MATHEMATICAL MODELING OF PERFECTLY ELASTO-PLASTIC PROBLEMS

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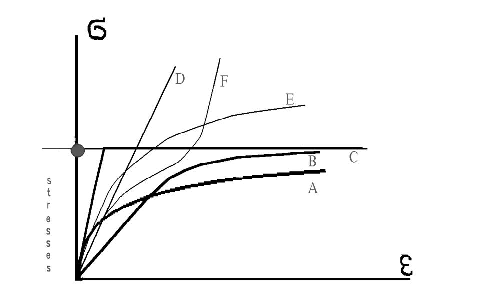
The plan

- **Constitutive relations and energy functionals.**
- **Linear growth functionals. How discontinuities in the displacements may appear?**
- **Modeling of the discontinuous solutions by special Finite Element Method.**

Stress-strain relations in plasticity and nonlinear elasticity



Stress–strain relations for various constitutive laws.

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}(\varepsilon)}$$

The group I

(A), (B) and (C) $\Rightarrow$ admissible stresses in some sense are **bounded**.

The group II

(D), (E) and (F) $\Rightarrow$ admissible stresses are **unbounded**.

In continuum media the internal energy is given by the relation

$$\mathbf{E}(\varepsilon) = \boldsymbol{\sigma}(\varepsilon) : \boldsymbol{\varepsilon} := \sum_{\mathbf{ij}} \boldsymbol{\sigma}_{\mathbf{ij}} \boldsymbol{\varepsilon}_{\mathbf{ij}} \tag{1}$$

For the group I $\mathbf{E}(\boldsymbol{\varepsilon})$ has **linear growth** with respect to components of the strain tensor $\boldsymbol{\varepsilon}$.

For the group II $\mathbf{E}(\boldsymbol{\varepsilon})$ has **superlinear growth**.

Energy variational formulation

Problem $\mathcal{P}$

. Find $\mathbf{u} \in \mathbb{V}$ such that

$$\mathbf{I}(\mathbf{u}) = \inf_{\mathbf{v}\in\mathbb{V}} \mathbf{I}(\mathbf{v}), \ \mathbf{I}(\mathbf{v}) = \int\limits_{\boldsymbol{\Omega}} \mathbf{E}(\varepsilon(\mathbf{v}))\mathbf{dx} + \ell(\mathbf{v})$$

where $\ell(\mathbf{v}) = \int\limits_{\boldsymbol{\Omega}} \mathbf{f} \cdot \mathbf{vdx} + \int\limits_{\boldsymbol{\Gamma}_2} \mathbf{F} \cdot \mathbf{vdl}$, Ω is a bounded domain in $\mathbb{R}^n$ with the boundary

$$\partial\boldsymbol{\Omega} = \boldsymbol{\Gamma}_1 \cup \boldsymbol{\Gamma}_2; \ \ \boldsymbol{\Gamma}_1 \cap \boldsymbol{\Gamma}_2 = \emptyset$$

and $\mathbb{V}$ is the space of admissible displacements.

The mathematical properties of Problem $\mathcal{P}$ for models of the groups I and II are quite different.

Group II

For any (regular enough) external data there exists a weak (generalized) solution as an element of some Sobolev space

Group I

1. For some (small) external data there exists a generalized solution in some reflexive Sobolev space.
2. For some data there is no weak solution in a Sobolev space. In this case Problem $\mathcal{P}$ is ill–posed. This problem should be relaxed and the minimizer should be understood in a very weak sense: as an element of the space $\mathrm{BD}(\Omega)$ of vector–functions whose deformations are Radon measures.
3. For other data the problem $\mathcal{P}$ is senseless because the functional J is unbounded from below.

The above differences are also lead to the following physically meaningful observation:

Solutions to problems with the energy **E** having superlinear growth have **no jumps** along lines (surfaces) inside Ω and along $\partial\Omega$.

Solutions to problems with energy **E** having linear growth **may have jumps** along some lines (surfaces).

The Deformation Theory of Perfect Plasticity gives various practically important examples of the group I models. In this theory stresses must satisfy the condition

$$\mathcal{F}(\boldsymbol{\sigma}) \leq \mathbf{0}, \tag{2}$$

The **von–Mises** condition

$$\mathcal{F}(\boldsymbol{\sigma}) = \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid - \sqrt{\mathbf{2}}\mathbf{k}_* \leq \mathbf{0}, \tag{3}$$

the **Tresca – Saint-Venant** condition

$$\mathcal{F}(\boldsymbol{\sigma}) = \max_{\mathbf{i} \neq \mathbf{j}} \mid \boldsymbol{\sigma}_{\mathbf{i}} - \boldsymbol{\sigma}_{\mathbf{j}} \mid - \sqrt{\mathbf{3}}\mathbf{k}_* \leq \mathbf{0}, \tag{4}$$

the **Drucker–Prager** condition

$$\mathcal{F}(\boldsymbol{\sigma}) = \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid + \alpha \mathbf{Sp} \boldsymbol{\sigma} - \sqrt{\mathbf{2}}\mathbf{k}_* \leq \mathbf{0}, \tag{5}$$

and the **"elliptic"** condition

$$\mathcal{F}(\boldsymbol{\sigma}) = \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid^{\mathbf{2}} + \alpha (\mathbf{Sp} \boldsymbol{\sigma})^{\mathbf{2}} - \mathbf{2}{\mathbf{k}_*}^{\mathbf{2}} \leq \mathbf{0} \tag{6}$$

Here $\boldsymbol{\sigma}_i$, $i = 1, 2, 3$ are the main stresses,
α and k_* are the material constants,
$\mathbf{Sp}\boldsymbol{\sigma}$ is the trace of $\boldsymbol{\sigma}$,
$\boldsymbol{\sigma}^D := \boldsymbol{\sigma} - \frac{1}{n}\mathbf{Sp}\boldsymbol{\sigma}\mathbf{1}$ is the deviator
and $\mathbf{1}$ denotes the unit tensor.

The relations (3), (4) are widely used for **metals** while (5) is often applied in **geomechanics**. The criterion (6) arises in some continual models of **porous media**.

The aim of our mathematical analysis is to represent **a computational technology able to answer the following two practically important questions**:

(a) **WHERE** the line (lines) of discontinuity may appear ?
(b) **WHEN** (for which values of the external data) it appears?

The approach introduced consists of the following steps:

1. We make a variational extension of Problem $\mathcal{P}$ on a wider set $\mathbb{V}^+$ which includes limits of all sequences converging in $\mathbb{V}$ and obtain a new relaxed variational problem and named $\mathcal{P}^+$.

Problem $\mathcal{P}^+$ has a solution $\mathbf{u}^+ \in \mathbb{V}^+$. This solution coincides with the minimiser of Problem $\mathcal{P}$ if the latter has a solution $\mathbf{u}$ in $\mathbb{V}$. In general the solution $\mathbf{u}^+$ to Problem $\mathcal{P}^+$ should be understood in a very weak sense – as an element of the functional space of all summable functions whose derivatives are bounded measures. For plasticity problems with the yield condition (4) the existence of a minimizer to Problem $\mathcal{P}^+$ was established in the so-called space of **bounded deformations BD**$(\mathbf{\Omega})$.

2. The mathematical analysis of the relaxed problem gives:
(a) some analytical conditions which shows where discontinuity lines may appear.
(b) new type finite element formulations adapted to the analysis of discontinuous solutions.

3. We investigate properties of the introduced finite element approximations, prove convergence and demonstrate numerical results.

It should be emphasized that the space $\mathbb{V}^+$ (unlike the space $\mathbb{V}$) contain discontinuous functions and, thus, the infimum of Problem $\boldsymbol{\mathcal{P}}^+$ may be attained on some discontinuous field of displacements. Therefore, via the analysis of Problem $\boldsymbol{\mathcal{P}}^+$ we can see how the classical (continuous) solution transforms into a weak (discontinuous) one. From the viewpoint of mechanics such transformation may be regarded as a precursor (initial phase) of **fracture**.

RELAXED VARIATIONAL PROBLEMS

The classical formulation of a boundary–value problem in the deformational plasticity theory consists of determining a stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x)$ and a displacement vector–function $\mathbf{u}$ satisfying the following system:

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \text{ in } \boldsymbol{\Omega},\ \mathbf{i} = \mathbf{1}, \ldots \mathbf{n}, \tag{7}$$

$$\boldsymbol{\sigma} \cdot \mathbf{e} = \mathbf{F} \text{ on } \boldsymbol{\Gamma}_{\mathbf{2}},\ \mathbf{u} = \mathbf{u_0} \text{ on } \boldsymbol{\Gamma}_{\mathbf{1}}, \tag{8}$$

$$\varepsilon(\mathbf{u}) = \mathbf{A}\boldsymbol{\sigma} + \boldsymbol{\lambda}, \tag{9}$$

$$\mathcal{F}(\boldsymbol{\sigma}) \leq \mathbf{0}, \tag{10}$$

$$\boldsymbol{\lambda} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leq \mathbf{0},\ \forall \boldsymbol{\tau} : \mathcal{F}(\boldsymbol{\tau}) \leq \mathbf{0}. \tag{11}$$

Here

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}}).$$

$\mathbf{A_{ijkl}}$ are the components of the elasticity tensor and $\mathbf{e}$ is a unit vector of the outward normal to Γ.

Suppose that

$$\mathbf{f} \in \mathrm{L}_2(\boldsymbol{\Omega}, \mathbb{R}^{\mathbf{n}}),\ \mathbf{F} \in \mathrm{L}_\infty(\boldsymbol{\Omega}, \mathbb{R}^{\mathbf{n}})$$
$$\mathbf{u_0} \in \mathrm{W}^1_2(\boldsymbol{\Omega}, \mathbb{R}^{\mathbf{n}}),$$

In order to give a functional statement of the problem we introduce the Lagrangian

$$\mathbf{L}(\boldsymbol{\sigma}, \mathbf{v}) = \int_{\boldsymbol{\Omega}} (\boldsymbol{\varepsilon} : \boldsymbol{\sigma} - \tfrac{1}{2}\mathbf{A}\boldsymbol{\sigma} : \boldsymbol{\sigma})\mathbf{dx} - \ell(\mathbf{v})$$

The Lagrangian $\mathbf{L}$ generates the following minimax problem

Minimax Problem

Find a pair of functions $(\sigma^*, \mathbf{v}^*) \in \mathbf{Q} \times (\mathbf{V_0} + \mathbf{u_0})$ such that

$$\mathbf{L}(\boldsymbol{\sigma}, \mathbf{v}^*) \leq \mathbf{L}(\sigma^*, \mathbf{v}^*) \leq \mathbf{L}(\sigma^*, \mathbf{v}) \qquad (12)$$

for all

$$\boldsymbol{\sigma} \in \mathbf{Q} = \{\boldsymbol{\sigma} \in \boldsymbol{\Sigma} := \mathrm{L}_2(\boldsymbol{\Omega}, \mathbb{M}_{\mathbf{s}}^{\mathbf{n}\times\mathbf{n}}) \mid \boldsymbol{\sigma} \in \mathbf{K}\}$$

and all $\mathbf{v} \in \mathbb{V} = \mathbf{V_0} + \mathbf{u_0}$, where

$$\mathbf{K} = \{\boldsymbol{\tau} \in \mathbb{M}_{\mathbf{s}}^{\mathbf{n}\times\mathbf{n}} \mid \mathcal{F}(\boldsymbol{\tau}) \leq \mathbf{0}\}$$

and

$$\mathbf{V_0} = \{\mathbf{v} \in \mathrm{W}_{\mathbf{2}}^{\mathbf{1}}(\boldsymbol{\Omega}, \mathbb{R}^{\mathbf{n}})\} \mid \mathbf{v} = \mathbf{0} \text{ on } \boldsymbol{\Gamma}_{\mathbf{1}}\}$$

.

If a saddle point of the Lagrangian **L** exists and is attained at sufficiently smooth functions σ^* and $\mathbf{v}^*$, then the latter are solutions of the classical problem (7) – (11). Conversely, if the classical problem has a solution, the latter corresponds to the saddle point of **L**.
Hence problem (7) – (11) can be investigated as a minimax problem. The following two variational problems are associated with it, namely they are the problems $\inf \sup \mathbf{L}$ and $\sup \inf \mathbf{L}$.

Problem $\mathcal{P}$ (Primal problem)

Find a vector $\mathbf{v}^* \in \mathbf{V}$ such that

$$\mathbf{I}(\mathbf{v}^*) = \inf\{\mathbf{I}(\mathbf{v}) \mid \mathbf{v} \in \mathbf{V_0} + \mathbf{u_0}\},$$

where $\mathbf{I}(\mathbf{v}) = \sup\{\mathbf{L}(\boldsymbol{\sigma}, \mathbf{v}) \mid \boldsymbol{\sigma} \in \mathbf{Q}\}$.

Problem $\mathcal{P}^*$ (Dual problem)

Find a tensor $\sigma^* \in \mathbf{Q} \cap \mathbf{M}$ such that

$$\boldsymbol{\Phi}(\sigma^*) = \sup\{\boldsymbol{\Phi}(\boldsymbol{\sigma}) \mid \boldsymbol{\sigma} \in \mathbf{Q} \cap \mathbf{M}\},$$

where $\boldsymbol{\Phi}(\boldsymbol{\sigma}) = \inf\{\mathbf{L}(\boldsymbol{\sigma}, \mathbf{v}) \mid \mathbf{v} \in \mathbf{V_0} + \mathbf{u_0}\}$.

In the above

$$\mathbf{\Phi}(\boldsymbol{\sigma}) = \int_{\Omega} (\boldsymbol{\sigma} : \varepsilon(\mathbf{u_0}) - \tfrac{1}{2}\mathbf{A}\boldsymbol{\sigma} : \boldsymbol{\sigma})\mathbf{dx} - \ell(\mathbf{u_0}),$$

$$\mathbf{I}(\mathbf{v}) = \int_{\Omega} \mathbf{E}(\varepsilon(\mathbf{v}))\mathbf{dx} - \ell(\mathbf{v})$$

and $\mathbf{E} : \mathbb{M}_{\mathbf{s}}^{\mathbf{n}\times\mathbf{n}} \rightarrow \mathbb{R}^{\mathbf{n}}$ is defined by

$$\mathbf{E}(\boldsymbol{\eta}) = \sup\{\boldsymbol{\tau} : \boldsymbol{\eta} - \tfrac{1}{2}\mathbf{A}\boldsymbol{\tau} : \boldsymbol{\tau} \mid \boldsymbol{\tau} \in \mathbf{K}\}.$$

For the von Mises condition and for an isotropic body

$$\mathbf{A_{ijkl}} = \boldsymbol{\lambda}\delta_{\mathbf{ij}}\delta_{\mathbf{kl}} + \boldsymbol{\mu}(\delta_{\mathbf{ik}}\delta_{\mathbf{jl}} + \delta_{\mathbf{il}}\delta_{\mathbf{jk}})$$

one has (see e.g. G.Duvaut, J.-L. Lions. Les inéquations en méchanique et physique)

$$\mathbf{E}(\varepsilon) = \begin{cases} \boldsymbol{\mu} \mid \varepsilon \mid^{\mathbf{2}} & \text{if } \mid \varepsilon \mid \leq \mathbf{t}_* := \frac{\mathbf{k}_*}{\sqrt{\mathbf{2}}\mu} \\ K_0(\sqrt{2}(\mid \varepsilon \mid - \frac{\mathbf{k}_*}{\mathbf{2}\boldsymbol{\mu}}) & \text{otherwise} \end{cases}$$

where $\mathbf{K_0} = \mathbf{3}\boldsymbol{\lambda} + \mathbf{2}\boldsymbol{\mu}$.

The major mathematical difficulty is that the above functional **I** is non-coercive on a reflexive Sobolev space (as e.g., $\mathbf{H^1}$) and coercive only on a non-reflexive functional space, so that it is impossible to guarantee existence of a minimizer to Problem $\mathcal{P}$. From the physical viewpoint this means that a minimizing sequence may converge to a function which does not belong to the original "admissible space $\mathbb{V}$".
Therefore, a relaxation (lower semicontinuous extension) of Problem $\mathcal{P}$ should be carried out in order to obtain a well–posed variational posing. Getting this relaxation for a variational problem related to Hencky plasticity model was the point of intensive investigations at the beginning of 80-s. It was established that a "generalized" solution to this problem exists in the extended space which is

$$\mathbf{BD}(\boldsymbol{\Omega}) := \{\mathbf{u} \in \mathsf{L}_\mathbf{1}(\boldsymbol{\Omega}; \mathbb{R}^\mathbf{n}) \mid \|\varepsilon_{\mathbf{ij}}\|_{\mathbf{M}(\boldsymbol{\Omega})} \leq +\infty\}$$

BD($\boldsymbol{\Omega}$) contains summable vector-valued functions **u** whose deformations $\varepsilon(\mathbf{u})$ are bounded Radon measures.

$$\|\mu\|_{\mathbf{M}(\boldsymbol{\Omega})} := \sup_{\phi \in \mathbf{C}_0^\infty(\boldsymbol{\Omega}),\, |\phi(\mathbf{x})| \leq 1} \int_{\boldsymbol{\Omega}} \boldsymbol{\mu} \phi \mathbf{dx}$$

This space was introduced in Suquet. P. (1978). Existence et regularite des solutions des equations de la plasticity parfaite. *C. R. Acad. Sc., Paris, Serie D, 1201–1204*, and its properties were investigated in the papers of R. Temam, G. Strang, R. Kohn, G. Anzellotti and others, see also
Temam R. (1983). *Problemes mathématiques en plasticité*.

Another definition of the "generalized" solution based on an extended minimax representation was given in
S. I. Repin, G. A. Seregin (1995). Existence of a weak solution of the minimax problem arising in Coulomb–Mohr plasticity. Amer. Math. Soc. Transl.
where it was proved that for plasticity models governed Drucker–Prager condition a "generalized" solution exists in some $\mathbb{V}^+ \subset \mathbb{D}(\Omega)$.

However, in general Problem $\boldsymbol{\mathcal{P}}^+$ is rather abstract to be directly applied for creating numerical methods. For this purpose in the cited above paper we also represented the so–called **partially extended** Problem $\hat{\boldsymbol{\mathcal{P}}}$, in which the extended functional is defined on a set $\hat{\mathbb{V}}$ such that

$$\mathbf{V_0} + \mathbf{u_0} \subset \hat{\mathbb{V}} \subset \mathbb{V}^+$$

and $\hat{\mathbb{V}}$ contains a selection of certain discontinuous displacement fields.

This problem also preserves the exact lower bound of Problem $\mathcal{P}$, so that we can replace Problem $\mathcal{P}$ by

Problem $\hat{\mathcal{P}}$ (partially relaxed)

Find $\hat{\mathbf{u}} \in \hat{\mathbb{V}}$ such that

$$\hat{\mathbf{I}}(\hat{\mathbf{u}}) = \inf\{\hat{\mathbf{I}}(\mathbf{v}) \mid \mathbf{v} \in \hat{\mathbb{V}}\}$$

Properties of the above problems are as follows:

$$\left.\begin{array}{l}\textbf{Problem}\,\mathcal{P}\\ \textbf{Problem}\,\hat{\mathcal{P}}\\ \textbf{Problem}\,\mathcal{P}^+\end{array}\right\} \Leftrightarrow \quad \text{have the same dual problem}\,\mathcal{P}^*$$

$$\inf\mathcal{P} = \inf\hat{\mathcal{P}} = \inf\mathcal{P}^+ = \sup\mathcal{P}^*$$

Below we consider the case when the set $\hat{\mathbb{V}}$ consists of functions that have jumps on the line(s) $\mathbf{\Gamma}$.

$$\hat{\mathbf{I}}(\mathbf{v}) = \int\limits_{\mathbf{\Omega}} \mathbf{g}(\varepsilon(\mathbf{v}))\mathbf{dx} - \ell(\mathbf{v}) + \mathbf{G}_{\mathbf{\Gamma}}(\mathbf{v}^{+} - \mathbf{v}^{-}),$$

where

$$\mathbf{G}_{\mathbf{\Gamma}}(\mathbf{v}^{+} - \mathbf{v}^{-}) = \sup_{\tau \in \mathbf{K}} \int\limits_{\mathbf{\Gamma}} \tau : \mathbf{S}(\boldsymbol{\nu}, \mathbf{v}^{+} - \mathbf{v}^{-})\, \mathbf{dl},$$

$$\mathbf{S}(\mathbf{a}, \mathbf{b}) = \tfrac{1}{2}(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}) \;\; \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^{\mathbf{n}}$$

and $\otimes$ denotes the tensor product in $\mathbb{R}^n$. In the simplest case the discontinuity line is a singleton, so that $\mathbf{\Gamma} = \{\gamma\}$,

$$\hat{\mathbb{V}} = \;\; \{\mathbf{v} = \mathbf{v}^{-} \text{ in } \mathbf{\Omega}^{-}, \, \mathbf{v} = \mathbf{v}^{+} \text{ in } \mathbf{\Omega}^{+}, \; \mathbf{v_i} \in \; \mathbf{V_0} + \mathbf{u_0} \, , \, \mathbf{i} = \mathbf{1}, \mathbf{2}\}$$

and $\boldsymbol{\nu}$ is the unit vector normal to the line γ which separates the domains Ω^{-} and Ω^{+} ($\boldsymbol{\nu}$ is the outer normal for Ω^{-}).

By

$$[\mathbf{v}(\mathbf{x})] = \mathbf{v}^{+}(\mathbf{x}) - \mathbf{v}^{-}(\mathbf{x}) \;\; \mathbf{x} \in \gamma$$

we denote the "jump" of **v** at the point x.
The problem $\hat{\mathcal{P}}$ also can be formulated as a minimax problem for the Lagrangian

$$\mathbf{L}_{\Gamma}(\mathbf{v}, \tau) = \mathbf{L}(\mathbf{v}, \tau) + \mathbf{G}_{\Gamma}([\mathbf{v}])$$

The consideration of a minimax condition

$$\mathbf{L}_{\Gamma}(\boldsymbol{\sigma}, \hat{\mathbf{v}}^{*}) \leq \mathbf{L}_{\Gamma}(\sigma^{*}, \hat{\mathbf{v}}^{*}) \leq \mathbf{L}_{\Gamma}(\sigma^{*}, \hat{\mathbf{v}}), \quad \hat{\mathbf{v}} \in \hat{\mathbb{V}} \tag{13}$$

yields the following **necessary condition for a discontinuous solution**

Theorem[S.R. 1991]. Suppose that Problem $\hat{\mathcal{P}}$ has a solution $\hat{\mathbf{v}}$. Then the pair of functions $(\sigma^*, \hat{\mathbf{v}})$, where σ^* is a solution of Problem $\mathcal{P}^*$ is a saddle point of the extended Lagrangian $\hat{L}$ on $\mathbf{\Sigma} \times \hat{\mathbb{V}}$ and the inequality

$$\int_{\mathbf{\Gamma}} (\sigma^* - \tau) : \mathbf{S}(\boldsymbol{\nu}, [\mathbf{v}]))\, \mathbf{ds} \leq \mathbf{0} \quad \forall \boldsymbol{\tau} \in \mathbf{K}. \tag{14}$$

Corollary. The above assertion gives the necessary conditions that must be satisfied along the discontinuity line. If $\mathcal{F}$ is a differentiable function, and $\hat{\mathbf{v}}$ is continuous in $\Omega \setminus \mathbf{\Gamma}$ then (14) means that

$$\mathbf{S}(\boldsymbol{\nu}, [\hat{\mathbf{v}}]) = \boldsymbol{\lambda} \mathbf{g}_{\mathbf{ij}}(\sigma^*), \ \boldsymbol{\lambda} > \mathbf{0}, \tag{15}$$

where $\mathbf{g}_{\mathbf{ij}} = \frac{\partial \mathcal{F}(\sigma^*)}{\partial \boldsymbol{\sigma}_{\mathbf{ij}}}$.

By virtue of (14) we arrive at the conclusion that a jump at some point $\mathbf{x} \in \mathbf{\Gamma}$ (i.e., $[\hat{\mathbf{v}}(\mathbf{x})] \neq \mathbf{0}$)

is possible only if the system

$$\begin{aligned} \boldsymbol{\nu} \otimes [\hat{\mathbf{v}}] + [\hat{\mathbf{v}}] \otimes \boldsymbol{\nu} = \mathbf{g_{ij}}(\boldsymbol{\sigma}^*), \\ \mid \boldsymbol{\nu} \mid = \mathbf{1} \\ \mathcal{F}(\boldsymbol{\sigma}^*(\mathbf{x})) = \mathbf{0} \end{aligned} \tag{16}$$

has a nontrivial solution at this point.

It should be emphasized that in general the above algebraic system has nontrivial solutions only **for some specific** $\boldsymbol{\sigma}^*$.

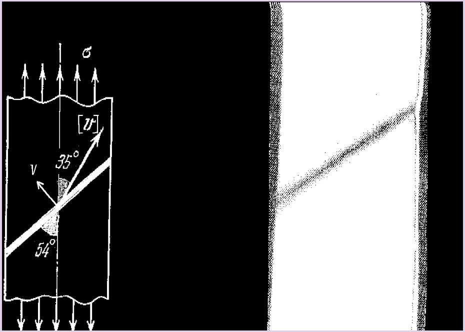
This yields some interesting conclusions concerning the sets

- **(a) where "jumps" (= initial fracture points?) may arise,**
- **(b) where "jumps" will never arise (safe points).**

This information may be used in the numerical modeling when the stress set is computed by a certain method. By the above criteria we can not only find the "plastic zone", but also mark out especially dangerous subzones in it where discontinuities may arise first of all.

By the analysis of the above "necessary discontinuity conditions" various interesting conclusions were obtained. In particular, it was shown that for 3-D elasto–plastic problems with von Mises condition the corresponding "dangerous" stresses form 6 parallel lines on the yield surface which is a cylinder in the space of main stresses.
It was also shown that for 3-D axially symmetric problems a crack may approach the lateral area at an angle of 54°.

Below we give only one example for a plane stress problem with von Mises yield condition. Here, $\boldsymbol{\nu}$ and the "jump"–vector $\mathbf{v}$ which are nontrivial solutions of (17) are shown at the left hand side and the physical experiment is at the right (slip–line is taken from L. M. Kachanov, Plasticity theory, M. Nauka, 1969).



Another necessary conditions that comes from the variational posing $\hat{\mathcal{P}}$ are imposed on the yield function $\mathcal{F}$. Prior giving them, however, we have to represent the explicit form of the functional $G_{\mathbf{\Gamma}}$ for the Drucker-Prager yield condition (6). This functional is

$$\mathbf{G}_{\mathbf{\Gamma}}([\mathbf{v}]) = \int\limits_{\mathbf{\Gamma}} \mathbf{g_0}(\mathbf{S}(\boldsymbol{\nu}, [\mathbf{v}]))\mathbf{dl}$$

where

$$\mathbf{g_0}(\mathbf{S}(\boldsymbol{\nu}, [\mathbf{v}])) = \begin{cases} \frac{\sqrt{2}\mathbf{k}_*}{\boldsymbol{\alpha}\mathbf{n}}[\mathbf{v}]_{\boldsymbol{\nu}} & \text{if } [\mathbf{v}]_{\boldsymbol{\nu}} \geq \mathbf{0} \text{ and } \beta[\mathbf{v}]^2_{\boldsymbol{\nu}} + [\mathbf{v}]^2_{\tau} \leq \mathbf{0}, \\ +\infty & \text{otherwise.} \end{cases}$$

Here

$$[\mathbf{v}]_{\boldsymbol{\nu}} = [\mathbf{v}] \cdot \boldsymbol{\nu}, \quad [\mathbf{v}]_{\tau} = [\mathbf{v}] - [\mathbf{v}]_{\boldsymbol{\nu}},$$
$$\beta = \tfrac{2}{\mathbf{n}^2}\frac{\boldsymbol{\alpha}^2 - \boldsymbol{\alpha}_*^2}{\boldsymbol{\alpha}^2\boldsymbol{\alpha}_*^2}, \qquad \boldsymbol{\alpha}_* = \mathbf{1}/\sqrt{\mathbf{n}^2 - \mathbf{n}}$$

$$\mathrm{tr}\mathbf{S}([\mathbf{v}],\nu) = [\mathbf{v}]\cdot\nu, \quad |\ \mathbf{S}^{\mathbf{D}}([\mathbf{v}],\nu)\ |^2 = \frac{\mathbf{n}-\mathbf{1}}{\mathbf{n}}\ |\ [\mathbf{v}]_\nu\ |^2 + \frac{\mathbf{1}}{\mathbf{2}}\ |\ [\mathbf{v}]_\tau\ |^2$$

The condition

$$|\ \mathbf{S}^{\mathbf{D}}\ | \leq \frac{\mathbf{1}}{\mathbf{n}\alpha}$$

can be replaced by the two inequalities

$$|\ \mathbf{S}^{\mathbf{D}}\ |^2 - \frac{\mathbf{1}}{\mathbf{n}^2\alpha^2}\ |\ \mathrm{tr}\mathbf{S}\ |^2 \leq \mathbf{0}, \ \mathrm{tr}\mathbf{S} \geq \mathbf{0}$$

The first of them can be rewritten as

$$\frac{\mathbf{2}}{\mathbf{n}}\left(\mathbf{n}-\mathbf{1}-\frac{\mathbf{1}}{\mathbf{n}\alpha^2}\right)\ |\ [\mathbf{v}]_\nu\ |^2 + \frac{\mathbf{1}}{\mathbf{2}}\ |\ [\mathbf{v}]_\tau\ |^2 \leq \mathbf{0}$$

It is easy to see that for

$$\boldsymbol{\alpha} > \ \boldsymbol{\alpha}_* \tag{17}$$

no jumps may arise! Indeed, the condition

$$\beta[\mathbf{v}]^2_{\boldsymbol{\nu}} + [\mathbf{v}]^2_{\tau} \leq \mathbf{0}$$

where

$$\beta = \frac{\mathbf{2}}{\mathbf{n}^2}\frac{\boldsymbol{\alpha}^2 - \boldsymbol{\alpha}^2_*}{\boldsymbol{\alpha}^2\boldsymbol{\alpha}^2_*}$$

cannot be fulfilled. Therefore, in such a case,

$$\mathbf{g_0}(\mathbf{S}([\mathbf{v}], \boldsymbol{\nu}) = \mathbf{0} \text{ if } [\mathbf{v}] = \mathbf{0},$$
$$\mathbf{g_0}(\mathbf{S}([\mathbf{v}], \boldsymbol{\nu}) = +\infty \text{ if } [\mathbf{v}] \neq \mathbf{0}.$$

Thus, the analysis of the relaxed problem implies an interesting conclusion

For plasticity problems with the yield criterion

$$\mathcal{F}(\mathbf{Sp}\boldsymbol{\sigma}, \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid) \leq \mathbf{0}$$

solutions containing discontinuities along certain lines (surfaces) can occur only when the yield function $\mathcal{F}$ satisfies the condition

$$\boxed{\mathcal{F}(\mathbf{Sp}\boldsymbol{\sigma}, \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid) \geq \mid \boldsymbol{\sigma}^{\mathbf{D}} \mid + \boldsymbol{\alpha}_* \mathbf{Sp}\boldsymbol{\sigma} - \mathbf{b},}$$

where b is some positive constant and

$$\boldsymbol{\alpha}_* = \begin{cases} 1/\sqrt{2} & \text{if } n = 2 \\ 1/\sqrt{6} & \text{if } n = 3. \end{cases}$$

FINITE ELEMENT APPROXIMATIONS OF DISCONTINUOUS SOLUTIONS

When using numerical methods based on *relaxed* posings there arise the following two groups of subproblems:
A.) MATHEMATICAL

1. *To represent new finite element approximations which explicitly describe discontinuous displacement fields,*
2. *To prove convergence of these FEM methods and to establish a priori rate convergence estimates,*
3. *To propose effective numerical methods for solving the corresponding finite dimensional problems and to prove their convergence.*

B.) COMPUTATIONAL

4. *To testify the above computational technology on a number of model problems,*
5. *To use this technology for numerical modelling of various industrial objects and to compare obtained results with the ones from actual (physical) experiments*

Item 1. New types of the finite–element approximations based on discontinuous approximations of the set $\hat{\mathbb{V}}$ have been proposed. They are *conforming* approximations of this set and their particular form depends on the chosen yield function $\mathcal{F}$.
See, e.g. S. R. (1988-1996)
Item 2. Methods of getting error estimates for plasticity problems based on DUALITY THEORY methods are presented in:

Rate convergence **a priori** error estimates
S. R. Errors of finite element methods for perfectly elasto-plastic problems. *Mathematical Models and Methods in Applied Sciences*, 6 (1996).

Item 3. Assume that $\mathbf{n} = \mathbf{2}$, Ω is a polygonal domain decomposed into a set of simplexes T_h of the diameter h. $\hat{\mathbb{V}}^h$ is a set of piecewise affine functions discontinuous along the interelement boundaries.

Problem $\hat{\mathcal{P}}^h$:

Find $w_h \in \hat{\mathbb{V}}^h \in V_0 + u_0$ **such that**

$$\hat{\mathbf{I}}_{\mathbf{h}}(\mathbf{w}_{\mathbf{h}}) = \inf \left\{ \hat{\mathbf{I}}_{\mathbf{h}}(\mathbf{v}_{\mathbf{h}}) \mid \mathbf{v}_{\mathbf{h}} \in \hat{\mathbb{V}}^{\mathbf{h}} \right\}. \tag{18}$$

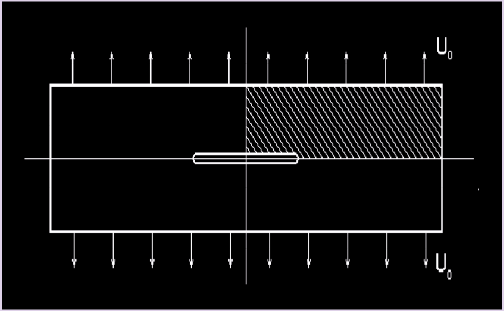
$$\hat{\mathbf{I}}_{\mathbf{h}}(\mathbf{v}_{\mathbf{h}}) = \sum_{\mathbf{i}=1}^{\mathbf{N}} \int_{\mathbf{T}_{\mathbf{h}}^{\mathbf{i}}} \mathbf{g}(\varepsilon(\mathbf{v}_{\mathbf{h}}^{\mathbf{i}}))\,\mathbf{dx} + \mathbf{k}_* \int_{\partial\boldsymbol{\Omega}} \mathbf{S}(\nu, \mathbf{u}_0 - \mathbf{v}_{\mathbf{h}})\,\mathbf{dl} - \ell(\mathbf{v}_{\mathbf{h}}) +$$

$$+ \sum^{\mathbf{N}} \int \mathbf{S}(\nu, \mathbf{v}_{\mathbf{h}}^{\mathbf{i}} - \mathbf{v}_{\mathbf{h}}^{\mathbf{j}})\mathbf{dl}$$

Here ν is the unit normal to the common face $\mathbf{r}(\mathbf{T}_h^i, \mathbf{T}_h^j)$ of the two simplexes $\mathbf{T}_h^i$, $\mathbf{T}_h^j$. This is a minimization problem of a **nondifferentiable functional of high dimensionality**. The minimization method for these type functionals is suggested see the list below.

Item 4.

Example The model plain strain problem for tension of the rectangular domain with a six-corner crack with free lateral sides.



On the next figure we represent the typical curve depicting the difference $\Delta = \mathbf{I}(\mathbf{v_h}) - \hat{\mathbf{I}}(\hat{\mathbf{v}}_\mathbf{h})$ computed for the standard piecewise–affine continuous approximations $\mathbf{v_h}$ of Problem $\mathcal{P}$ and discontinuous approximations $\hat{\mathbf{v}}_h$ of Problem $\hat{\mathcal{P}}^h$.

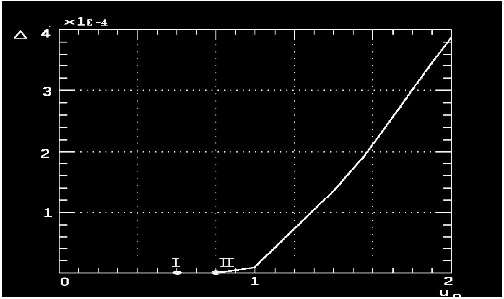
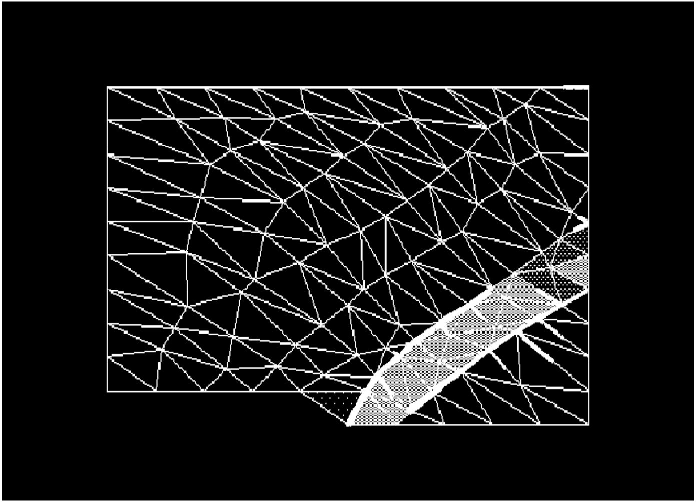
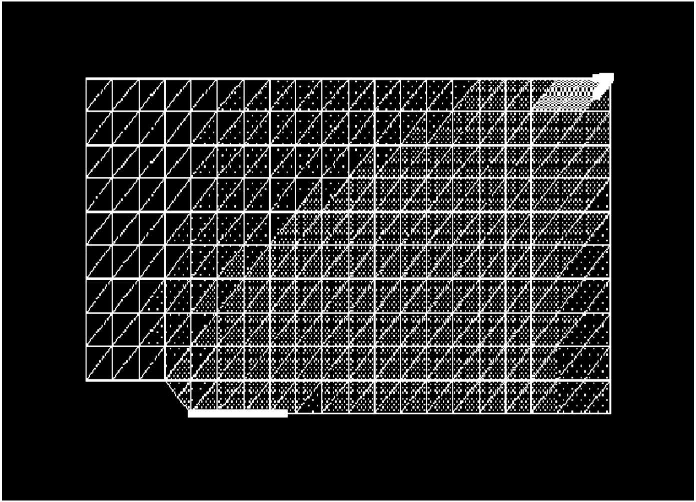
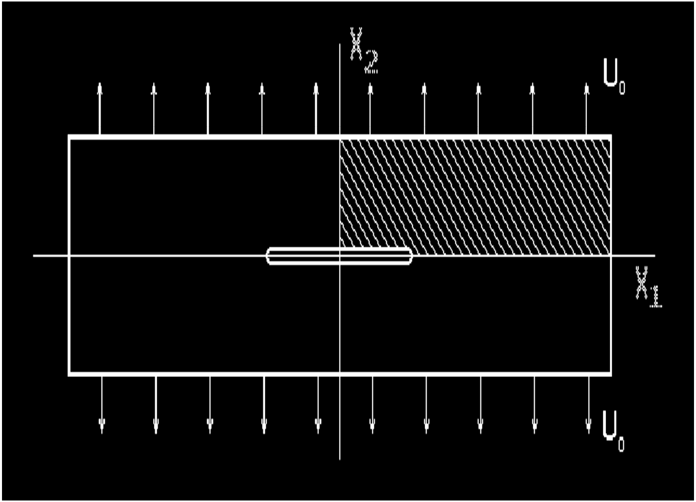


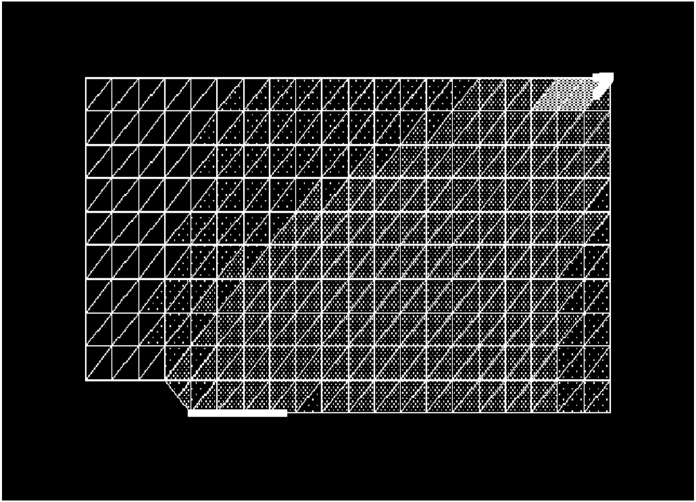
Figure 4: The difference $\Delta = \mathbf{I}(\mathbf{v_h}) - \hat{\mathbf{I}}(\hat{\mathbf{v_h}})$.

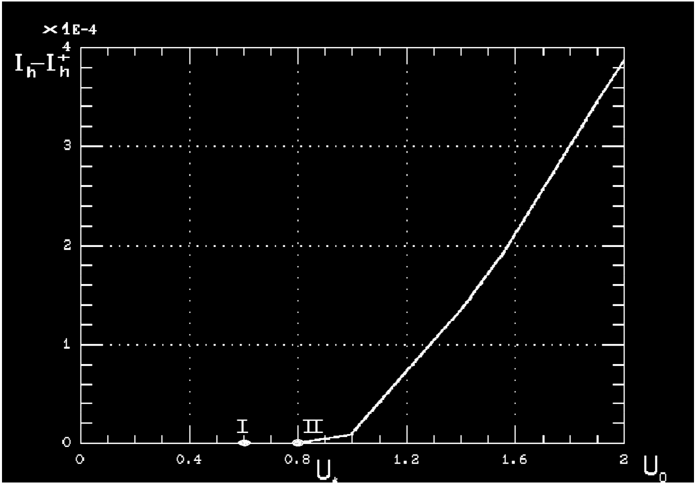
CONCLUSION: Standard FEM is able to give approximate configuration of the plastic zone only. In opposite, our method shows the location of discontinuities and, thus, provide an important information to be used in the strength analysis.






X2
U0
X1
U0




×1E-4
$I_h^- - I_h^+$
0
1
2
3
4
0
0.4
0.8
1.2
1.6
2
U_*
U_0
I
II

Some additional references

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