D}_{\text{plate}} \varepsilon(\vartheta)\, d\Omega + \frac{1}{t^2} \int_\Omega |\nabla\, \omega - \vartheta|^2\, d\Omega$$

- **Appropriate mixed formulation**

$$\gamma := t^{-2}(\nabla\, \omega - \vartheta) \quad \text{shear term}$$

$$\begin{aligned} a(\vartheta, \psi) + \frac{1}{h^2}(\nabla\, \omega - \vartheta, \nabla\, v - \psi) + (\nabla\, v - \psi, \gamma) &= (f, v) \\ (\nabla\, \omega - \vartheta, \eta) - t^2(\gamma, \eta) &= 0 \end{aligned}$$

Computation of thin (flat) Structures (II)

- **Instead of mixed formulation, solve**

$$\begin{aligned} a(\vartheta_h, \psi) &+ k\alpha(\nabla\, \omega_h - \vartheta_h, \nabla\, v - \psi) \\ &+ k^2(t^{-2} - \alpha)\,(\pi_h[\nabla\, \omega_h - \vartheta_h], \pi_h[\nabla\, v - \psi]) \\ &= (f, v) \end{aligned}$$

Computes mean value over each element

- **Arnold & Brezzi**

$$\alpha = \frac{1}{t^2} - \alpha > 0 \;, \quad \alpha = const.$$

- **Chapell & Stenberg**

$$\alpha = \frac{\mu}{t^2 + h^2} \;, \quad \mu = const.$$

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Computation of thin (flat) Structures (III)

3D formulation

$$\int_\Omega \varepsilon_{13}(u)^2 \, d\Omega \approx t \qquad \int_\Omega \varepsilon_{11}(u)^2 \, d\Omega \approx t^3$$

$$\mathbf{D} := \mathbf{D}_b + \mathbf{D}_s$$

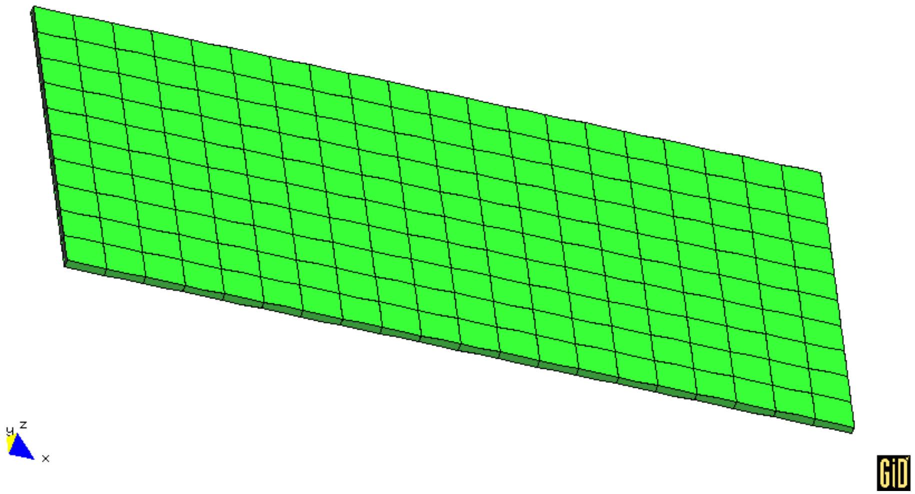
$$\mathbf{D}_b = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha * c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha * c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha * c_6 \end{pmatrix} \quad \mathbf{D}_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta * c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta * c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta * c_6 \end{pmatrix}$$

$$\alpha = \frac{t^2}{h^2 + t^2} \qquad \beta = \frac{h^2}{h^2 + t^2}$$

$$\mathbf{k}^e = \underbrace{\int_{\Omega^e} \mathcal{B}\mathbf{D}_b\mathcal{B} \, d\Omega}_{\text{standard integration}} + \underbrace{\int_{\Omega^e} \mathcal{B}\mathbf{D}_s\mathcal{B} \, d\Omega}_{\text{reduced integraion}}$$

Computation of thin (flat) Structures (IV)

- **Plate: 2a x 2b, a= 20mm, b=10mm, t=0.2mm**



h/t	lin	lin-SRI	lin-BK	quad	quad-BK	Kirchhoff
25	0.072 μm	9.531 μm	7.49 μm	9.17 μm	12.74 μm	13.47 μm
10	0.426 μm	11.70 μm	9.23 μm	13.32 μm	13.77 μm	13.47 μm
5	1.528 μm	12.04 μm	9.43 μm	13.75 μm	13.83 μm	13.47 μm
2	5.586 μm	12.12 μm	9.80 μm	13.84 μm	13.84 μm	13.47 μm
1	9.003 μm	12.13 μm	9.002 μm	13.84 μm	13.84 μm	13.47 μm

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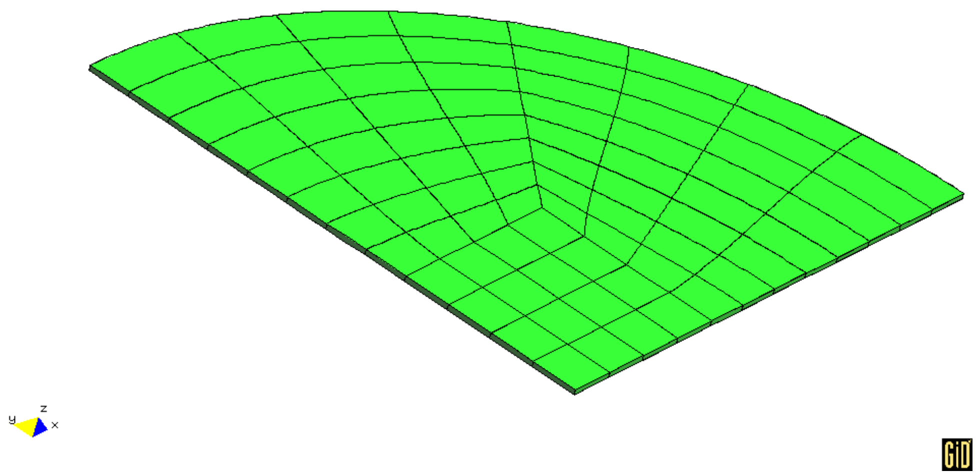
Computation of thin (flat) Structures (V)

❒ **Plate: R=,20mm t=0.2mm**



h/t	lin	lin-SRI	lin-BK	quad	quad-BK	Kirchhoff
16.25	1.16 μm	99.11 μm	72.92 μm	99.08 μm	105.06 μm	108.62 μm
8.125	4.39 μm	103.59 μm	75.49 μm	106.87 μm	108.27 μm	108.62 μm
4	15.18 μm	105.00 μm	76.58 μm	108.40 μm	108.65 μm	108.62 μm
2	39.89 μm	105.55 μm	78.87 μm	108.70 μm	108.73 μm	108.62 μm
1	67.47 μm	105.85 μm	83.78 μm	108.78 μm	108.79 μm	108.62 μm

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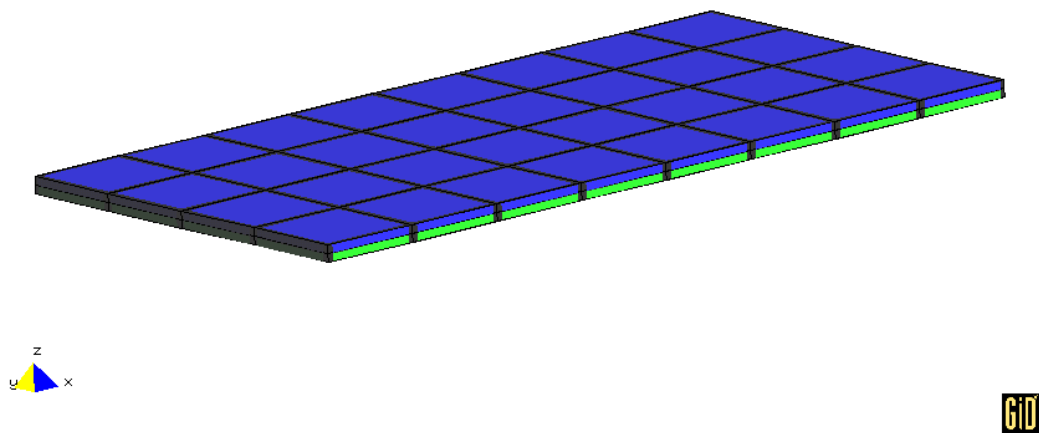
Computation of thin (flat) Structures (VI)

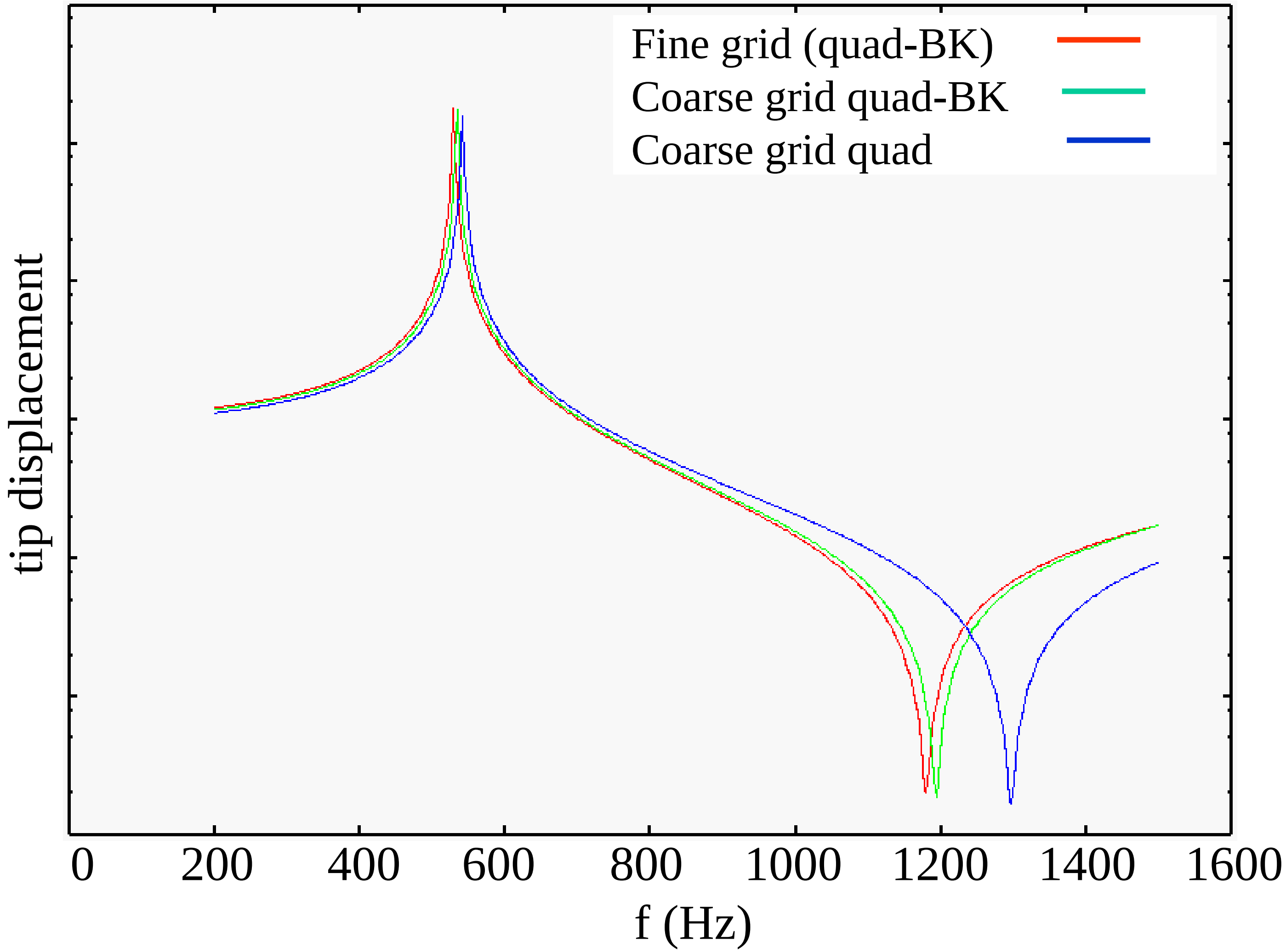
- Plate with piezoelectric layer: a=20mm, b=10mm, $t_{1,2}$=0.1mm



h/t	lin	lin-SRI	lin-BK	quad	quad-BK
50	0.081 μm	12.30 μm	11.00 μm	11.32 μm	11.75 μm
25	0.32 μm	12.67 μm	11.28 μm	11.57 μm	11.70 μm
12.5	1.18 μm	12.77 μm	11.35 μm	11.65 μm	11.70 μm

Computation of thin (flat) Structures (VII)

- Plate with piezoelectric layer



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Perefectly Matched Layer (I)

- Idea of Berenger: Splitting of physical quantities

  - PDE: linear acoustics $p' = \rho_0 c^2 \rho'$

$$\frac{\partial p'}{\partial t} = -\rho_0 c^2 \nabla \mathbf{v}' \qquad \frac{\partial \mathbf{v}'}{\partial t} = -\frac{1}{\rho_0} \nabla p'$$

  - Splitted formulation $p' = p_x + p_y + p_z$

$$\frac{\partial p_x}{\partial t} + \sigma_x p_x = -\rho_0 c^2 \frac{\partial v_x}{\partial x} \qquad \frac{\partial v_x}{\partial t} + \sigma_x v_x = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial p_y}{\partial t} + \sigma_y p_y = -\rho_0 c^2 \frac{\partial v_y}{\partial y} \qquad \frac{\partial v_y}{\partial t} + \sigma_y v_y = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial p_z}{\partial t} + \sigma_z p_z = -\rho_0 c^2 \frac{\partial v_z}{\partial z} \qquad \frac{\partial v_z}{\partial t} + \sigma_z v_z = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

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Perfectly Matched Layer (II)

- Analytic continuation of solution (Teixeira, Chew)

$$x_i \Rightarrow \tilde{x}_i = \int_0^{x_i} \gamma(\xi) d\xi$$

$$\gamma(x_1) = 1 + \frac{\sigma(x_i)}{j\omega}$$

$$\frac{\partial}{\partial x_i} \Rightarrow \frac{\partial}{\partial \tilde{x}_i} = \frac{1}{\gamma(x_1)} \frac{\partial}{\partial x_i}$$

- Acoustic PDE (Helmholtz)

$$\gamma(x_2)\gamma(x_3)\frac{\partial}{\partial x_1}\left(\frac{1}{\gamma(x_1)}\frac{\partial p}{\partial x_1}\right)$$
$$+\gamma(x_1)\gamma(x_3)\frac{\partial}{\partial x_2}\left(\frac{1}{\gamma(x_2)}\frac{\partial p}{\partial x_2}\right)$$
$$+\gamma(x_1)\gamma(x_2)\frac{\partial}{\partial x_3}\left(\frac{1}{\gamma(x_3)}\frac{\partial p}{\partial x_3}\right) = \gamma(x_1)\gamma(x_2)\gamma(x_3)\, k^2\, p$$

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Perfectly Matched Layer (III)

- **Reflection coefficient**

$$R = e^{-\frac{\cos\vartheta}{c}\int\limits_0^L \sigma(x_i)dx_i}$$

- **Choice of damping factor ($R \approx 10^{-3}$)**
  - **Constant**

$$\sigma(x_i) = \sigma_0 \qquad \sigma_0 = \frac{c\ln R}{2L\cos\vartheta}$$

  - **Quadratic distance weighting**

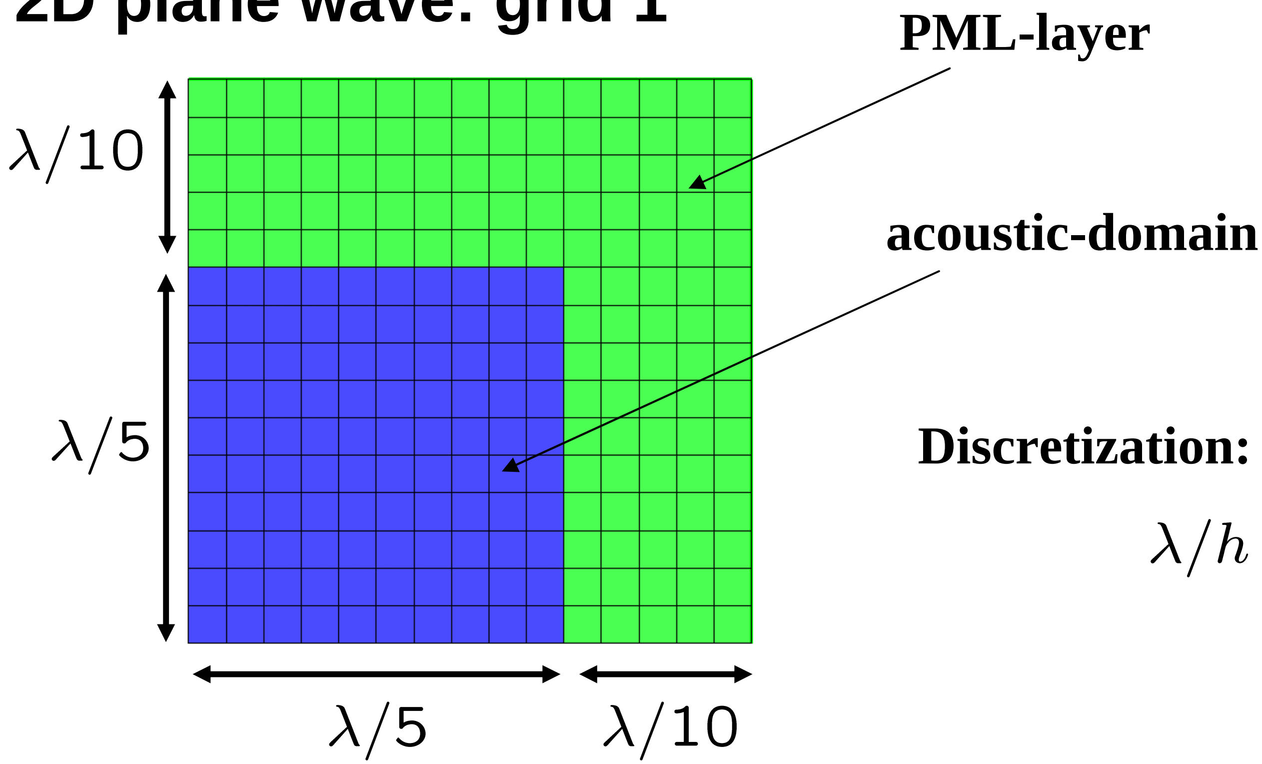
$$\sigma(x_i) = \sigma_0 \frac{x_i^2}{L^2} \qquad \sigma_0 = \frac{3c\ln R}{2L\cos\vartheta}$$

  - **Inverse distance weighting**

$$\sigma(x_i) = \frac{c}{L - |x_i|} \qquad \int\limits_0^L \sigma(x_i)dx_i = \infty$$

Perfectly Matched Layer (IV)

❒ 2D plane wave: grid 1



Discretization: linear FE

$$\lambda/h = 57$$

	ABC 1st order	PML const. $(\sigma_0 = 14)$	PML quadDist $(\sigma_0 = 42)$	PML inverseDist
L2-error	0.104	0.0072	0.0070	0.0040
rel. Error (0.5,0.5)	17%	0.63%	0.35%	0.34%

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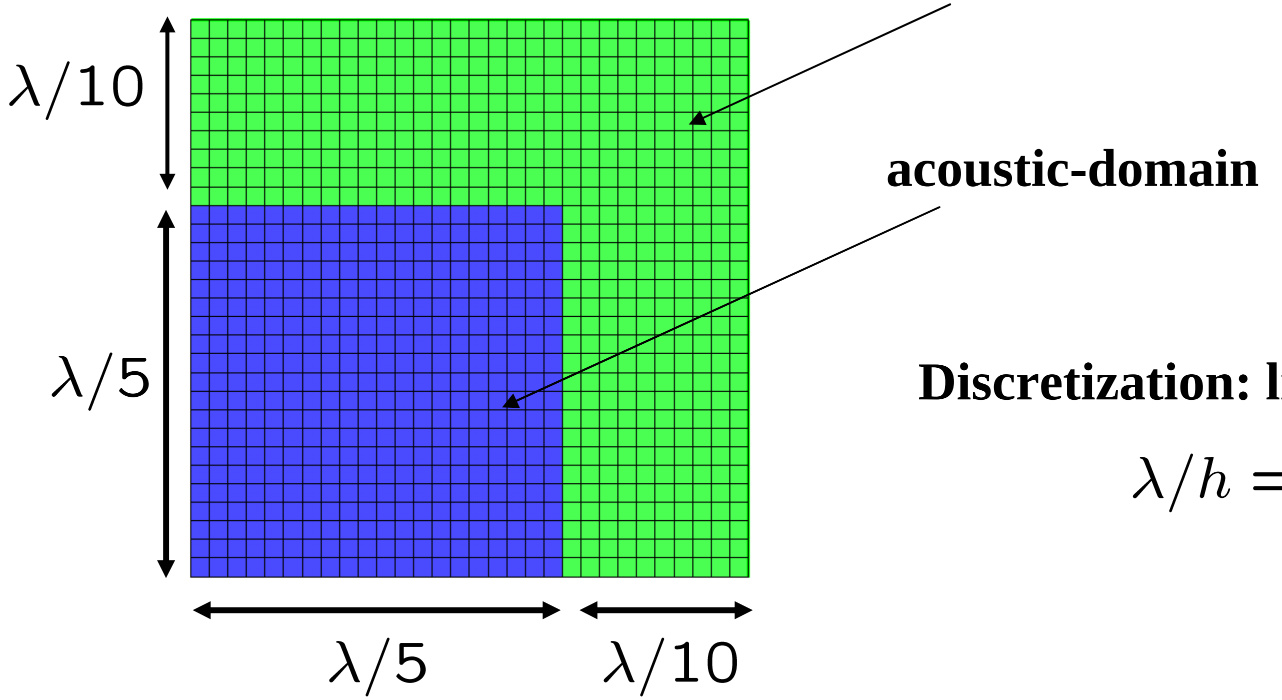
Perfectly Matched Layer (V)

❐ 2D plane wave: grid 2



Discretization: linear FE

$$\lambda/h = 114$$

	ABC 1st order	PML const. ($\sigma_{max} = 14$)	PML quadDist ($\sigma_{max} = 42$)	PML inverseDist
L2-error	0.105	0.001613	0.001351	0.001193
rel. Error (0.5,0.5)	17%	0.22%	0.135%	0.11%

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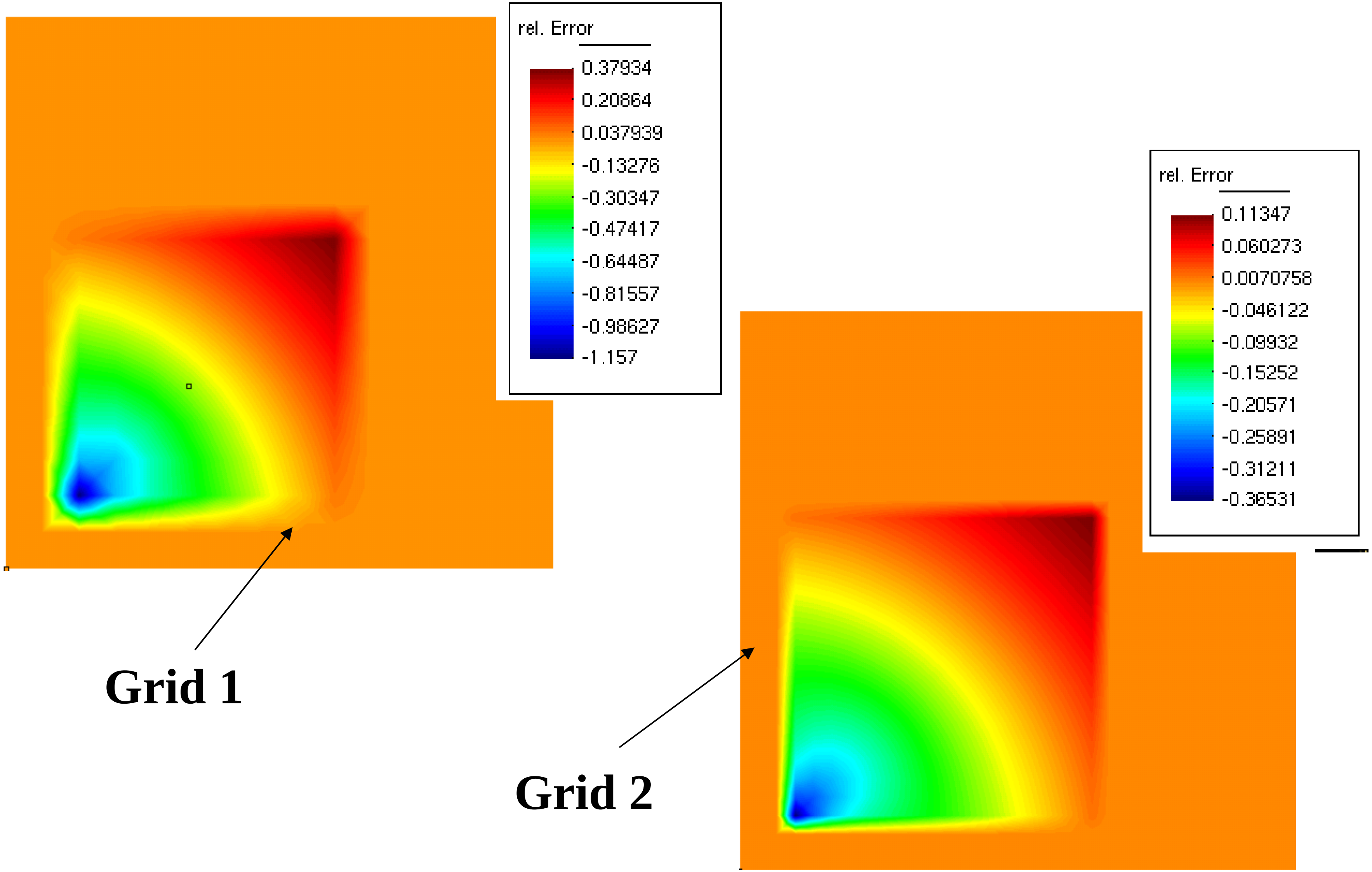
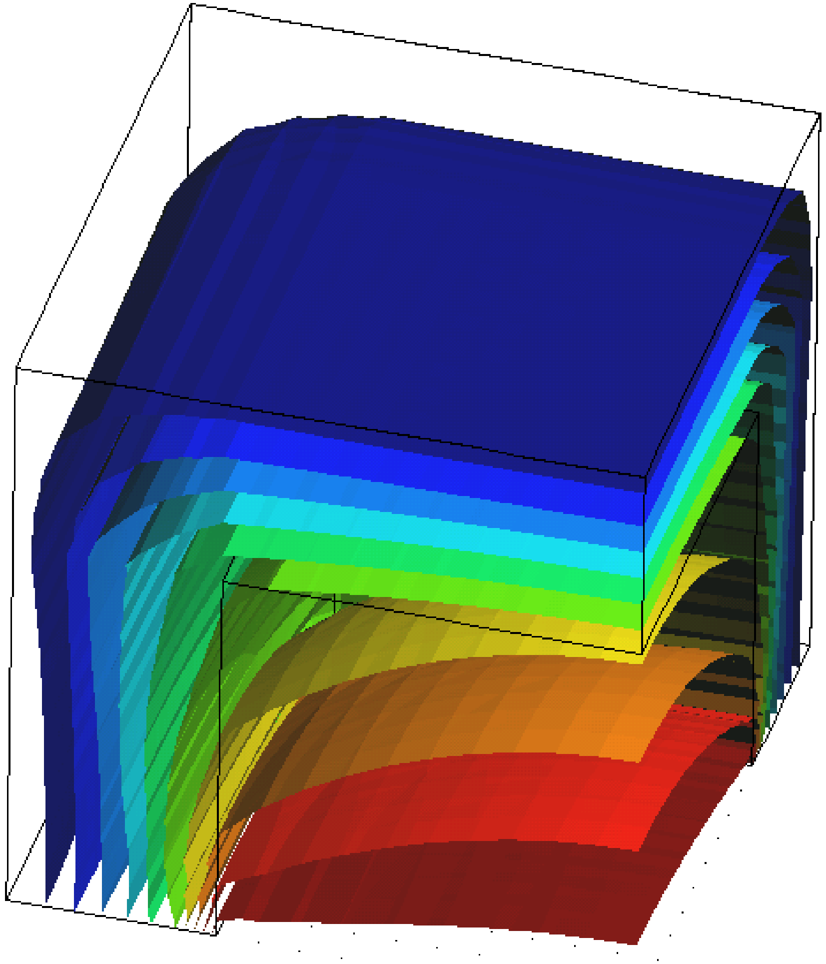
Perfectly Matched Layer (VI)

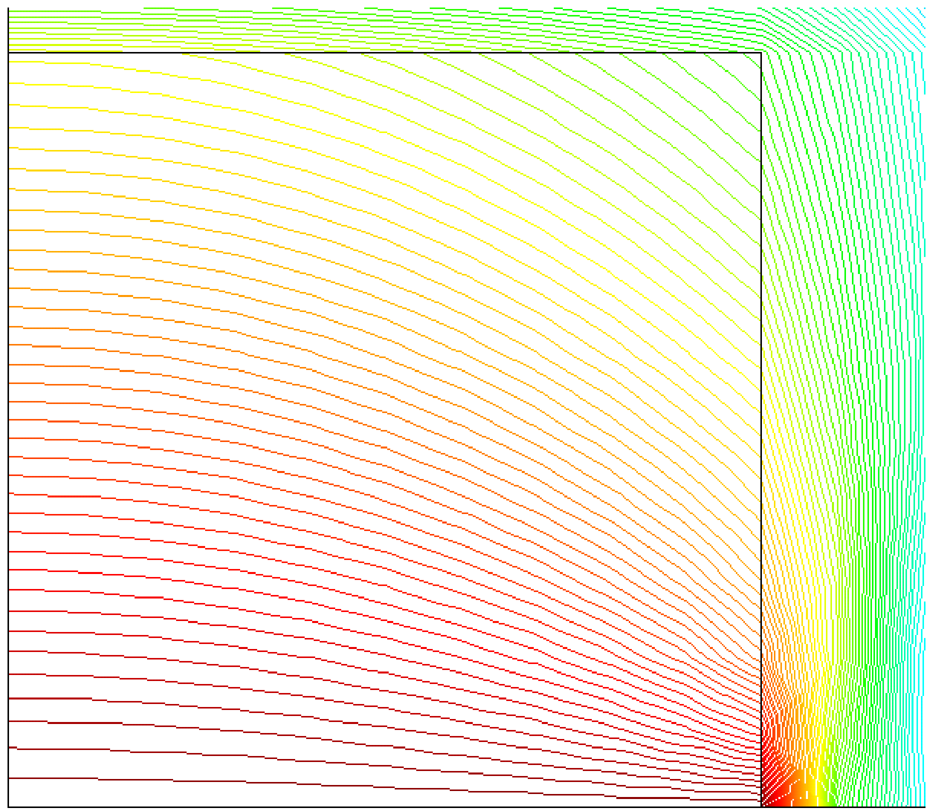
- Error plots



Perfectly Matched Layer (VII)

- 3D example:





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Perfectly Matched Layer (VIII)

- **Time domain ansatz:**

$$\int \frac{\partial w}{\partial x_1}\frac{\gamma_{x_2}\gamma_{x_3}}{\gamma_{x_1}}\frac{\partial p}{\partial x_1} + \frac{\partial w}{\partial x_2}\frac{\gamma_{x_1}\gamma_{x_3}}{\gamma_{x_2}}\frac{\partial p}{\partial x_2} + \frac{\partial w}{\partial x_3}\frac{\gamma_{x_1}\gamma_{x_2}}{\gamma_{x_3}}\frac{\partial p}{\partial x_3} + \gamma(x_1)\gamma(x_2)\gamma(x_3)\, k^2\, p = 0$$

$$\frac{a_1}{s(s+b_1)}; \quad \frac{a_2}{s+b_2}; \quad \frac{sa_3}{s+b_3}$$

$$\frac{a_1 s}{s(s+b_1)}\frac{1}{s} \qquad a_i e^{-b_i t}$$

$$c_1 s^2; \quad c_2 s; \quad c_3; \quad \frac{c_4}{s}$$

s...Laplace variable

Convolution: $\psi(x,t) = a_i e^{-b_i t} * \mathcal{L}(p(x,t))$

$$\psi(x,t) = e^{-b_i(t-t_1)}\psi(x,t_1) + a e^{-bt}\int_{t_1}^{t} e^{b\tau}\mathcal{L}(p(x,\tau))\,d\tau$$

Further Cooperations (I)

- **G. Of:**
  - **Paper on** *Fast Boundary Element Methods for Electrostatic Field Computations*
  - **FE/BE for Maxwell (eddy current case)**
    - Magnetic vector potential in solid parts (FE)

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

    - Magnetic scalar potential in air-regions (BE)

$$\mathbf{H} = -\boldsymbol{\nabla} \psi$$

We follow ideas of M. Kuhn and O. Steinbach

- **M. Liebmann**
  - **Explicit time integration algorithm** for acoustic wave equation

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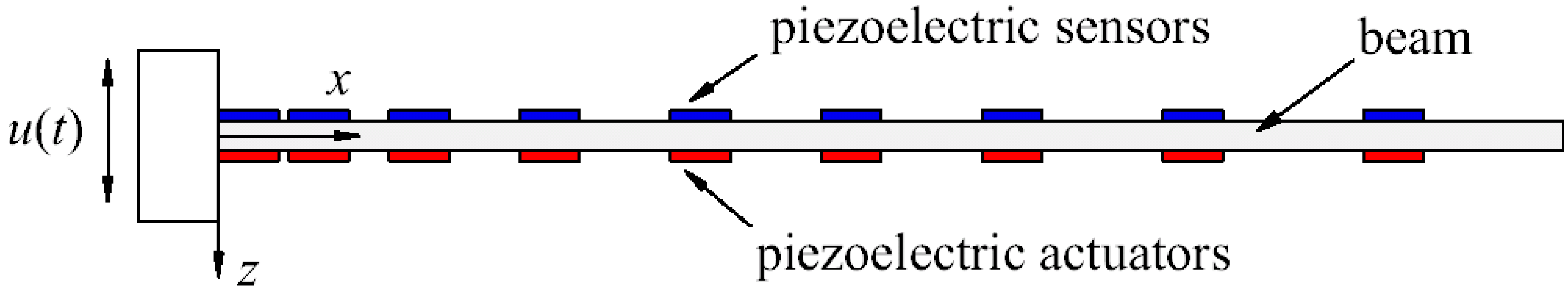
Further Cooperations (II)

- **J. Kraus:**
  - **Test of his AMG for large mechanical problems**
  - **Adapt his AMG for Maxwell** (edge element discretization)

- **M. Nader:**
  - **Control of flexible structures by piezoelectric patches**



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