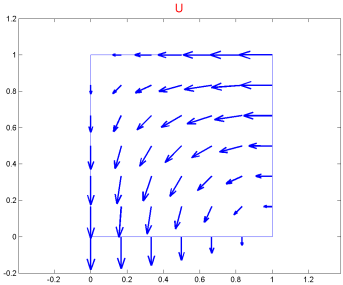
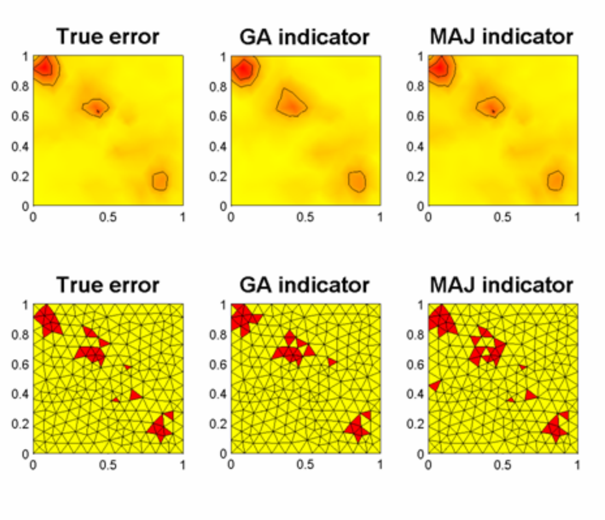
Example 2. Problem Statement.

D.Braess, R. Sarazin, "An efficient smoother for the Stokes problem". Applied Numerical Mathematics 23 (1997) 3-19

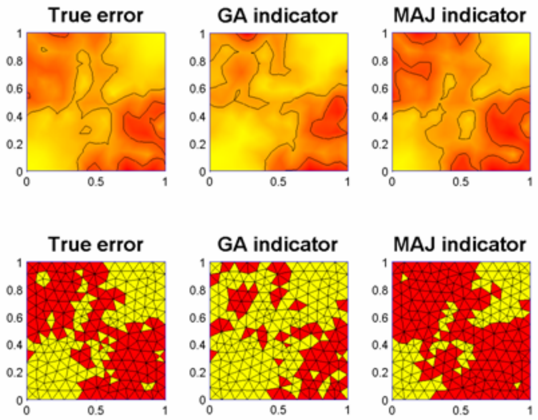
$$\Omega = (0..1) \times (0..1)$$
$$f_1 = 0 \quad f_2 = -\pi^2 \cos(\tfrac{\pi}{2} x) \cos(\tfrac{\pi}{\,} y);$$
$$u_1 = -\sin(\tfrac{\pi}{2} x) \sin(\tfrac{\pi}{2} y)$$
$$u_2 = -\cos(\tfrac{\pi}{2} x) \cos(\tfrac{\pi}{2} y)$$
$$p = \pi \cos(\tfrac{\pi}{2} x) \sin(\tfrac{\pi}{2} y)$$



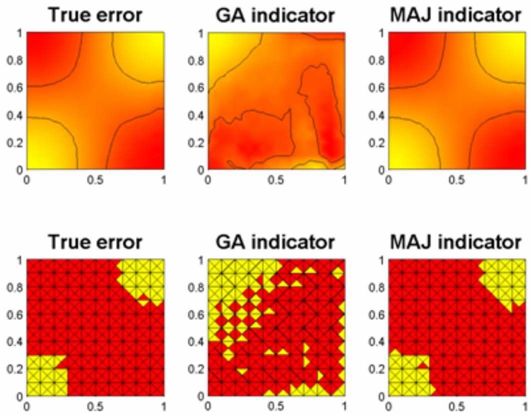
Taylor Hood elements, Penalty method



Taylor Hood elements, Uzawa algorithm



Taylor Hood elements, Hestenes-Powel algorithm



Way 3. What is the computational cost?

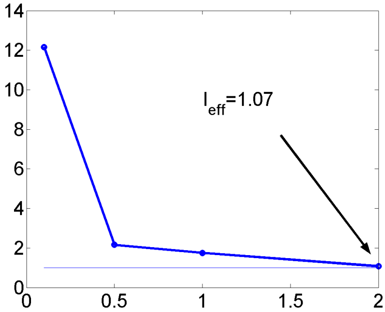
$$\nu \| \nabla(u-v) \| \leq \| \nu\nabla v - \tau \| + C_\Omega \| \operatorname{div} \tau + f - \nabla q \| + \frac{2}{\mathbb{C}_{LBB}} \nu \| \operatorname{div} v \|$$

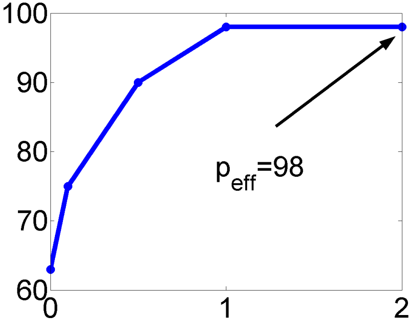
	t=0	t=0.5 TU	t=1 TU	t=2 TU
$\| \nabla(u-v) \|$	0.0014	0.0014	0.0014	0.0014
error majorant	0.036	0.003	0.0024	0.0016
primary term	0.0001	0.0015	0.0014	0.0015
reliability term	0.0356	0.0015	0.0011	0.00018
div term	1.1e-5	1.1e-5	1.1e-5	1.1e-5
I_{eff}	24	2.16	1.71	1.07
p_{eff}	63 %	90 %	98 %	98 %

Way 3. What is the computational cost?

Effectivity Index

Mesh refinement effectivity (P_{eff})





1 TU - time spend on the numerical solution of the original problem.

Way 3. What is the computational cost?

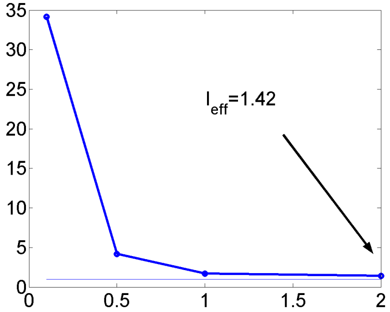
Overestimation of C_Ω

Let $C_\Omega = 3 * C_\Omega^{exact}$

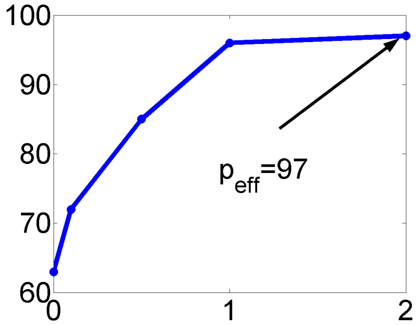
$$\nu \parallel \nabla(u-v) \parallel \leq \parallel \nu\nabla v - \tau \parallel + C_\Omega \parallel \text{div } \tau + f - \nabla q \parallel + \frac{2}{\mathbb{C}_{LBB}} \nu \parallel \text{div } v \parallel$$

	t=0	t=0.5 TU	t=1 TU	t=2 TU
$\parallel \nabla(u-v) \parallel$	0.0014	0.0014	0.0014	0.0014
error majorant	0.107	0.059	0.0041	0.00197
primary term	0.0001	0.0015	0.0015	0.0015
reliability term	0.1068	0.0044	0.003	0.0005
div term	1.1e-5	1.1e-5	1.1e-5	1.1e-5
I_{eff}	76	4.2	3.14	1.41
p_{eff}	63 %	85 %	96 %	97 %

Effectivity Index



Mesh refinement effectivity (P_{eff})

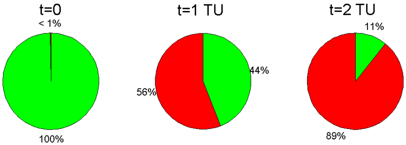


1 TU - time spend on the numerical solution of the problem.

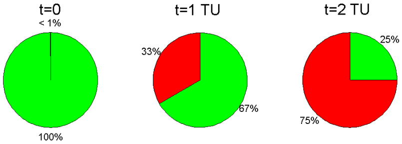
Impact of the different terms

$$\nu \| \nabla(u-v) \| \le \| \nu\nabla v - \tau \| + C_\Omega \| \operatorname{div} \tau + f - \nabla q \| + \frac{2}{\mathbb{C}_{LBB}} \nu \| \operatorname{div} v \|$$

Sharp estimation of C_Ω



Overestimation of C_Ω



Way 2. Cheap algorithm with one step retardation

Way 2. Cheap algorithm: Some adaptive refinement process:
Meshes $T_1, T_2, T_3, ...T_k, T_{k+1}, ...$ To estimate error on mesh T_k:
take $\tau_k = (\mathbb{G}\nabla v)_{k+1}$, $q = q_{k+1}$ from finer mesh.

Macro elements, algorithm with one step retardation (uniform refinement)

N	error (%)	I_{eff}
253	3.74	2.9
468	2.0	3.2
590	1.82	2.8
854	1.34	3.1
1278	0.98	3.45