LECTURE NOTES

Domain Decomposition Methods

Sergey Nepomnyaschikh

1 Domain Decomposition

Let Ω be L-shaped domain. Decompose Ω into two rectangles Ω_1 and Ω_2 with the common boundary γ. Consider the following p.d.e.

$$-\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on} \Gamma.$$

Let $H_0^1(\Omega) = \{u \in H^1(\Omega) \,|\, u(x) = 0,\, x \in \Gamma\}$. Then the weak formulation is : Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u, \nabla v) d\Omega = \int_\Omega f v \, d\Omega \;, \forall v \in H_0^1(\Omega).$$

Assume that Ω^h be uniform triangulation. Let $H_h(\Omega) = \{u^h \in H_0^1(\Omega) \,|\, u^h =$ PW-linear$\}$. For the function $u^h \in H_h(\Omega)$, it can be identified as a vector $\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$, where $u_i = u^h(z_i)$.

Define a bilinear form A by

$$A(\bar{u}, \bar{v}) = \int_\Omega (\nabla u^h, \nabla v^h) d\Omega.$$

Then the weak formulation can be expressed as a algebraic(matrix) form.

$$A\bar{u} = \bar{f}.$$

The vector $\bar{u}$ can be decomposed into three groups, that is, $\bar{u} = \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$, where $\bar{u}_0$, $\bar{u}_1$, and $\bar{u}_2$ are corresponding to γ, Ω_1, and Ω_2, respectively.

$$A\bar{u} = \begin{bmatrix} A_0 & A_{01} & A_{02} \\ A_{10} & A_1 & 0 \\ A_{20} & 0 & A_2 \end{bmatrix} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} \bar{f}_0 \\ \bar{f}_1 \\ \bar{f}_2 \end{bmatrix} = \bar{f}.$$

Observe that A_i corresponds to the Dirichlet problem in Ω_i, that is, $A_i \longleftrightarrow -\Delta_{\Omega_i}$, for $i = 1, 2$.
From the second and the third equation, we obtain

$$\bar{u}_1 = A_1^{-1} \bar{f}_1 - A_1^{-1} A_{10} \bar{u}_0,$$

$$\bar{u}_2 = A_2^{-1}\bar{f}_2 - A_2^{-1}A_{20}\bar{u}_0.$$

Substitute into the first equation to get

$$(A_0 - A_{01}A_1^{-1}A_{10} - A_{02}A_2^{-1}A_{20})\bar{u}_0 = \bar{f}_0 - A_{01}A_1^{-1}\bar{f}_1 - A_{02}A_2^{-1}\bar{f}_2.$$

Let

$$\begin{aligned} S &= A_0 - A_{01}A_1^{-1}A_{10} - A_{02}A_2^{-1}A_{20} \\ \phi &= \bar{u}_0 \\ \psi &= \bar{f}_0 - A_{01}A_1^{-1}\bar{f}_1 - A_{02}A_2^{-1}\bar{f}_2. \end{aligned}$$

S is called by Schur-complement matrix and we have to solve the equation

$$S\phi = \psi.$$

If we can find a preconditioner Σ for S, then the solution ϕ is obtained by a iterative method, e.g.

$$\Sigma(\phi^{k+1} - \phi^k) = -\tau_k(S\phi^k - \psi).$$

Lemma 1.1 *If* $\|\phi^n - \phi\|_S = \epsilon$ *and* $\bar{u}_0^n = \varphi^n, \quad \bar{u}_1^n = A_1^{-1}(\bar{f}_1 - A_{10}\phi^n),\ \bar{u}_2^n = A_2^{-1}(\bar{f}_2 - A_{20}\phi^n)$*, then* $\|\bar{u}^n - \bar{u}\|_A = \epsilon$.

Proof.

$$\begin{aligned} \|\bar{u}^n - \bar{u}\|_A^2 &= (A(\bar{u}^n - \bar{u}), \bar{u}^n - \bar{u}) \\ &= \left(\begin{bmatrix} A_0 & A_{01} & A_{02} \\ A_{10} & A_1 & 0 \\ A_{20} & 0 & A_2 \end{bmatrix} \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} A_0\bar{u}_0^n + A_{01}\bar{u}_1^n + A_{02}\bar{u}_2^n - \bar{f}_0 \\ A_{10}\bar{u}_0^n + A_1\bar{u}_1^n - \bar{f}_1 \\ A_{20}\bar{u}_0^n + A_2\bar{u}_2^n - \bar{f}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} S\bar{u}_0^n - \psi \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \bar{u}_0^n - \bar{u}_0 \\ \bar{u}_1^n - \bar{u}_1 \\ \bar{u}_1^n - \bar{u}_2 \end{bmatrix} \right) = \begin{pmatrix} S\bar{u}_0^n - \psi, & \varphi^n - \varphi \\ 0, & 0 \\ 0, & 0 \end{pmatrix} \\ &= \epsilon. \end{aligned}$$

□

There are some interesting facts about S. The first is that though, S is an interface problem, it is closely related to the entire problem. The second is that the quadratic form $(S\phi, \phi)$ is equivalent to some trace norm.

$$\begin{aligned} (A\bar{u}, \bar{u}) &= \int_\Omega |\nabla u^h|^2 \, d\Omega \\ &= \int_{\Omega_1} |\nabla u^h|^2 \, d\Omega_1 + \int_{\Omega_2} |\nabla u^h|^2 \, d\Omega_2 \\ &= \left(A^{(1)} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix}, \begin{bmatrix} \bar{u}_0 \\ \bar{u}_1 \end{bmatrix} \right) + \left(A^{(2)} \begin{bmatrix} \bar{u}_0 \\ \bar{u}_2 \end{bmatrix}, \begin{bmatrix} \bar{u}_0 \\ \bar{u}_2 \end{bmatrix} \right), \end{aligned}$$

where $A^{(i)} = \begin{bmatrix} A_0^{(i)} & A_{0i} \\ A_{i0} & A_i \end{bmatrix}$.
$A^{(i)}$, $i = 1, 2$, is just the discrete Laplacian $-\Delta_{\Omega_i}$, which satisfies the Dirichlet condition on $\partial\Omega_i \setminus \gamma$ and the Neumann condition on γ.
Let $S_i = A_0^{(i)} - A_{i0}A_i^{-1}A_{i0}$. Then $S = S_1 + S_2$ and $A_0 = A_0^{(1)} + A_0^{(2)}$.

$$\begin{aligned}
&\inf_{u_1} \left(A^{(1)} \begin{bmatrix} \phi \\ u_1 \end{bmatrix}, \begin{bmatrix} \phi \\ u_1 \end{bmatrix} \right) \\
=&\inf_{u_1}((A_0^{(1)}\phi, \phi) + (A_1 u_1, u_1) + 2(A_{10}\phi, u_1)) \\
=&(A_0^{(1)}\phi, \phi) + \inf_{u_1}((A_1 u_1, u_1) - 2(-A_{10}\phi, u_1)).
\end{aligned}$$

The quadratic form $(A_1 u_1, u_1) - 2(-A_{10}\phi, u_1)$ has its minimum at $A_1 u_1 = -A_{10}\phi$, that is, $u_1 = -A_1^{-1}A_{10}\phi$. So

$$\begin{aligned}
&\inf_{u_1} \left(A^{(1)} \begin{bmatrix} \phi \\ u_1 \end{bmatrix}, \begin{bmatrix} \phi \\ u_1 \end{bmatrix} \right) \\
=&(A_0^{(1)}\phi, \phi) + (A_{10}\phi, A_1^{-1}A_{10}\phi) - 2(A_{10}\phi, A_1^{-1}A_{10}\phi) \\
=&(A_0^{(1)}\phi, \phi) - (A_{01}A_1^{-1}A_{10}\phi, \phi) \\
=&(S_1\phi, \phi).
\end{aligned}$$

Hence we obtain the following.

$$(S\phi, \phi) = \inf_{u_h \in H_h(\Omega_1), u_h|_\gamma = \phi_h} |u^h|^2_{H^1(\Omega_1)} + \inf_{u_h \in H_h(\Omega_2), u_h|_\gamma = \phi_h} |u^h|^2_{H^1(\Omega_2)}.$$

In fact, the infimum occurs at u^h which solves the discrete Laplacian problem :

$$\begin{aligned}
-\Delta_h u^h &= 0, \text{ in } \Omega_i \\
u^h &= 0, \text{ on } \partial\Omega_i \setminus \gamma \\
u^h &= \phi^h, \text{ on } \gamma
\end{aligned}$$

Then we obtain the following theorem.

Theorem 1.1 *Suppose that there exist c_1 and c_2 such that for any $u_i^h \in H_h(\Omega_i)$ with $u_i^h = \phi^h$ on γ,*

$$\|\phi^h\|_{H_h^{\frac{1}{2}}(\partial\Omega_i)} \le c_1 \|u_i^h\|_{H^1(\Omega_i)}$$

holds and for any $\phi^h \in H_h^{\frac{1}{2}}(\partial\Omega_i)$, there exist $u_i^h \in H_h(\Omega_i)$ with $u_i^h = \phi^h$ such that

$$\|u_i^h\|_{H^1(\Omega_i)} \le c_2 \|\phi^h\|_{H_h^{\frac{1}{2}}(\partial\Omega_i)}.$$

Then

$$(S_i\phi, \phi) \simeq \|\phi^h\|^2_{H_h^{\frac{1}{2}}(\partial\Omega_i)}.$$

Here ϕ is matrix representation of ϕ^h.

We treated just the Laplacian equation. For the general elliptic case, S is more complicated so that it is difficult to find a precondioner for S. But the following lemma shows that it is enough to find a precondioner for the Laplacian equation.

Lemma 1.2 *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *and* $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. *Assume that* $A = A^T \geq 0$, $B = B^T \geq 0$, $A_{11} > 0$, *and* $B_{11} > 0$. *Let* $S_A = A_{22} - A_{21}A_{11}^{-1}A_{12}$ *and* $S_B = B_{22} - B_{21}B_{11}^{-1}B_{12}$. *If*

$$c_1(Au, u) \leq (Bu, u) \leq c_2(Au, u), \qquad \forall u$$

then

$$c_1(S_A u, u) \leq (S_B u, u) \leq c_2(S_A u, u), \qquad \forall u.$$

Proof.

$$\begin{aligned}
(S_B u_2, u_2) &= \inf_{u_1} \left(B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \\
&= \left(B \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \right) \qquad\qquad \text{(for some } v_1) \\
&\geq c_1 \left(A \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} \right) \\
&\geq c_1 \inf_{u_1} \left(A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \\
&= c_1(S_A u_2, u_2).
\end{aligned}$$

□

Let

$$Lu = -\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + a_0(x)u = f(x), \ x \in \Omega.$$

Assume that

$$(Lu, u) \simeq \|u\|^2_{H^1(\Omega)}.$$

Let S_L and S_1 be Schur-complement matrices for L and -Δ, respectively. The above lemma implies that it is enough to construct a precondtioner for S_1 in place of S_L.

2 Finite Element Trace Theorem

Theorem 2.1 *Let* Ω *be a bounded domain with the piecewise smooth boundary* Γ, *and let* Ω^h *(* Ω^h *is a polygonal approximation of* Ω *whose vertex may not lie on* Γ*) be a shape-regular triangulation of* Ω *such that*

i) we have

$$\frac{\text{diam } \tau_i}{r_i} \leq c \neq c(h),$$

where r_i *denotes the radius of the largest ball inscribed in* τ_i, *ii) there exists a mapping $T : \tau_i \to \tilde{\tau}_i$ such that $T(z_i) = \tilde{z}_i$ (z_i and $\tilde{z}_i$ are the vertices of τ_i and $\tilde{\tau}_i$, respectively) and*

- *$T(\tau_i) = \tilde{\tau}_i$ is also shape regular,*
- *$z_i \in \Gamma^h \Longrightarrow \tilde{z}_i \in \Gamma$ (The map T moves $z_i \in \Gamma^h$ to $\tilde{z}_i \in \Gamma$.*
- *$\exists c_1, c_2 \neq c(h),\ c_1|z_i - z_j| \leq |\tilde{z}_i - \tilde{z}_j| \leq c_2|z_i - z_j|$.*

Then

(1) There exists $c_3 \neq c_3(h)$ such that

$$\|\varphi^h\|_{H_h^{1/2}(\Gamma^h)} \leq c_3\|u^h\|_{H^1(\Omega^h)}, \quad \forall u^h \in H_h(\Omega^h) \text{ with } u^h|_{\Gamma^h} = \varphi^h.$$

(2) There exists $c_4 \neq c_4(h)$ such that for any given $\varphi^h \in H_h(\Gamma^h)$, a $u^h \in H_h(\Omega^h)$ exists satisfying $u^h = \varphi^h$ on Γ_h and

$$\|u^h\|_{H^1(\Omega^h)} \leq c_4\|\varphi^h\|_{H^{1/2}(\Gamma^h)}.$$

Remark 2.1 *Such T's exist if $\Gamma^h \approx \Gamma$ in $O(h^2)$.*

In the paper of Korneev (1970) the special finite element space $\tilde{u}^h \in H_h(\widetilde{\Omega}^h)$ on the curvelinear tirangulation $\widetilde{\Omega}^h$ was suggested that $\tilde{u}^h(\tilde{z}_i) = u^h(z_i)$ where $\tilde{u}^h \in H_h(\widetilde{\Omega}^h)$ $u^h \in H_h(\Omega^h)$ and the following lemma holds

Lemma 2.1 *There exists $c_5, c_6 \neq c(h)$ such that*

$$c_5\|\tilde{u}^h\|_{L^2(\tilde{\tau}_i)} \leq \|u^h\|_{L^2(\tau_i)} \leq c_6\|\tilde{u}^h\|_{L^2(\tilde{\tau}_i)},$$
$$c_5|\tilde{u}^h|_{H^1(\tilde{\tau}_i)} \leq |u^h|_{H^1(\tau_i)} \leq c_6|\tilde{u}^h|_{H^1(\tilde{\tau}_i)},$$
$$c_5\|\tilde{\varphi}^h\|_{L^2(\tilde{I}_i)} \leq \|\varphi^h\|_{L^2(I_i)} \leq c_6\|\tilde{\varphi}^h\|_{L^2(\tilde{I}_i)},$$

and

$$c_5 \int_{\tilde{I}_i}\int_{\tilde{I}_j} \frac{(\tilde{\varphi}^h(x) - \tilde{\varphi}^h(y))^2}{|x-y|^2}\, dxdy \leq \int_{I_i}\int_{I_j} \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2}\, dxdy$$
$$\leq c_6 \int_{\tilde{I}_i}\int_{\tilde{I}_j} \frac{(\tilde{\varphi}^h(x) - \tilde{\varphi}^h(y))^2}{|x-y|^2}\, dxdy,$$

where $\Gamma^h = \bigcup_i I_i$ and $\Gamma = \bigcup_i \tilde{I}_i$.

Proof. (Existence of c_3) There exists c_7 such that for any given $u^h \in H_h(\Omega^h)$ there is a $\tilde{u}^h \in H^1(\tilde{\Omega}^h)$ satisfying $\|\tilde{u}^h\|_{H^1(\tilde{\Omega}^h)} \leq c_7\|u^h\|_{H^1(\Omega^h)}$.

Letting $\tilde{\varphi}^h = \tilde{u}^h|_\Gamma \in H_h(\Gamma)$, define $\varphi^h \in H_h(\Gamma_h)$ as a linear combinations of vertex values of $\tilde{\varphi}^h$. We have from the trace theorem

$$\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)} \leq c_8\|\tilde{u}^h\|_{H^1(\Omega)}.$$

By Lemma 2.1 it follows that

$$\|\varphi^h\|_{H_h^{1/2}(\Gamma^h)} \leq c_9 \|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}.$$

We remark that this is immediate in the case $\Omega^h = \overline{\Omega}$. □

Proof. **(Existence of c_4)** For a given $\varphi^h \in H_h(\Gamma^h)$, let $\tilde{\varphi}^h \in H_h(\Gamma)$ be such that $\tilde{\varphi}^h(\tilde{z}_i) = \varphi^h(z_i)$. Then we have by Lemma 2.1

$$\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)} \leq c \|\varphi^h\|_{H_h^{1/2}(\Gamma^h)}.$$

By inverse trace theorem, there exists $u \in H^1(\Omega)$ such that $u|_\Gamma = \tilde{\varphi}^h$ and $\|u\|_{H^1(\Omega)} \leq c\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}$. But $u \notin H_h(\Omega)$. How can we construct $\tilde{u}^h \in H_h(\Omega)$? It's enough to have values at $\tilde{z}_i$. Let

$$\tilde{u}^h(\tilde{z}_i) = \begin{cases} \tilde{\varphi}^h(\tilde{z}_i), & \text{if } \tilde{z}_i \in \Gamma, \\ \dfrac{1}{\pi r_i} \displaystyle\int_{B(\tilde{z}_i, r_i)} u(x)\, dx, & \text{otherwise}, \end{cases}$$

where r_i is the radius of the largest ball $B(\tilde{z}_i, r_i)$ inscribed in the union of all elements sharing the vertex $\tilde{z}_i$ which is denoted by K_i. Then we take $u^h \in H_h(\Omega^h)$ with $u^h(z_i) = \tilde{u}^h(\tilde{z}_i)$. By Lemma 2.1 it follows that $\|u^h\|_{H^1(\Omega^h)} \leq c\|\tilde{u}^h\|_{H^1(\Omega)}$.

It remains to show that $\|\tilde{u}^h\|_{H^1(\Omega)} \leq c\|\tilde{\varphi}^h\|_{H^{1/2}(\Gamma)}$. (Note $\varphi^h \to \tilde{\varphi}^h \to u \to \tilde{u}^h \to u^h$.) By Friedrich's inequality we obtain

$$\|\tilde{u}^h\|_{L^2(\Omega)} \leq c(|\tilde{u}^h|_{H^1(\Omega)} + \|\tilde{\varphi}^h\|_{L^2(\Gamma)}),$$

and since $\|\tilde{\varphi}^h\|_{L^2(\Gamma)}) \leq C\|\varphi\|_{H^{1/2}(\Gamma)}$ it is enough to estimate $|\tilde{u}^h|_{H^1(\Omega)}$. Note

$$|\tilde{u}^h|^2_{H^1(\Omega)} \leq c \sum_{l_i \in \tilde{\Omega}^h} (\tilde{u}^h(\tilde{z}_{i_1}) - \tilde{u}^h(\tilde{z}_{i_2}))^2,$$

where $\tilde{z}_{i_1}$ and $\tilde{z}_{i_2}$ are the vertices of the edge l_i. We consider the following three cases separately:

Case 1) $\tilde{z}_{i_1}, \tilde{z}_{i_2} \in \Gamma$

$$\begin{aligned} \sum (\tilde{u}^h(\tilde{z}_{i_1}) - \tilde{u}^h(\tilde{z}_{i_2}))^2 &= \sum (\tilde{\varphi}^h(\tilde{z}_{i_1}) - \tilde{\varphi}^h(\tilde{z}_{i_2}))^2 \\ &\leq \sum_{\tilde{z}_i} \sum_{\tilde{z}_j} \frac{(\tilde{\varphi}^h(\tilde{z}_i) - \tilde{\varphi}^h(\tilde{z}_i))^2}{|z_i - z_j|^2} h_i h_j \\ &\leq c|\tilde{\varphi}^h|^2_{H^{1/2}(\Gamma)} \end{aligned}$$

Case 2) $\tilde{z}_{i_1}, \tilde{z}_{i_2} \in \Omega$

Lemma 2.2 *Let $0 < h_1 \le h_2$. Then we have for all $u \in H^1(B(0,h_2))$*

$$\left(\frac{1}{\pi h_2^2}\int_{B(0,h_2)} u(x)\,dx - \frac{1}{\pi h_1^2}\int_{B(0,h_1)} u(x)\,dx\right)^2 \le \frac{h_2}{\pi h_1}|u|^2_{H^1(B(0,h_2))}.$$

Proof. Let (r,θ) be the radial coordinate system given by

$$x = (x_1, x_2) = (r\cos\theta, r\sin\theta).$$

Then we obtain

$$\begin{aligned}
&\left(\frac{1}{\pi h_2^2}\int_{B(0,h_2)} u(x)\,dx - \frac{1}{\pi h_1^2}\int_{B(0,h_1)} u(x)\,dx\right)^2 \\
&= \left(\frac{1}{\pi h_2^2}\int_0^{h_2}\int_0^{2\pi} u(r,\theta)\,rd\theta dr - \frac{1}{\pi h_1^2}\int_0^{h_1}\int_0^{2\pi} u(r,\theta)\,rd\theta dr\right)^2 \\
&= \left(\frac{1}{\pi h_2^2}\int_0^{h_2}\int_0^{2\pi} (u(r,\theta) - u(r/a,\theta))\,rd\theta dr\right)^2 \qquad (a = h_2/h_1 \ge 1) \\
&= \frac{1}{\pi^2 h_2^4}\left(\int_0^{h_2}\int_0^{2\pi}\left[\int_{\frac{r}{a}}^{r} r^{1/2}\frac{\partial u(t,\theta)}{\partial t}\,dt\right] r^{1/2} d\theta dr\right)^2 \\
&\le \frac{1}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_{\frac{r}{a}}^{r}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 r\,dtd\theta dr \qquad \text{(by C-B inequality)} \\
&\le \frac{a}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_{\frac{r}{a}}^{r}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta dr \qquad (\because r \le at) \\
&\le \frac{a}{\pi h_2}\int_0^{h_2}\int_0^{2\pi}\int_0^{h_2}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta dr \\
&= \frac{a}{\pi}\int_0^{2\pi}\int_0^{h_2}\left(\frac{\partial u(t,\theta)}{\partial t}\right)^2 t\,dtd\theta \\
&\le \frac{a}{\pi}|u|^2_{H^1(B(0,h_2))}.
\end{aligned}$$

The last inequality follows from the fact that $(\frac{\partial u}{\partial r})^2 \le (\frac{\partial u}{\partial x_1})^2 + (\frac{\partial u}{\partial x_2})^2$. □

Does there exist c_4 such that $\forall \phi_h \in H_h(\Gamma^h)$ $\exists u^h \in H_h(\Omega^h)$ such that $u^h(x) = \phi^h(x), \quad x \in \Gamma^h$ and $\|u^h\|_{H^1(\Omega^h)} \le c_4|\phi^h|_{H_h^{1/2}(\Gamma^h)}$?

$\phi^h \to \tilde{\phi}^h \in \tilde{H}_h(\Gamma) \to u \in H^1(\Omega) \to \tilde{u}^h \in \tilde{H}_h(\Omega)$

$$\tilde{u}^h(\tilde{z}_i) = \frac{1}{\pi r_i^2}\int_{B(\tilde{z}_i,r_i)} u(x)dx \tag{1}$$

There are two cases:

1) $\tilde{z}_i, \tilde{z}_j \in \Gamma$

2) $\tilde{z}_i, \tilde{z}_j \in \Omega$

Let r denote the radius satisfying the following inclusion:

$$r\colon B(x,\sqrt{2}r)\subset K_{i_1}\cup K_{i_2},\quad x\in l_i.$$

Now we estimate $(\tilde{u}^h(\tilde{z}_{i_2})-\tilde{u}^h(\tilde{z}_{i_1}))^2$.

$$(\tilde{u}^h(\tilde{z}_{i_2})-\tilde{u}^h(\tilde{z}_{i_1}))^2 \le 3\left(\left(\tilde{u}^h(\tilde{z}_{i_2})-\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)}u(x)dx\right)^2\right.$$

$$\left.+\left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)}u(x)dx-\tilde{u}^h(\tilde{z}_{i_1})\right)^2+\left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)}u(x)dx-\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)}u(x)dx\right)^2\right)$$

For the first two terms we can use Lemma 6.2 Now we will have the estimation for the third term.

$$\begin{aligned}
&\left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)}u(x)dx-\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)}u(x)dx\right)^2\\
&=\frac{1}{\pi^2 r^4}\left(\int_{B(\tilde{z}_{i_1},r)}(u(x+y)-u(x))\cdot 1dx\right)^2\\
&\le\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)}(u(x+y)-u(x))^2dx\\
&\le\frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r}(u(s+h,t)-u(s,t))^2dsdt\\
&\le\frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r}\int_{s}^{s+h}\left(\frac{\partial u(\xi,t)}{\partial\xi}\right)^2d\xi dsdt\\
&\le\frac{1}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r}\int_{-r}^{r+h}\left(\frac{\partial u(\xi,t)}{\partial\xi}\right)^2d\xi dsdt\\
&=\frac{2h}{\pi r^2}\int_{-r}^{r}\int_{-r}^{r+h}\left(\frac{\partial u(\xi,t)}{\partial\xi}\right)^2d\xi dt\\
&\le\frac{2h}{\pi r^2}|u|_{H^1(K_{i_1}\cup K_{i_2})}.
\end{aligned}$$

Case 3) $\tilde{z}_{i_1}\in\Gamma,\tilde{z}_{i_2}\in\Omega$

Next, let $\tilde{z}_{i_1}=(0,0),\tilde{z}_{i_1}^{+}=(h_1,0),\tilde{z}_{i_1}^{-}=(-h_2,0),\tilde{z}_{i_2}=(0,h_3)$, and $r\colon B(\tilde{z}_i,r)\subset S$, where $S=\{(s,h)|-h_2\le s\le h_1,0\le h\le 2h_3\}$.

$$(\tilde{u}^h(\tilde{z}_{i_2})-\tilde{u}^h(\tilde{z}_{i_1}))^2\le 2\left(\tilde{u}^h(\tilde{z}_{i_2})-\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)}u(x)dx\right)^2+\left(\frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_2},r)}u(x)dx-\tilde{u}^h(\tilde{z}_{i_1})\right)^2$$

The the second term is estimated as follows:

$$\frac{1}{\pi^2 r^4}\left(\int_{B(\tilde{z}_{i_1},r)}(u(x)-\tilde{u}^h(\tilde{z}_{i_2}))dx\right)^2 \le CB \le \frac{1}{\pi r^2}\int_{B(\tilde{z}_{i_1},r)}(u(x)-\tilde{u}^h(\tilde{z}_{i_2}))^2 dx$$
$$\le \frac{1}{\pi r^2}\int_{-h_2}^{h_1}\int_0^{2h_3}(u(s,t)-\phi^h(0))^2 dtds$$
$$\le \frac{2}{\pi r^2}\left(\int_{-h_2}^{h_1}\int_0^{2h_3}(u(s,t)-\phi^h(s))^2 dtds+\int_{-h_2}^{h_1}\int_0^{2h_3}(\phi(s)-\phi^h(0))^2 dtds\right)$$
$$\le \frac{2}{\pi r^2}\left(\int_{-h_2}^{h_1}\int_0^{2h_3}\left(\int_0^t\frac{\partial u(s,\xi)}{\partial\xi}d\xi\right)^2 dtds\right.$$
$$\left.+\,2\left(\int_0^{h_1}\left(\tilde{\phi}(s)-\tilde{\phi}^h(0)\right)^2 ds+\int_{-h_2}^0\left(\tilde{\phi}(s)-\tilde{\phi}^h(0)\right)^2 ds\right)\right)$$
$$\le C\left(|u|^2_{H^1(S)}+(\tilde{\phi}^h(z^+_{i_1})-\tilde{\phi}^h(z_{i_1})^2+(\tilde{\phi}^h(z^-_{i_1})-\tilde{\phi}^h(z_{i_1})^2\right)^2.$$

We have

$$\|u^h\|_{H^1(\Omega^h)}\le C_4\|\phi^h\|_{H_h^{1/2}(\Gamma^h)}.$$

□

Theorem 2.2 *(Sobolev) Let* $l\colon H^1(\Omega)\to R$ *be a linear bounded functional such that* $l(c)=0$ *and* c *is constant* $\Longrightarrow c=0$. *Then* $\|u\|_{H^1(\Omega)}\approx|u|_{H^1(\Omega)}+|l(u)|$.

For example of this theorem, Poincaré inequality.

$$\|u\|_{L^2(\Omega)}\le C\left(|u|^2_{H^1(\Omega)}+\left(\int_\Omega u(x)dx\right)^2\right)$$

If $\int_\Omega u\,dx=0$ we have usual Poincaré inequality.

Lemma 2.3 (Poincaré inequality in $H^{1/2}(\Gamma)$)

$$\int_\Gamma\phi^2(x)dx\le C\left(\int_\Gamma\int_\Gamma\frac{|\phi(x)-\phi(y)|^2}{|x-y|^2}dxdy+\left(\int_\Gamma\phi(x)dx\right)^2\right)\tag{2}$$

Proof. Let $x,y\in\Gamma$ and $x\ne y$ then

$$(\phi(x)-\phi(y))^2\le C_0\frac{(\phi(x)-\phi(y))^2}{|x-y|^2},$$

where $C_0=\text{diam}\Omega$. So we have

$$\int_\Gamma\int_\Gamma(\phi(x)-\phi(y))^2dxdy\le C_0\int_\Gamma\int_\Gamma\frac{(\phi(x)-\phi(y))^2}{|x-y|^2}dydy.$$

But,

$$\int_\Gamma\int_\Gamma(\phi(x)-\phi(y))^2dxdy = \int_\Gamma\int_\Gamma \phi(x)^2dxdy - 2\int_\Gamma\int_\Gamma \phi(x)\phi(y)^2dxdy + \int_\Gamma\int_\Gamma \phi(y)^2dxdy$$
$$= 2\cdot \text{meas}(\Gamma)\int_\Gamma \phi^2(x)dx - 2\left(\int_\Gamma \phi(x)dx\right)^2.$$

Substituting the integral into above equation we have the result. □

Theorem 2.3 (Trace theorem with semi-norm) *There are two positive constant C_1 and C_2 satisfying the following conditions: There exists C_1 such that $\forall u \in H^1(\Omega), \phi(x) = u(x), \quad x \in \Gamma$ exists and*

$$|\phi|_{H^{1/2}(\Gamma)} \leq C_1|u|_{H^1(\Omega)}.$$

There exists C_2 such that $\forall\phi \in H^{1/2}(\Gamma), \exists u \in H^1(\Omega)$ such that $u(x) = \phi(x) \quad x \in \Gamma$ and

$$|u|_{H^1(\Omega)} \leq C_2|\phi|_{H^{1/2}(\Gamma)}$$

Proof. For the first proof: Let $u \in H^1(\Omega)$. Then u can be split into two parts as follows:

$$u = u_0 + u_1, \quad u_0 = \text{ constant } = \frac{1}{\text{meas}(\Omega)}\int_\Omega ud\Omega, \quad \int_\Omega u_1 d\Omega = 0$$

Let us split ϕ as following:

$$\phi_0 = u_0|_\Gamma, \quad \phi_1 = u_1|_\Gamma, \quad \phi = \phi_0 + \phi_1.$$

Then we have

$$|\phi|_{H^{1/2}(\Gamma)} = |\phi_1|_{H^{1/2}(\Gamma)} \leq C_3\|u_1\|_{H^1(\Omega)}$$
$$\leq C_4|u_1|_{H^1(\Omega)} = C_4|u|_{H^1(\Omega)}$$

For the second proof: Let $\phi \in H^{1/2}(\Gamma)$ be decomposed as

$$\phi = \phi_0 + \phi_1, \quad \phi_0 = \text{ constant } = u_0, \quad \int_\Gamma \phi_1 d\Gamma = 0$$

By Theorem 3.1(Trace theorem), there exists u_1 s.t. $u_1(x) = \phi_1(x)$ and

$$\|u_1\|_{H^1(\Omega)} \leq C_5\|\phi_1\|_{H^{1/2}(\Gamma)}$$

Set $u = u_0 + u_1$. Then $u(x) = \phi(x), \quad x \in \Gamma$ and

$$|u|^2_{H^1(\Omega)} = |u_1|^2_{H^1(\Omega)} \leq C_5\|\phi_1\|^2_{H^{1/2}(\Gamma)} \leq C_6|\phi|^2_{H^{1/2}(\Gamma)}$$

where the last one by Poincaré. □

Remark 2.2 *We have the same theorem for finite element space because FEM space contains constant function.*

Let $H_0 \subset H$ be two Hilbert spaces and $a : H \times H$ be a semi positive symmetric bilinear form. Assume a is positive definite only on H_0. For a given $u \in H$ consider the problem of finding $u \in H$ s.t.

$$\inf_{v_0 \in H_0} a(v_0 + u, v_0 + u)$$

Define $u_0 \in H_0$ by

$$a(u + u_0, w_0) = 0, \quad \forall w_0 \in H_0$$

Let $H = H^1, H_0 = H_0^1$ and consider the problem: Find $v \in H^1(\Omega)$ s.t $v(x) = \phi(x),\ x \in \Gamma$ and

$$\|v\|_{H^1(\Omega)} = \inf_{w|_\Gamma = \phi} \|w\|_{H^1(\Omega)}.$$

$$a(u, v) = \int \nabla u \cdot \nabla v + uv\, dx$$

$u \in H^1, u|_\Gamma = \phi$

$$\inf_{v_0 \in H_0^1} \|u + v_0\|_{H^1(\Omega)} = \inf_{w|_\Gamma = \phi} \|w\|_{H^1(\Omega)}$$

$v = u + u_0, a(u + u_0, u_0) = 0$ Solves

$$\begin{aligned} -\Delta v + v &= 0 \\ v|_\Gamma &= \phi \end{aligned}$$

Remark 2.3 *Let* $a'(u, v) = \int \nabla u \cdot \nabla v\, dx$ *Then*

$$v' = \min_{w|_\Gamma = \phi} a'(w, w)$$

is equivalent to(Harmonic extension)

$$\begin{aligned} -\Delta v &= 0 \\ v|_\Gamma &= \phi \end{aligned}$$

$\|v\|_{H^1(\Omega)} \leq \|v'\|_{H^1(\Omega)}$ *because left hand side is extension with minimal norm.*

$$\begin{aligned} \|v'\|_{H^1(\Omega)} &\leq |v'|^2_{H^1(\Omega)} + \|v'\|^2_{L^2} \leq |v|^2_{H^1(\Omega)} + \|v'\|^2_{L^2} \\ &\leq |v|^2_{H^1(\Omega)} + C(|v'|^2_{H^1(\Omega)} + \|\phi\|^2_{H^{1/2}(\Gamma)}) \\ &\leq (1 + C)|v|^2_{H^1(\Omega)} + c\|\phi\|^2_{H^{1/2}(\Gamma)} \leq C\|v\|^2_{H^1(\Omega)} \leq C\|v'\|^2_{H^1(\Omega)} \end{aligned}$$

So extension by min with semi norm is equivalent to min with norm.

Now FEM. With $H_h(\Omega^h), H_h^{1/2}(\Gamma^h)$, The solution of the minimization problem

$$\|v^h\|_A = \inf_{w^h|_\Gamma = \phi^h} \|w^h\|_A$$

is the solution of

$$\begin{aligned} -\Delta v^h &= 0 \\ v^h|_\Gamma &= \phi^h. \end{aligned}$$

In variational form

$$\left\{ \begin{aligned} a(v^h, w_0^h) &= 0, \quad w_0^h \in \overset{0}{H}_h(\Omega^h) \\ v^h|_\Gamma &= \phi^h \end{aligned} \right.$$

a form can be written as

$$a(v^h, v^h) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \left(\begin{bmatrix} v_0 \\ \phi \end{bmatrix}, \begin{bmatrix} v_0 \\ \phi \end{bmatrix} \right)$$

Solution is equivalent to

$$0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} v_0, & w_0 \\ \phi, & 0 \end{pmatrix} = (A_{11} v_0 + A_{12}\phi, w_0), \quad \forall w_0$$

This implies

$$A_{11} v_0 = -A_{12}\phi, \quad v_0 = -A_{11}^{-1} A_{12} \phi$$

Then consider

$$\inf a(w^h, w^h)$$

where $a(w^h, w^h)$ is

$$\begin{aligned} & \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \left(\begin{bmatrix} -A_{11}^{-1} A_{12}\phi \\ \phi \end{bmatrix}, \begin{bmatrix} -A_{11}^{-1} A_{12}\phi \\ \phi \end{bmatrix} \right) \\ = \; & (-A_{12}\phi + A_{12}\phi, -A_{11}^{-1} A_{12}\phi) + (-A_{21} A_{11}^{-1} A_{12}\phi + A_{22}\phi, \phi) \\ = \; & ((A_{22} - A_{21} A_{11}^{-1} A_{12})\phi, \phi) \approx \|\phi\|^1_{H^{1/2}(\Gamma)} \end{aligned}$$

Here the $S = (A_{22} - A_{21} A_{11}^{-1} A_{12})$ is the Schur complement. How to construct an equivalent norm ?

Note $\|\phi\|_{H^{1/2}(\Gamma)}$ is very complicated.

If $\phi \in H^1(-1, 1)$ then

$$\|\phi\|^2_{H(-1,1)} = \|\phi\|^2_{H(-1,0)} + \|\phi\|^2_{H(0,1)}$$

This is only true for H^α, $0 < \alpha \le 1$, $\alpha \ne 1/2$.

$$\|\phi\|^2_{H^\alpha} = \|\phi\|^2_{L^2} + \int_{-1}^1 \int_{-1}^1 \frac{(\phi(x) - \phi(y))^2}{|x - y|^{1+2\alpha}} \, dx$$

Lemma 2.4 *There exist c_1, c_2 such that*

$$c_1 \|\phi\|^2_{H^{1/2}(-1,1)} \le \|\phi\|^2_{H^{1/2}(-1,0)} + \|\phi\|^2_{H^{1/2}(0,1)} + \int_0^1 \frac{(\phi(x) - \phi(-x))^2}{x} \, dx \le c_2 \|\phi\|^2_{H^{1/2}(-1,1)} \tag{3}$$

We denote the third term as $I(\phi)$.

Proof.

$$\int_0^1 \frac{dy}{(x+y)}^2 = \int_x^{1+x} \frac{dt}{t^2} = \frac{1}{x(1+x)}$$

Thus

$$\frac{1}{2x} \leq \int_0^1 \frac{dy}{(x+y)}^2 \leq \frac{1}{x}$$

Consider

$$\begin{aligned}
I(\phi) &= \int_0^1 \frac{(\phi(x)-\phi(-x))^2}{x}\, dx \\
&\leq \int_0^1 \int_0^1 \frac{(\phi(x)-\phi(-x))^2}{(x+y)^2}\, dydx \\
&\leq 4\int_0^1 \int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x+y)^2}\, dydx + 4\int_0^1 \int_0^1 \frac{(\phi(y)-\phi(-x))^2}{(x+y)^2}\, dydx \\
&\leq 4(|\phi|^2_{H^{1/2}(0,1)} + |\phi|^2_{H^{1/2}(-1,1)})
\end{aligned}$$

So C_2 is proved. For C_1 we only need to consider semi norm. Since

$$|\phi|^2_{H^{1/2}(0,1)} = \int_0^1 \int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\, dydx$$

$$\begin{aligned}
|\phi|^2_{H^{1/2}(-1,1)} &= \int_0^1 \int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\, dydx + \int_0^1 \int_{-1}^0 + \int_{-1}^0 \int_0^1 + \int_{-1}^0 \int_{-1}^0 \\
&= I + II + III + IV
\end{aligned}$$

Since $I = |\phi|^2_{H^{1/2}(0,1)}, IV = |\phi|^2_{H^{1/2}(-1,0)}$ only $II = III$ matters. So consider one of them.

$$\begin{aligned}
&\int_{-1}^0 \int_0^1 \frac{(\phi(x)-\phi(y))^2}{(x-y)^2}\, dydx \\
&= 2\left(\int_{-1}^0 \int_0^1 \frac{(\phi(x)-\phi(-x))^2}{(x-y)^2} + \int_{-1}^0 \int_0^1 \frac{(\phi(-x)-\phi(y))^2}{(x-y)^2}\right) \\
&\leq 2\int_0^1 (\phi(-x)-\phi(y))^2 \int_0^1 \frac{dy}{(x+y)}^2 (y \to -y') \\
&\quad + \int_{-1}^0 \int_{-1}^0 \frac{(\phi(x)-\phi(y))^2}{(x+y)^2}\, dydx (x \to -x')
\end{aligned}$$

Use the estimate of the integral

$$\frac{1}{2x} \leq \int_0^1 \frac{dy}{(x+y)}^2 \leq \frac{1}{x}$$

to see the third term is less than $|\phi|^2_{H^{1/2}(-1,0)}$

The whole thing is less than

$$2(I(\phi) + |\phi|^2_{H^{1/2}(-1,0)})$$

□

Now divide the boundary(like circle) by two point a, b on Γ and left hand side is called Γ_1 Consider

$$H^{1/2}_{00}(\Gamma_1)$$

Assume ϕ is equal to zero on $\Gamma_0 = \Gamma \backslash \Gamma_1$ and equivalent to harmonic extension into interior, i.e,

$$\|\phi\|^2_{H^{1/2}(\Gamma)} \approx \|\phi\|^2_{H^{1/2}(\Gamma_1)} + \|\phi\|^2_{H^{1/2}(\Gamma_0)}(=0) + \int_{\Gamma_1} \frac{\phi^2}{|x-a|^2} + \int_{\Gamma_1} \frac{\phi^2}{|x-b|^2}.$$

With this motivation, define

$$\|\phi\|^2_{H^{1/2}_{00}(\Gamma_1)} = \|\phi\|^2_{H^{1/2}(\Gamma_1)} + \int_{\Gamma_1} \frac{\phi^2}{|x-a|^2} + \int_{\Gamma_1} \frac{\phi^2}{|x-b|^2} \tag{4}$$

Similarly define

$$\|\phi\|_{\overset{00}{H}{}^{1/2}(0,1)} = \|\phi\|^1_{H^{1/2}(0,1)} + \int_0^1 \frac{\phi^2}{x(1-x)} \tag{5}$$

Meanwhile a function in $H^{1/2}(0,1)$ does not have anything to do with the value outside $(0,1)$.

Now a FEM case. Let Ω be triangularized by Ω^h. Some part of it is denoted by Γ^h_1 some other by Γ^h_0 define

$$\|\phi^h\|_{\overset{00}{H}{}^{1/2}_h(\Gamma^h_1)} = \|\phi^h\|^1_{H^{1/2}_h(\Gamma^h_1)} + \sum_{z_i \in \Gamma^h_1} \frac{(\phi^h(z_i))^2}{|z_i - a|} h_i + \sum_{z_i \in \Gamma^h_1} \frac{(\phi^h(z_i))^2}{|z_i - b|} h_i$$

Let $\overset{0}{H}_h(\Gamma^h_1) = \{\phi^h \in H^{1/2}_h(\Gamma^h_1) |\ \phi^h(a) = \phi^h(b) = 0\}$.

$\phi^h \to \phi \in \mathbb{R}^n$

$$(S\phi, \phi) \approx \|\phi^h\|_{H^{1/2}(\Gamma^h)} \approx \|\phi^h\|_{\overset{00}{H}{}^{1/2}(\Gamma^h_1)} \approx \|\tilde{\phi}^h\|_{\overset{00}{H}{}^{1/2}(I)}$$

Here I is straightened boundary. Now $\phi^h(z_i) = \tilde{\phi}(\tilde{z}_i)$ by mapping and extend into the unit square and consider on uniform grid

$$\|\tilde{\phi}^h\|_{\overset{00}{H}{}^{1/2}(I)} \approx (\tilde{S}\phi, \phi)$$

Finally, we have

$$(S\phi, \phi) \approx (\tilde{S}\phi, \phi)$$

Hence a preconditioner for $\tilde{S}$ suffices for the original problem. In summary, Schur complement S is equivalent to the interface norm which is in turn equivalent to Schur complement $\tilde{S}$. On good domain, Schur complement $\tilde{S}$ can be found analytically.

A detailed study on the space $H_{00}^{\frac{1}{2}}(\Gamma_1)$

A review on the Schur complement as norm:

Since

$$(S\varphi,\varphi) = \inf_{w^h|_\Gamma=\varphi^h} \|w^h\|^2_{H^1(\Omega)} = \|u^h\|^2_{H^1(\Omega)} \tag{6}$$

Here u^h satisfying $u^h|_\Gamma = \varphi$ is the minimizer. Then

$$\|\varphi^h\|^2_{H^{1/2}(\Gamma)} \le C_3\|u^h\|^2_{H^1(\Omega)} = (S\varphi,\varphi)$$

For φ^h, there exists c^h such that

$$\|v^h\|^2_{H^1(\Omega)} \le C_4\|\varphi^h\|^2_{H^{1/2}(\Gamma)}$$

Thus

$$(S\varphi,\varphi) = \inf_{w^h|_\Gamma=\varphi^h} \|w^h\|^2_{H^1(\Omega)} = \|u^h\|^2_{H^1(\Omega)} \le \|v^h\|^2_{H^1(\Omega)} \le C_4\|\varphi^h\|^2_{H^{1/2}(\Gamma)}$$

We see that Schur complement norm is equivalent to $H^{1/2}(\Gamma)$ norm. Let $\dot{C}^\infty(0,1)$ be the subspace of $C^\infty(0,1)$ with compact support. Then it is well known that for L^2 case

$$\overline{(C^\infty(0,1))}_{L^2} = L^2(0,1) \text{ and } \overline{(\dot{C}^\infty(0,1))}_{L^2} = L^2(0,1)$$

However, for H^1

$$\overline{(C^\infty(0,1))}_{H^1(0,1)} = H^1(0,1) \qquad \text{and } \overline{(\dot{C}^\infty(0,1))}_{H^1(0,1)} = H^1_0(0,1)$$

For H^α for $\alpha \le 1/2$ follows from L^2 case and for $\alpha > 1/2$ follows H^1 case:

<u>Def</u>. $\|\varphi\|^2_{H^\alpha(0,1)} = \|\varphi\|^2_{L^2(0,1)} + \int_0^1\int_0^1 \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{1+2\alpha}}\,dx\,dy$

If $\alpha \le \frac{1}{2}$

$$\overline{(C^\infty(0,1))}_{H^\alpha} = H^\alpha(0,1) \qquad \overline{(\dot{C}^\infty(0,1))} = H^\alpha(0,1) \tag{7}$$

When $\alpha < 1/2$, for $u \in H^\alpha(0,1)$, its extension by zero outside $(0,1)$ belongs to $H^\alpha(-1,2)$ like L^2 space. But for $\alpha = 1/2$, a function in $H^{1/2}(0,1)$ cannot be extended by zero. (Note: $H_0^{\frac{1}{2}} = H^{\frac{1}{2}}(0,1)$)

If $\alpha > \frac{1}{2}$, $\qquad \overline{(C^\infty(0,1))}_{H^\alpha} = H^\alpha(0,1) \qquad \overline{(\dot{C}^\infty(0,1))}_{H^\alpha} = H^\alpha_0(0,1)$

Let $\alpha = \frac{1}{2}$. If we extend $\dot{C}^\infty(0,1)$ by the norm $\|\cdot\|_{H_{00}^{\frac{1}{2}}}$, then we obtain $H_{00}^{\frac{1}{2}}(0,1)$ and we can extend the function in $H_{00}^{\frac{1}{2}}(0,1)$ to a function in $H^{\frac{1}{2}}(-1,2)$ by zero.

Hence
$$H^{\frac{1}{2}}(0,1) \supsetneq H_{00}^{\frac{1}{2}}(0,1)$$

Note: $\|\varphi\|^2_{H^{\frac{1}{2}}(-1,1)} \approx \|\varphi\|^2_{H^{\frac{1}{2}}(-1,0)} + \|\varphi\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{(\varphi(x) - \varphi(-x))^2}{x}\,dx$

For $\varphi \in H_{00}^{\frac{1}{2}}(0,1)$

$$\|\varphi\|^2_{H_{00}^{\frac{1}{2}}(0,1)} = \|\varphi\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{(\varphi(x))^2}{x(1-x)}\,dx$$

Hence $\varphi \to 0$ as $x \to 0, 1$.

Define $\tilde{\varphi} \in H^{\frac{1}{2}}(-1,2)$

$$\tilde{\varphi}(x) = \begin{cases} 0 & x \in (-1,0), \\ \varphi(x) & x \in (0,1), \\ 0 & x \in (1,2) \end{cases}$$

$$\begin{aligned}
\|\tilde{\varphi}\|^2_{H^{\frac{1}{2}}(-1,2)} &\approx \|\varphi\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{(\tilde{\varphi}(x) - \tilde{\varphi}(-x))^2}{x}\,dx + \int_0^1 \frac{(\tilde{\varphi}(x) - \tilde{\varphi}(2-x))^2}{1-x}\,dx \\
&= \|\varphi\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{\varphi(x)^2}{x} + \frac{\varphi(x)^2}{1-x}\,dx \\
&\approx \|\varphi\|^2_{H^{\frac{1}{2}}(0,1)} + \int_0^1 \frac{\varphi(x)^2}{x(1-x)}\,dx \\
&\approx \|\varphi\|^2_{H_{00}^{\frac{1}{2}}}(0,1)
\end{aligned}$$

where in the first equivalence, we omitted $\|\tilde{\varphi}\|^2_{H^{\frac{1}{2}}(-1,0)}$ and $\|\tilde{\varphi}\|^2_{H^{\frac{1}{2}}(1,2)}$ because they are zero by extension.

$\varphi \in H^\alpha(0,1), \quad \alpha > \frac{1}{2}, \quad \lim_{x\to x_0} \varphi(x) = \varphi(x_0)$
$(H^\alpha(\alpha > \frac{1}{2}) \hookrightarrow C^0(0,1))$

Example 2.1 *Let the boundary Γ of Ω be divided by three points a, b, c and call the resulting pieces $\Gamma_1, \Gamma_0, \tilde{\Gamma}_1$*

Consider the problem:

$$\inf_{w \in H^1(\Omega), w|_{\Gamma_1} = \varphi, w|_{\Gamma_0} = 0} \|w\|^2_{H^1(\Omega)}$$

The above problem is equivalent to

$$\begin{cases} -\triangle w + w = 0 \\ w|_{\Gamma_1} = \varphi \\ w|_{\Gamma_0} = 0 \\ \dfrac{\partial w}{\partial n}|_{\check{\Gamma}_1} = 0 \end{cases}$$

What is correct norm of $\check{H}^{\frac{1}{2}}(\Gamma_1) =?$

$$\|\varphi\|^2_{\check{H}^{\frac{1}{2}}(\Gamma_1)} = \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma_1)} + \int_{\Gamma_1} \frac{(\varphi(x))^2}{|x-a|}\, dx$$

It is like $H^{1/2}_{00}(\Gamma_1)$, *but only one side norm because integral near the point* b *is missing.*

FEM case, we use discrete norm:

$$H_h(\Omega), \qquad H_h(\Gamma), \qquad H_h(\Gamma_1)$$

$$\|\varphi^h\|^2_{\check{H}^{\frac{1}{2}}_h(\Gamma_1)} = \|\varphi^h\|^2_{H^{\frac{1}{2}}_h(\Gamma_1)} + \sum_{z_i \in \Gamma_1} \frac{(\varphi^h(z_i))^2}{|z_i - a|} h_i$$

3 Domain Decomposition Method : *Strip Case

*It does not have any cross point

$$\bar{\Omega} = \cup_{i=1}^n \bar{\Omega}_i, \qquad \gamma = \cup_{i=1}^n \partial\Omega_i \backslash \Gamma,$$

$$\gamma_i \cap \gamma_j = \emptyset, \quad \gamma = \cup_{i=1}^{n-1} \gamma_i$$

$$\begin{cases} -\triangle u = f(x) & x \in \Omega, \\ u(x) = 0 & x \in \Gamma \end{cases}$$

$$\Longrightarrow Au = f$$

$$\sim S\varphi = \psi$$

$$\varphi^{k+1} = \varphi^k - \hat{\tau}_k \Sigma^{-1}(S\varphi^k - \psi) \quad \text{where } C_1(\Sigma\varphi,\varphi) \le (S\varphi,\varphi) \le C_2(\Sigma\varphi,\varphi)$$

$$Au = \begin{pmatrix} A_0 & A_{01} & \cdot & \cdot & \cdot & A_{0n} \\ A_{10} & A_1 & & & & \\ \cdot & & \cdot & & 0 & \\ \cdot & & & \cdot & & \\ \cdot & & 0 & & \cdot & \\ A_{n0} & & & & & A_n \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$$

$$A_{i0}u_0 + A_i u_i = f_i,$$

$$u_i = -A_i^{-1}A_{i0}u_0 + A_i^{-1}f_i$$

Substitute and get

$$\underbrace{(A_0 - \sum_{i=1}^{n} A_{0i}A_i^{-1}A_{i0})}_{S}u_0 = f_0 - \sum_{i=1}^{n} A_{0i}A_i^{-1}f_i$$

We get

$$S\varphi = \psi$$

Consider an iterative method

$$\varphi^{k+1} = \varphi^k - \tau_k \Sigma^{-1}(S\varphi^k - \psi)$$

where Σ is a preconditioner satisfying

$$c_1(\Sigma\varphi, \varphi) \leq (S\varphi, \varphi) \leq C_2(\Sigma\varphi, \varphi) \tag{8}$$

Split u_0 as

$$u_0 = \begin{pmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{n-1} \end{pmatrix}$$

where φ_i corresponds to γ_i. Then

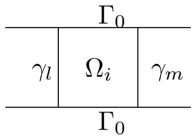
$$S = S_1 + \cdots + S_{n-1}$$

$$\Omega_i \to S_i u_0 = \begin{pmatrix} 0 & 0 \\ S_{11}^{(i)} & S_{12}^{(i)} \\ S_{21}^{(i)} & S_{22}^{(i)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \gamma_l \\ \gamma_m \\ 0 \end{pmatrix}$$

$$\tilde{S}_i \approx \begin{pmatrix} \Sigma_l & 0 \\ 0 & \Sigma_m \end{pmatrix}$$

Problem: Exnted by mini norm with φ_l, φ_m on γ_l, γ_m. $(\Sigma_l \varphi_l, \varphi_l) \approx \|\varphi_l\|^2_{H^{\frac{1}{2}}_{00}}(\gamma_l)$

$(\Sigma_m \varphi_m, \varphi_m) \approx \|\varphi_m\|^2_{H^{\frac{1}{2}}_{00}}(\gamma_m)$



$$\left(S_i \begin{pmatrix} \varphi_l \\ \varphi_m \end{pmatrix}, \begin{pmatrix} \varphi_l \\ \varphi_m \end{pmatrix} \right) = \inf_{w^h \in H_h(\Omega_i), w^h|_{\gamma_l} = \varphi_l, w^h|_{\gamma_m} = \varphi_m, w^h|_{\Gamma \cap \partial\Omega_i} = 0} |w^h|^2_{H^1(\Omega_i)}$$

$$\approx \|\varphi_l^h\|^2_{H^{\frac{1}{2}}_{00}(\gamma_l)} + \|\varphi_m^h\|^2_{H^{\frac{1}{2}}_{00}(\gamma_m)}$$

by previous analysis. Hence we have preconditioner for global Schur complement as block diagonal. Consider some one substructure γ_ℓ and omit the subindex ℓ.

$$\begin{aligned} \|\varphi^h\|^2_{H^{\frac{1}{2}}_{00}(\gamma)} &= \sum_{z_i \in \gamma} (\varphi^h(z_i))^2 \cdot h + \sum_{z_i, i \neq j} \sum_{z_j} \frac{(\varphi^h(z_i) - \varphi^h(z_j))^2}{|z_i - z_j|^2} h_i h_j + \sum_{z_i \in \gamma} \frac{(\varphi^2(z_i))^2}{(z_i - a)(z_j - b)} h_i \\ &\approx \sum_{\tilde{z}_i \in \tilde{\gamma}} \cdots + \cdots + \cdots \\ &= \|\tilde{\varphi}^h\|^2_{H^{\frac{1}{2}}_{00}(\tilde{\gamma})} \end{aligned}$$

where in the second equation, we have everything replaced by its "tilde" (map it onto $[0, 1]$) which is for a curved boundary.

Hence consider the square domain. For example, consider the domain with 4 subdomains(Ω_i, i=1,2,3,4) whose interfaces(γ_i, i=1,2,3) do not meet each other. Then, we have

$$S = \begin{bmatrix} S_1 + S_2^{(1,1)} & S_2^{(1,2)} & \\ S_2^{(2,1)} & S_2^{(2,2)} + S_3^{(1,1)} & S_3^{(1,2)} \\ & S_3^{(2,1)} & S_3^{(2,2)} + S_4 \end{bmatrix}$$

where the submatrix S_i is the Schur-Complement matrix corresponding to the subdomain Ω_i. For instance,

$$S_2 = \begin{bmatrix} S_2^{(1,1)} & S_2^{(1,2)} \\ S_2^{(2,1)} & S_2^{(2,2)} \end{bmatrix}$$

and $S_2^{(i,j)}$ is the S-C corresponding to Ω_2 and γ_i and γ_j. Here, we may write

$$S = \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 + \tilde{S}_4$$

where $\tilde{S_i}$ is just the extension of S_i by zero elements. Now, in terms of the norm equivalence we have

$$S_1 \approx \Sigma_1, \qquad S_2 \approx \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}, \qquad S_3 \approx \begin{bmatrix} \Sigma_2 & \\ & \Sigma_3 \end{bmatrix}, \qquad S_4 \approx \Sigma_3$$

Note that $(\Sigma_i \phi_i, \phi_i) \approx \|\phi\|^2_{H^{1/2}_{00}(\gamma_i)}$. Hence, we have

$$S \approx \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & \Sigma_3 \end{bmatrix}$$

Given a vertical interface -line segment, we introduce an artificial uniform domain and consider the problem with zero boundary condition on three side except Γ_1 on the left. And consider the Schur complement of this problem, denote it by S.

For given γ_i interface suppose we have mapping from γ_i onto one side of rectangular domain with uniform mesh of size $h = 1/n$, thus now we can consider our interface problem as the rectangular model. The Shur-Complement of this model satisfies

$$(S\phi, \phi) \approx \|\phi^h\|^2_{H^{1/2}_{00}(\Gamma_1)}$$

In the rectangular domain, we have

$$\begin{aligned} A_\Omega &= \begin{bmatrix} A_0 + 2I & -I & & & & \\ -I & A_0 + 2I & -I & & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & A_0 + 2I & -I \\ & & & & -I & \frac{1}{2}A_0 + I \end{bmatrix} \\ &:= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \end{aligned}$$

where

$$A_0 = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \end{bmatrix},$$

and

$$\bar{A}_{12} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -I \end{bmatrix}^t = (\bar{A}_{21})^t, \qquad \bar{A}_{22} = \frac{1}{2}A_0 + I$$

Now we have

$$S = \bar{A}_{22} - \bar{A}_{21}(\bar{A}_{11})^{-1}\bar{A}_{12}$$

and

$$\begin{aligned}(S\phi,\phi) &= \inf_{u^h|_{\Gamma_1}=\phi^h,u^h|_{\partial\Omega\setminus\Gamma_1}=0} \|u^h\|^2_{H^1(\Omega)} \\ &= \inf_{u^h|_{\Gamma_1}=\phi^h,u^h|_{\partial\Omega\setminus\Gamma_1}=0} (A_\Omega u,u)\end{aligned}$$

By the diagonalization, we decompose A_0 as

$$A_0 = Q\Lambda Q^t$$

where $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_{n-1}\end{bmatrix}$, Λ is the diagonal matrix with the diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$, and $A_0 q_i = \lambda_i q_i$. Note that it is known that the eigenvalue $\lambda_i = 4sin^2\frac{\pi}{2n}$,the jth component of the eigenvector q_i is $\sqrt{\frac{2}{n}}sin(\frac{i\pi}{n}j)$, and $QQ^t = I$. Using this, we get

$$(\bar{A}_{11})^{-1} = \begin{bmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & Q\end{bmatrix} \begin{bmatrix} \Lambda+2I & -I & & \\ -I & \Lambda+2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda+2I\end{bmatrix}^{-1} \begin{bmatrix} Q^t & & & \\ & Q^t & & \\ & & \ddots & \\ & & & Q^t\end{bmatrix}$$

and

$$\begin{aligned}\bar{A}_{21}(\bar{A}_{11})^{-1}\bar{A}_{12} &= \begin{bmatrix} 0 & 0 & \dots & -Q\end{bmatrix} \begin{bmatrix} \Lambda+2I & -I & & \\ -I & \Lambda+2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda+2I\end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -Q^t\end{bmatrix} \\ &= QB_{22}Q^t\end{aligned}$$

where

$$B := \begin{bmatrix} \Lambda+2I & -I & & \\ -I & \Lambda+2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda+2I\end{bmatrix}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{bmatrix}$$

Now let's compute the matrix B_{22}. Let $e_i = \begin{bmatrix} 0 & \cdots & 1\cdots & 0\end{bmatrix}^T$, where 1 is in the i-th position. Consider the following matrix equation.

$$\begin{bmatrix} \Lambda+2I & -I & & \\ -I & \Lambda+2I & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & \Lambda+2I\end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_{n-1}^{(i)}\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_i\end{bmatrix}$$

Then the $(n-1)$-th solution vector of the above matrix equation is the i-th column of the matrix B_{22}, that is,

$$B_{22} = \begin{bmatrix} x_{n-1}^{(1)} & x_{n-1}^{(2)} & \cdots & x_{n-1}^{(n-1)} \end{bmatrix}$$

Denote the vector $x_k^{(i)}$ by

$$x_k^{(i)} = \begin{bmatrix} x_k^{(i)}(1) \\ x_k^{(i)}(2) \\ \vdots \\ x_k^{(i)}(n-1) \end{bmatrix}$$

Consider j-th component. Then we obtain the following matrix equation

$$\begin{bmatrix} \lambda_j + 2 & -1 & & \\ -1 & \lambda_j + 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & \lambda_j + 2 \end{bmatrix} \begin{bmatrix} x_1^{(i)}(j) \\ x_2^{(i)}(j) \\ \vdots \\ x_{n-1}^{(i)}(j) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \delta_{ij} \end{bmatrix}$$

(Some detail) Consider the vector equation with i fixed:

$$\begin{array}{ccccc} (\Lambda + 2I)x_1^{(i)} & -x_2^{(i)} & & & = 0 \\ -x_1^{(i)} & +(\Lambda + 2I)x_1^{(i)} & -x_3^{(i)} & & = 0 \\ & & \ddots & & \\ & -x_{n-3}^{(i)} & +(\Lambda + 2I)x_{n-2}^{(i)} & -x_{n-1}^{(i)} & = 0 \\ & & -x_{n-2}^{(i)} & +(\Lambda + 2I)x_{n-1}^{(i)} & = e_i \end{array}$$

First block corresponds to

$$\begin{array}{ccc} (\lambda + 2)x_1^{(i)}(1) & -x_2^{(i)}(1) & = 0 \\ +(\lambda + 2)x_1^{(i)}(2) & -x_2^{(i)}(2) & = 0 \\ & \cdots & \\ +(\lambda + 2)x_1^{(i)}(n-1) & -x_2^{(i)}(n-1) & = 0 \end{array}$$

Collect j-th line. Then when $i = j$ Hence

$$x_{n-1}^{(i)} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_{n-1}^{(i)}(i) \\ \vdots \\ 0 \end{bmatrix}$$

Combining the vectors $x_{n-1}^{(i)}$, we can obtain the matrix B_{22}

$$B_{22} = \begin{bmatrix} x_{n-1}^{(1)}(1) & 0 & \cdots & 0 \\ 0 & x_{n-1}^{(2)}(2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x_{n-1}^{(n-1)}(n-1) \end{bmatrix}$$

Then

$$\tilde{S} = \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12} = \frac{1}{2}A + I - QB_{22}Q_T = Q(\frac{1}{2}\Lambda + I - B_{22})Q_T$$

and the i-th eigenvalue of S is

$$\lambda_i(\tilde{S}) = \frac{1}{2}\lambda_i + 1 - x_{n-1}^{(i)}(i).$$

To compute $x_{n-1}^{(i)}(i)$, we have to solve the following matrix equation.

$$\begin{bmatrix} \lambda_i + 2 & -1 & & \\ -1 & \lambda_i + 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & \lambda_i + 2 \end{bmatrix} \begin{bmatrix} x_1^{(i)}(i) \\ x_2^{(i)}(i) \\ \vdots \\ x_{n-1}^{(i)}(i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\alpha_i = \frac{1}{2}\lambda_i + 1$. By using Gauss-elimination technique, (multiply $2\alpha_i$ to j-th row and add $j-1$-th to j-th row) we obtain the following

$$\begin{bmatrix} d_1 & -d_0 & & 0 \\ & d_2 & -d_1 & \\ & & \ddots & -d_{n-3} \\ 0 & & & d_{n-1} \end{bmatrix} \begin{bmatrix} x_1^{(i)}(i) \\ x_2^{(i)}(i) \\ \vdots \\ x_{n-1}^{(i)}(i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_{n-2} \end{bmatrix}$$

where $d_0 = 1$, $d_1 = 2\alpha_i$, and $d_{j+1} = 2\alpha_i d_j - d_{j-1}$, for $j = 1, 2, \cdots, n-2$. Let $U_n(x)$ be the second kind Chebyshev polynomial of degree n, that is,

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}((x + \sqrt{x^2-1})^{n+1} - (x + \sqrt{x^2-1})^{-(n+1)}).$$

(Note the first kind is determined by the same condition with different I.C. $d_0 = 1, d_1 = \alpha_i$)

Then $d_j = U_j(\alpha_i)$ and

$$x_{n-1}^{(i)}(i) = \frac{d_{n-2}}{d_{n-1}} = \frac{U_{n-2}(\alpha_i)}{U_{n-1}(\alpha_i)}.$$

Hence

$$
\begin{aligned}
\lambda_i(\tilde{S}) &= \alpha_i - \frac{d_{n-2}}{d_{n-1}} \\
&= \alpha_i - \frac{U_{n-2}(\alpha_i)}{U_{n-1}(\alpha_i)} \\
&= \alpha_i - \frac{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n-1} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n+1}}{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}} \\
&= \sqrt{\alpha_i^2 - 1}\, \frac{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} + (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}}{(\alpha_i + \sqrt{\alpha_i^2 - 1})^{n} - (\alpha_i + \sqrt{\alpha_i^2 - 1})^{-n}} \\
&= \sqrt{\alpha_i^2 - 1} f(x)
\end{aligned}
$$

where

$$
f(x) = \frac{x + 1/x}{x - 1/x}, \quad x = \left(\alpha_i + \sqrt{\alpha_i^2 - 1}\right)^n
$$

Using $\sqrt{\alpha_i^2 - 1} = \sqrt{\lambda_i}\sqrt{1 + \frac{\lambda_i}{4}}$, we have the following estimates for $\lambda_i(\tilde{S})$.

$$
\sqrt{\lambda_i} \le \lambda_i(\tilde{S}) \le \sqrt{\lambda_i} C(\lambda_{min}, \lambda_{max})
$$

where

$$
C(\lambda_{min}, \lambda_{max}) = \sqrt{1 + \frac{\lambda_{max}}{4}} \cdot \frac{\beta^n + \beta^{-n}}{\beta^n - \beta^{-n}},
$$

$$
\beta = 1 + \frac{1}{2}\lambda_{min} + \sqrt{\lambda_{min} + \frac{1}{4}\lambda_{min}^2}
$$

Since $\lambda_{min} = 4\sin^2(\frac{\pi}{2n}) \simeq \frac{1}{n^2}$,

$$
\beta^n \ge (1 + \sqrt{\lambda_{min}})^n \simeq O(1).
$$

Hence by letting $\Sigma := A^{\frac{1}{2}} = Q\Lambda^{\frac{1}{2}}Q^T$, we have the following inequality

$$
(\Sigma\phi, \phi) \le (\tilde{S}\phi, \phi) \le C(\Sigma\phi, \phi)
$$

Thus

$$
\tilde{S} = QJQ^T \approx A^{1/2}, \quad J = diag(\lambda_i(\tilde{S}))
$$

$\Sigma^{-1} = Q\Lambda^{-\frac{1}{2}}Q^T, \quad Q = (q_1, \cdots, q_{n-1}), q_i(j) = \sqrt{\frac{2}{n}} \sin\frac{i\pi j}{n}$

Since $\lambda_{max} \le 4$, we have $C \le \frac{5\sqrt{2}}{3}$. If we use FFT algorithm, then the cost to compute $\Sigma^{-1}\phi$ is of order $h^{-1}\log(h^{-1})$. What is $\|\phi^h\|^2_{H^{\frac{1}{2}}}$? Since(as a discrete inner product on Γ)

$$
(\phi^h, \psi^h)_{L_{2,h}} = h(\phi, \psi),
$$

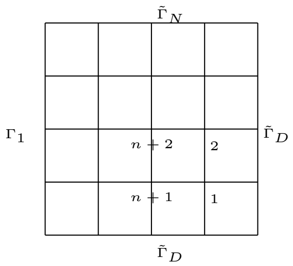


Figure 1: Domain and Grid Numbering

where the right hand side is vector inner product.

$$\|\phi^h\|^2_{H^1} = (\frac{1}{h^2}A\phi, \phi)_{L_{2,h}} = \frac{1}{h}(A\phi, \phi),$$

we have

$$\|\phi^h\|^2_{H^{\frac{1}{2}}} \simeq \left(\left(\frac{1}{h^2}A\right)^{1/2}\phi, \phi\right)_{L_{2,h}} = (A^{\frac{1}{2}}\phi, \phi).$$

So we are done with Dirichlet B.C. Next we consider Mixed B.C.

For Dirichlet boundary conditions we have

$$S(\varphi, \varphi) \approx \sum_l (\Sigma_l \varphi, \varphi), \qquad \Sigma_l = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}^{1/2}.$$

For mixed boundary conditions we have

$$S(\varphi, \varphi) \approx \sum_l \|\varphi^h\|^2_{\breve{H}^{\frac{1}{2}}(\gamma_l)},$$

where a_l is the endpoint of the interface γ_l lying on the Dirichlet boundary, and

$$\|\varphi\|^2_{\breve{H}^{\frac{1}{2}}(\gamma_l)} = \|\varphi\|^2_{H^{\frac{1}{2}}(\gamma_l)} + \int_{\gamma_l} \frac{\varphi^2(x)}{|x - a_l|}\, dx.$$

Let

$$A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & & & \\ & \ddots & 0 & \\ & & \ddots & \\ 0 & & 1 & \\ & & & 1/2 \end{bmatrix},$$

and

$$A_\Omega = \begin{bmatrix} A_1+2D & -D & & & \\ -D & A_1+2D & -D & & \\ & \ddots & \ddots & \ddots & \\ & & -D & A_1+2D & -D \\ & & & -D & \frac{1}{2}A_1+D \end{bmatrix}.$$

We note that A_1 corresponds to the first right vertical block.

$$\begin{aligned}(A_\Omega u, u) = \sum_{x(i,j)\in\Omega} &\{(u_{i,j}-u_{i-1,j})^2 + (u_{i,j}-u_{i,j-1})^2\} \\ &+ \frac{1}{2}\sum_{j=1}^{n}(u_{n,j}-u_{n,j-1})^2 + \frac{1}{2}\sum_{i=1}^{n}(u_{i,n}-u_{i-1,n})^2\end{aligned}$$

where the third sum corresponds to the left vertical and fourth sum corresponds to top horizontal line.

$$\begin{aligned}&\approx \sum_{x(i,j)\in\Omega}(\quad)^2 + \sum_{j=1}^{n}(\quad)^2 + \sum_{i=1}^{n}(\quad)^2 \\ &= (B_\Omega u, u),\end{aligned}$$

where

$$B_\Omega = \begin{bmatrix} A_1+2I & -I & & & \\ -I & A_1+2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & A_1+2I & -I \\ & & & -I & \frac{1}{2}A_1+I \end{bmatrix}.$$

Lemma 3.1 $A \sim B \Longrightarrow S_A \sim S_B$

$\lambda_{\min}(A_1) = O(h^2)$
$\lambda_{\max}(A_1) = O(1)$

$$S \approx \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}^{1/2}.$$

In this case the eigenvectors are not easy constructed. So consider

$$S = \left(\frac{1}{2}A_1 + D\right) - [0 \cdots 0 \ -D] \begin{bmatrix} \cdot & \cdot & \cdot & \\ \ddots & \ddots & \ddots & \\ & -D & A_1 + 2D & -D \\ & \ddots & \ddots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -D \end{bmatrix}$$

$$= D\left(\frac{1}{2}D^{-1}A_1 + I\right) - D[0 \cdots 0 \ -I] \left(\begin{bmatrix} D & & \\ & \ddots & \\ & & D \end{bmatrix} \begin{bmatrix} D^{-1}(A_1 + 2I) & -I & \\ -I & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -D \end{bmatrix}$$

$$= D\left(\frac{1}{2}D^{-1}A_1 + I\right) - D[0 \cdots 0 \ -I] \begin{bmatrix} D^{-1}(A_1 + 2I) & -I & \\ -I & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -I \end{bmatrix}$$

The following matrix corresponds to the finite difference version for Neumann problem by Samarsky:

$$A_2 = D^{-1}A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{bmatrix}.$$

We obtain

$$A_2 = Q_2 \Lambda_2 Q_2^{-1} = Q_2 \Lambda_2 Q_2^T D,$$

where

$$Q_2 = [q_1,\ q_2, \cdots], \qquad q_i(j) = \sqrt{\frac{2}{n}} \sin \frac{(2i-1)\pi j}{2n}, \qquad \lambda_i = 4 \sin^2 \frac{(2i-1)\pi}{2n},$$

for $i, j = 1, \cdots, n$. Here $\tilde{\Lambda}_2$ is obtained from Chebysheff polynomial.

$$S = DQ_2 \tilde{\Lambda}_2 Q_2^T D \approx DQ_2 \Lambda_2^{1/2} Q_2^T D = \Sigma_{DN}, \qquad \Sigma_{DN}^{-1} = Q_2 \Lambda_2^{-1/2} Q_2^T.$$

For implementation, use FFT for Q_2.

Fact: D-orthogonal basis.

$$\left(\begin{array}{l} A_2 q = D^{-1}A_1 q = \lambda q \qquad \Longrightarrow \qquad A_1 q = \lambda D q \\ (Dq_i, q_j) = \delta_{ij} \\ (D^{-1/2} A_1 D^{-1/2}) D^{1/2} q = \lambda D^{1/2} q \\ \tilde{q} = D^{1/2} q \\ (Dq_i, q_j) = (\tilde{q}_i, \tilde{q}_j) = \delta_{ij} \qquad \Longrightarrow \qquad Q_2^T D Q = I \qquad \Longrightarrow \qquad Q_2^{-1} = Q^T D. \end{array} \right)$$

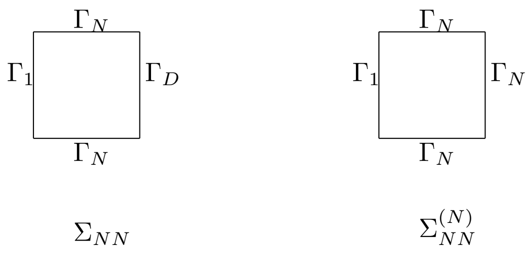


Figure 2: Two possibilities

Neumann B.C. both on top and bottom of boundary

In this case, we have

$$A_3 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 1/2 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1/2 \end{bmatrix}$$

Repeat the same analysis and we have two possibilities:

$$\Sigma_{NN} = A_3^{1/2} + \frac{1}{n} I, \qquad \Sigma_{NN}^{(N)} = A_3^{1/2}$$

$$\begin{aligned} \Omega_1 &\longrightarrow & S^{(1)} &\approx \Sigma_{DD}^{(1)}, \\ \Omega_2 &\longrightarrow & S^{(2)} &= \begin{bmatrix} S_{11}^{(2)} & S_{12}^{(2)} \\ S_{21}^{(2)} & S_{22}^{(2)} \end{bmatrix} \approx \begin{bmatrix} \Sigma_{DD}^{(1)} & \\ & \Sigma_{DD}^{(2)} \end{bmatrix} \\ \Omega_3 &\longrightarrow & S^{(3)} &= \begin{bmatrix} S_{11}^{(3)} & S_{12}^{(3)} \\ S_{21}^{(3)} & S_{22}^{(3)} \end{bmatrix} \approx \begin{bmatrix} \Sigma_{DN}^{(1)} & \\ & \Sigma_{DN}^{(3)} \end{bmatrix} \\ \Omega_4 &\longrightarrow & S^{(4)} &\approx \Sigma_{DN}^{(3)}, \end{aligned}$$

$$\Sigma = \begin{bmatrix} \Sigma_{DD}^{(1)} & & \\ & \Sigma_{DD}^{(2)} + \Sigma_{DN}^{(2)} & \\ & & \Sigma_{DN}^{(3)} + \Sigma_{ND}^{(3)} \end{bmatrix} \approx \begin{bmatrix} \Sigma_{DD}^{(1)} & & \\ & \Sigma_{DD}^{(2)} & \\ & & \Sigma_{DD}^{(3)} \end{bmatrix}$$

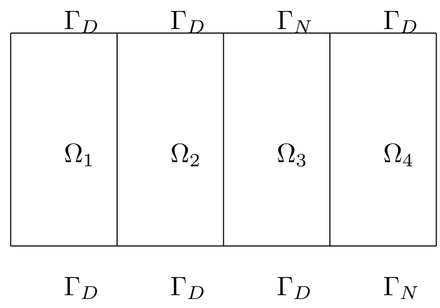


Figure 3: Domain partition with B.C.

Here $\Sigma_{DN}^{(2)}$ is smaller than $\Sigma_{DD}^{(2)}$ and $\Sigma_{ND}^{(3)} = \Sigma_{ND}^{(3)} \approx \Sigma_{DD}^{(3)}$.

$$\because \quad \Sigma_{DD}^{(2)} \leq \Sigma_{DD}^{(2)} + \Sigma_{DN}^{(2)} \leq c\Sigma_{DD} \qquad \text{on } \gamma_2$$
$$\Sigma_{DN}^{(3)} + \Sigma_{ND}^{(3)} \approx \Sigma_{DD}^{(3)} \qquad \text{on } \gamma_3$$
$$\text{Note } (\Sigma_{DN}^{(3)})^{-1} + (\Sigma_{ND}^{(3)})^{-1} \neq (\Sigma_{DN} + \Sigma_{ND})^{-1}.$$

$$(\Sigma_{DN}\varphi, \varphi) \approx \|\varphi^h\|_{H^{1/2}}^2 + \int \frac{(\varphi^h(x))^2}{x - a_3}\, dx$$

$$\Sigma = \begin{bmatrix} \Sigma_{DD}^{(1)} & & \\ & \Sigma_{NN}^{(2)} & \\ & & \Sigma_{DD}^{(3)} \end{bmatrix} \not\approx S$$

This is only semi norm. What to do in this case ? Use $-\Delta \approx -\Delta + I$ and construct a preconditioner for $-\Delta u + u$. Hence we have $\Sigma_{NN}^{(2)}$ in the second block of above expression.

Lemma 3.2 *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Then $B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

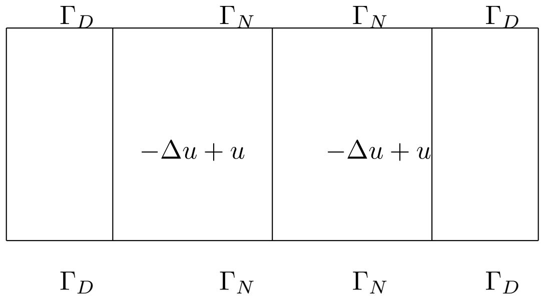


Figure 4: Another B.C.

Proof.

$$
\begin{aligned}
&\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \\
\Longrightarrow \quad & B_{11}A_{11} + B_{12}A_{21} = I_1,\ B_{11}A_{12} + B_{12}A_{22} = 0 \\
& B_{21}A_{11} + B_{22}A_{21} = 0,\ \ B_{21}A_{12} + B_{22}A_{22} = I_2 \\
\Longrightarrow \quad & B_{12} = -B_{11}A_{12}A_{22}^{-1},\ B_{11}A_{11} - B_{11}A_{12}A_{22}^{-1}A_{21} = I_1 \\
\Longrightarrow \quad & B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}
\end{aligned}
$$

□

Note: Similar result holds for B_{22}.
On artificial domain before,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$[0 \quad I]A^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} = S^{-1}$$

$$u = \begin{bmatrix} 0 \\ \varphi \end{bmatrix},\ \psi = \tilde{S}^{-1}\varphi \quad \Longrightarrow \quad v = A^{-1}u = \begin{bmatrix} \times \\ \psi \end{bmatrix},$$

$$\varphi^{k+1} = \varphi^k - \tau_k \Sigma^{-1}(S\varphi^k - \psi), \qquad \Sigma^{-1} = \tilde{S}^{-1}$$

$$\Sigma \longleftarrow \text{Neumann}, \qquad S \longleftarrow \text{Dirichlet}$$

$$S\varphi = (A_{22} - A_{21}A_{11}^{-1}A_{12})\varphi \longleftarrow \text{Orignal problem}$$

$$\Sigma = \begin{bmatrix} \Sigma_{**}^{(1)} & & \\ & \Sigma_{**}^{(2)} & \\ & & \Sigma_{**}^{(3)} \end{bmatrix} \approx S$$

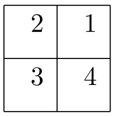


Figure 5: Cross Point

$$\Sigma^{-1} = \begin{bmatrix} (\Sigma_{**}^{(1)})^{-1} & & \\ & (\Sigma_{**}^{(2)})^{-1} & \\ & & (\Sigma_{**}^{(3)})^{-1} \end{bmatrix}$$

$$\Sigma_{**}^{(l)} \approx \tilde{S}_{**}^{(l)}, \qquad (\tilde{S}_{**}^{(l)})^{-1} = [0 \quad I_l]\tilde{A}_l^{-1} \begin{bmatrix} 0 \\ I_l \end{bmatrix}$$

Domain Decomposition: Cross-point Case

$$S = S^{(1)} + S^{(2)} + S^{(3)} + S^{(4)} \approx \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)} + \Sigma^{(4)} = \Sigma$$

On each angle, construct preconditioner. Inversion of each $\Sigma^{(i)}$ is meaningless. Problem: What is Σ^{-1}?

$$\Sigma^{-1} \not\approx (\Sigma^{(1)})^{+} + (\Sigma^{(2)})^{+} + (\Sigma^{(3)})^{+} + (\Sigma^{(4)})^{+}$$

Trace theorem is not enough. We need ASM.

4 Schwarz alternating method (H.A.Schwarz,1870)

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u(x) = 0 & x \in \Gamma \end{cases} \tag{9}$$

$$u^{2k+1} = u^{2k} + u_{2k+1}$$

$$\begin{cases} -\Delta u^{2k+1} = f & \text{in } \Omega_1 \\ u_{2k+1} = 0 & \text{on } \Gamma_1 = \partial\Omega_1 \end{cases} \tag{10}$$

(10) is equivalent to solving

$$\begin{cases} -\Delta u_{2k+1} = f - \Delta u^{2k} \\ u_{2k+1} = 0 \quad \text{in } \Gamma_1 \end{cases}$$

and set

$$u^{2k+1} = u^{2k} + u_{2k+1}$$

In subdomain Ω_2

$$u^{2k+2} \quad = \quad u^{2k+1} + u_{2k+2}$$

$$\begin{cases} -\Delta u^{2k+2} = f & \text{in } \Omega_2 \\ u_{2k+2} = 0 & \text{on } \Gamma_2 = \partial\Omega_2 \end{cases}$$

$$k = 0, u^1 = u^0 + u_1$$

$$\begin{cases} -\Delta u^1 = f & \text{in } \Omega \\ u^1|_{\Gamma_1 \cup \Gamma} = 0 \\ u^1|_{\Gamma_1 \setminus \Gamma} = u^0|_{\Gamma_1 \setminus \Gamma} \end{cases}$$

$$\Longleftrightarrow \begin{cases} -\Delta u^1 = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma_1. \end{cases}$$

$$a(u, v) = \int_\Omega (\nabla u, \nabla v)\, d\Omega$$

$$l(v) = \int fv\, d\Omega$$

$$u \in H_0^1(\Omega) : a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega)$$

(Considered by S.L. Sobolev, 1936)

$$u_{2k+1} \in H_0^1(\Omega_1) :$$

$$a(u^{2k} + u_{2k+1}, v) = l(v) \quad \forall v \in H_0^1(\Omega_1)$$

$$u^{2k+1} = u^{2k} + u_{2k+1}$$

$$u_{2k+2} \in H_0^1(\Omega_2) :$$

$$a(u^{2k+1} + u_{2k+2}, v) = l(v) \quad \forall v \in H_0^1(\Omega_2)$$

$$u^{2k+2} = u^{2k+1} + u_{2k+2}$$

$$H = H_0^1(\Omega) \quad H_1 = H_0^1(\Omega_1), \quad H_2 = H_0^1(\Omega_2)$$

$P_i : H \to H_i$, orthogonal projection in $a(u, v)$

$$
\begin{aligned}
a(u_{2k+1}, v) &= l(v) - a(u^{2k}, v) = a(u, v) - a(u^{2k}, v) \\
&= a(u - u^{2k}, v) \quad \forall v \in H_1
\end{aligned}
$$

$$
\begin{cases}
u_{2k+1} = P_1(u - u^{2k}) \\
u^{2k+1} = u^{2k} + P_1(u - u^{2k}) \\
u^{2k+2} = u^{2k+1} + P_2(u - u^{2k+1})
\end{cases}
\Longleftarrow
\begin{cases}
u^{2k+1} - u \\
= u^{2k} - u + P_1(u - u^{2k}) \\
= (I - P_1)(u - u^{2k})
\end{cases}
$$

$$
\psi^k = u^k - u
$$

$$
Q_i : H \to H_1^{\perp}, \quad Q_i = I - P_i
$$

$$
\psi^{2k+1} = (I - P_1)\psi^{2k} = Q_1\psi^{2k}
$$

$$
\psi^{2k+2} = (I - P_2)\psi^{2k+1} = Q_2\psi^{2k+1}
$$

$$
\begin{aligned}
k &\geq 1 \\
a(\psi^{2k+1}, \psi^{2k+1}) &= \|\psi^{2k+1}\|_a^2 = \|Q_1\psi^{2k}\|_a^2 \\
&= \|Q_1Q_2\psi^{2k}\|_a^2 = a(Q_1Q_2\psi^{2k}, Q_1Q_2\psi^{2k}) \\
&= a(Q_2Q_1Q_1Q_2\psi^{2k}, \psi^{2k}) = a(Q_2Q_1Q_2\psi^{2k}, \psi^{2k}) \\
&= a((I - (P_1 + P_2) + P_1P_2 + P_2P_1 - P_2P_1P_2)\psi^{2k}, \psi^{2k}) \\
&= a(\psi^{2k}, \psi^{2k}) - a((P_1 + P_2)\psi^{2k}, \psi^{2k}) + a(P_1P_2\psi^{2k}, \psi^{2k}) \\
&\quad + a(P_2P_1\psi^{2k}, \psi^{2k}) - a(P_2P_1P_2\psi^{2k}, \psi^{2k}) \\
&= a(\psi^{2k}, \psi^{2k}) - a((P_1 + P_2)\psi^{2k}, \psi^{2k})
\end{aligned}
$$

Assume $\alpha a(u, u) \leq a((P_1 + P_2)u, u) \quad \forall u \in H$

$$
\begin{cases}
\|\psi^{2k+1}\|_a \leq (1 - \alpha)^{1/2}\|\psi^{2k}\|_a \\
\|\psi^{2k+2}\|_a \leq (1 - \alpha)^{1/2}\|\psi^{2k+1}\|_a
\end{cases}
$$

$$
\Longrightarrow \|\psi^{2k+2}\|_a \leq (1 - \alpha)\|\psi^{2k}\|_a
$$

5 Additive Schwarz Method (A.Matsokin,SN 1985)

$$
\alpha a(u, u) \leq a((P_1 + P_2)u, u) \leq 2a(u, u) \quad \forall u \in H
$$

$$
u^0 \in H
$$

$$u^{k+1} = u^k - \tau_k(P_1 + P_2)(u^k - u) \quad k = 0, 1, 2, \cdots$$

Theorem 5.1 *Let H be a Hilbert space with (u, v)*

$$H = H_1 + H_2 + \cdots + H_m$$
$$A : H \to H \quad 0 < A = A^* < \infty$$
$$a(u, v) = (Au, v)$$

$P_i : H \to H_i$ orthogonal projection in $a(u, v)$

a) $\exists\alpha : \forall u \in H \quad \exists u_i \in H_i$

$$u_1 + u_2 + \cdots + u_m = u$$
$$\alpha(a(u_1, u_1) + a(u_2 + u_2) + \cdots + a(u_m, u_m)) \leq a(u, u)$$

b) $\alpha a(u, u) \leq a((P_1 + P_2 + \cdots + P_m)u, u) \quad \forall u \in H$

Then a) is equivalent to b)

Proof. b)$\Longrightarrow$ a)

$$P = P_1 + P_2 + \cdots + P_m$$
$$\infty > P = P^* > 0$$

$\forall u$, $\exists v$ s.t. $u = Pv = \sum_{i=1}^m P_i v$

Let $u_i = P_i v$

$$\begin{aligned}\sum_{i=1}^m a(u_i, u_i) &= \sum_{i=1}^m a(P_i v, P_i v) = \sum_{i=1}^m a(P_i v, v) = a(\sum_{i=1}^m P_i v, v) \\ &= a(u, v) = a(u, P^{-1}u) \leq \frac{1}{\alpha} a(u, u)\end{aligned}$$

a)$\Longrightarrow$ b)

Lemma : $\|u\|_a = \sup_{v\in H} \frac{a(u,v)}{\|v\|_a}$

(proof) :

$$\sup_{v\in H} \frac{a(u, v)}{\|v\|_a} \leq \text{C.B} \leq \sup_{v\in H} \frac{\|u\|_a\|v\|_a}{\|v\|_a} = \|u\|_a$$

Take $v = u$

$$\sup \frac{a(u, v)}{\|v\|_a} \geq \frac{a(u, u)}{\|u\|_a} = \|u\|_a \quad \square$$

$$
\begin{aligned}
& u \in H \\
\|u\|_a &= \sup_{v\in H} \frac{a(u,v)}{\|v\|_a} = \sup_{v\in H} \frac{a(u,\sum_{i=1}^m v_i)}{\|v\|_a} \\
&= \sup_{v\in H} \sum_{i=1}^m \frac{a(u,P_i v_i)}{\|v\|_a} = \sup_{v\in H} \sum_{i=1}^m \frac{a(P_i u, v_i)}{\|v\|_a} \\
&\le \text{C.B.} \le \sup_{v\in H} \frac{\sum_{i=1}^m \|P_i u\|_a \|v_i\|_a}{\|v\|_a} \le \text{C.B.} \\
&\le \sup_{v\in H} \frac{\sqrt{\sum_{i=1}^m \|P_i u\|_a^2}\cdot\sqrt{\sum_{i=1}^m \|v_i\|_a^2}}{\|v\|_a} \\
&\le \; \le \frac{1}{\sqrt{\alpha}} \sqrt{\sum_{i=1}^m \|P_i u\|_a^2}
\end{aligned}
$$

by (a). □

We want to show for any $u \in H_0^1(\Omega)$ there exists $u_i \in H_0^1(\Omega)$: such that $u_1 + u_2 = u$ and

$$\|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)} \le \frac{1}{\alpha}\|u\|^2_{H^1(\Omega)}$$

$$u_1(x) = \begin{cases} u(x), & x \in \Omega_1 \backslash \Omega_2 \\ \text{extension} & x \in \Omega_1 \cap \Omega_2 \end{cases}$$

$$\|u_1\|_{H^1(\Omega_1)} \le C\|u\|_{H^1(\Omega)} \quad u_1 \in H_0^1(\Omega_1)$$

$$u_2 = u - u_1, \quad u_2 \in H_0^1(\Omega_2)$$

$$\|u_2\|_{H^1(\Omega_2)} \le \|u\|_{H^1(\Omega)} + \|u_1\|_{H^1(\Omega_1)} \le (1+C)\|u\|_{H^1(\Omega)}$$

convergence depends on the extension.

$$a(Pu,u) \le ? \le m \cdot a(u,u) \qquad m : \sharp \text{ of subspace}$$

Theorem 5.2 *a)* $a(Pu,u) \le \beta a(u,u) \quad ,\forall u$

b) $a(u,u) \le \beta \inf_{u_1+\dots+u_m=u, u_i\in H_i} \sum_{i=1}^m a(u_i,u_i)$

Then a) $\Longleftrightarrow$ *b).*

Proof. Let $u \in H$.

Put $u_i = P_i P^{-1} u$.

$$u_1 + \cdots + u_m = P_1 P^{-1} u + \cdots + P_m P^{-1} u = u$$

Let

$$v_i \in H_i : v_1 + \cdots + v_m = u$$

be another decomposition with $v_i = u_i + w_i$ then

$$\sum_{i=1}^{m} w_i = 0$$

$$\begin{aligned}
\sum_{i=1}^{m} a(v_i, v_i) &= \sum_{i=1}^{m} a(u_i, u_i) + 2a(u_i, w_i) + a(w_i, w_i) \\
&= \sum_{i=1}^{m} a(u_i, u_i) + 2a(P_i P^{-1} u, w_i) + a(w_i, w_i) \\
&= \sum_{i=1}^{m} a(u_i, u_i) + 2a(P^{-1} u, \sum_{i=1}^{m} w_i) + \sum_{i=1}^{m} a(w_i, w_i)
\end{aligned}$$

$$\begin{aligned}
\inf_{u = v_1 + \cdots + v_m, v_i \in H_i} \sum_{i=1}^{m} a(v_i, v_i) &= \sum_{i=1}^{m} a(u_i, u_i) = \sum_{i=1}^{m} a(P_i P^{-1} u, P_i P^{-1} u) \\
&= \sum_{i=1}^{m} a(P^{-1} u, P_i P^{-1} u) = a(P^{-1} u, P P^{-1} u) = a(P^{-1} u, u)
\end{aligned}$$

$$a(Pu, u) \le \beta a(u, u), \forall u \Longleftrightarrow a(u, u) \le \beta a(P^{-1} u, u), \forall u$$
$$\text{(proof: Take } u = P^{1/2} v\text{)}$$

□

Lemma 5.1 *With*

$$a(u, v) = A(u, v)$$

define

$$A_i : H_i \to H_i, \quad (A_i u_i, v_i) = A(u_i, v_i) \quad u_i, v_i \in H_i$$

and

$$Q_i : H \to H_i \quad \textit{in } (\cdot, \cdot)$$

Then $P_i = A_i^{-1} Q_i A$

Proof. $\forall u \in H, u_i = P_i u$

Denote $w_i = A_i^{-1} Q_i A u \quad , i.e., A w_i = Q_i A u$

$$\begin{array}{lll} \forall v_i & (A_i w_i, v_i) & = (Q_i A u, v_i) = (Au, v_i) = a(u, v_i) \\ & \quad \| & \\ & (A w_i, v_i) & = a(w_i, v_i) \end{array}$$

Hence $w_i = u_i$ □

$$a(u, v) = (u, v)_{H^1}$$

$$(u, v) = (u, v)_{L_2}$$

The previous Lemma gives the relation between the projection P_i corresponding to the given bilinear form $a(\cdot, \cdot)$ and L_2 projection Q_i.

That is, we began with

$$H = H_1 + H_2 + \cdots + H_m$$

$$a(u, v) = (Au, v)$$

$$P_i : H \to H_i$$

$$\alpha a(u, u) \leq a(Pu, u) \leq \beta a(u, u) \quad , \forall u \in H$$

$$P = P_1 + P_2 + \cdots + P_m$$

and then we have proved that

$$P_i = A_i^{-1} Q_i A$$

where $Q_i : H \to H_i \quad$ in $(\cdot, \cdot)$. Now, we have

$$\alpha (Au, u) \leq (A(\sum_{i=1}^{m} A_i^{-1} Q_i A) u, u) \leq \beta (Au, u)$$

and this is equivalent to

$$\alpha (Au, u) \leq (AB^{-1} Au, u) \leq \beta (Au, u) \quad \forall u \in H$$

where $B^{-1} = \sum_{i=1}^{m} Q_i A_i^{-1} Q_i$. Putting $Au = v$,

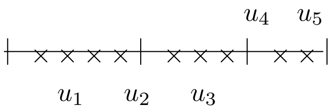
$$\alpha (A^{-1} v, v) \leq (B^{-1} v, v) \leq \beta (A^{-1} v, v) \quad \forall v \in H$$

or

$$\alpha (Bu, u) \leq (Au, u) \leq \beta (Bu, u) \quad \forall v \in H$$

Thus, we constructed B so far, which is equivalent to A and we now use B as a preconditioner so that

$$u^{k+1} = u^k - \tau_k B^{-1} (Au^k - f)$$



Example 5.1 *(Simple 1 dim'l example)*

Simply consider the equation $-u'' = f$ *in* Ω *with the boundary condition* $u(0) = u(1) = 0$. *Then we have*

$$Au = f$$

with

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Note that A *is* $(n-1)\times(n-1)$ *matrix and* $H = R^{n-1}$. *As above figure we define*

$$H = H_1 + H_2 \quad \Omega = \Omega_1 \cup \Omega_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

where

$$H_1 = \{(u_1, u_2, u_3, 0, 0)^t\}, H_2 = \{(0, 0, u_3, u_4, u_5)^t\}$$

Here we can take $\beta = 2$ *by the property of the projection. For* $u \in H$, *we want to find* α *such that*

$$\alpha \sum_{i=1}^{2} a(u_i, u_i) \le a(u, u), \quad u = u_1 + u_2$$

where $u_i \in H_i$. *From the figure,*

$$(Au_1, u_1) \le c(Au, u), \quad c \quad \text{is independent of} \quad h$$

Letting $u_2 = u - u_1$, *then we have such* α *independent of* h. *In this case*

$$Q_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

and

$$Q_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ 0 \end{bmatrix}, \quad Q_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & 0 \\ & & & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & & & \\ 0 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Note that A_1 has zero entries that correspond to u_4, u_5-multiplication and A_2 do to u_1, u_2-multiplication. Now, we obtain the preconditioner as following

$$B^{-1} = Q_1 A_1^+ Q_1 + Q_2 A_2^+ Q_2 = A_1^+ + A_2^+$$

where A_i^+ is the pseudo-inverse of A_i.

Theorem 5.3

$$H = H_1 + H_2 + \cdots + H_m$$

$$a(u,v) = (Au, v)$$

Let $P_i : H \longrightarrow H_i$ be the orthogonal projection w.r.t $a(\cdot,\cdot)$ and A is symmetric and positive definite. Furthermore, we have following three conditions:

(1) $\alpha(a(u_1,u_1) + \cdots + a(u_m,u_m)) \le a(u,u), \quad u_1 + \cdots + u_m = u$

$$(\text{ it is } \Longleftrightarrow \alpha a(u,u) \le a(Pu,u), \quad P = P_1 + \cdots + P_m)$$

(2) $a(Pu,u) \le \beta a(u,u)($ *it is* $\Longleftrightarrow a(u,u) \le \beta \inf_{u_1+\cdots+u_m=u} \sum_{i=1}^m a(u_i,u_i))$

(3) There are local preconditioners $B_i : H \longrightarrow H_i$ with $B = B^*$ satisfying that for some constant c_1 and c_2

$$c_1(B_i u, u) \le (Au, u) \le c_2(B_i u, u), \quad \forall u \in H_i$$

Then, we have

$$\alpha c_1(A^{-1}u, u) \le (B^{-1}u, u) \le \beta c_2(A^{-1}u, u), \quad \forall u \in H$$

where $B^{-1} = B_1^+ + \cdots + B_m^+$. (For matrix C, $C^+ :=$ pseudo-inverse of C).

Proof.

Note that $P_i = Q_i A_i^{-1} Q_i A$. We have a pseudo-inverse

$$(Q_i A Q_i)^+ = Q_i A_i^{-1} Q_i$$

since $(Q_i A Q_i) Q_i A_i^{-1} Q_i = Q_i A Q_i A_i^{-1} Q_i = Q_i$.

¿From (1) and (2) we have

$$\alpha(A^{-1}v, v) \le (((Q_1 A Q_1)^+ + \cdots + (Q_m A Q_m)^+)v, v) \le \beta(A^{-1}v, v)$$

and from (3)

$$c_1((Q_i A Q_i)^+ u, u) \le (B_i^+ u, u) \le c_2(c(Q_i A Q_i)^+ u, u), \quad \forall u \in H_i$$

Combining above two inequalities, we get the result of the theorem. □

Lemma 5.2 *Let* $\varphi \in H^{1/2}(-1,0)$. *Define*

$$\varphi = \begin{cases} (1-x)\varphi(-x), & x \in [0,1] \\ 0, & x \in [1,2] \end{cases}$$

Then, $\exists\ C$ *such that* $\|\varphi\|_{H^{1/2}(-1,2)} \le C\|\varphi\|_{H^{1/2}(-1,0)}$.

Proof.

By previous lemma 6.4, we have

$$\|\varphi\|^2_{H^{1/2}(-1,2)} \le C_1(\|\varphi\|^2_{H^{1/2}(-1,0)} + \|\varphi\|^2_{H^{1/2}(0,1)} + \|\varphi\|^2_{H^{1/2}(1,2)} + I_1(\varphi) + I_2(\varphi))$$

Note that $\|\varphi\|_{H^{1/2}(1,2)} = 0$. Now

$$\|\varphi\|_{L^2(0,1)} \le \|\varphi\|_{L^2(-1,0)}$$

$$\begin{aligned}
|\varphi|^2_{H^{1/2}(0,1)} &= \int_0^1\int_0^1 \frac{|\varphi(-x)(1-x) - \varphi(-y)(1-y)|^2}{|x-y|^2}\,dxdy \\
&\le 2\int_0^1\int_0^1 \frac{|\varphi(-x)(1-x) - \varphi(-x)(1-y)|^2}{|x-y|^2}\,dxdy \\
&\qquad + 2\int_0^1\int_0^1 \frac{|\varphi(-x) - \varphi(-y)|^2|(1-y)|^2}{|x-y|^2}\,dxdy \\
&\le 2\left(\int_0^1\int_0^1 \frac{|\varphi(-x)(x-y)|^2}{|x-y|^2}\,dxdy + \int_{-1}^0\int_{-1}^0 \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^2}\,dxdy\right) \\
&= 2(|\varphi|^2_{H^{1/2}(-1,0)} + \|\varphi\|^2_{L^2(-1,0)})
\end{aligned}$$

On the other hand,

$$I_1(\varphi) = \int_0^1 \frac{(\varphi(-x) - \varphi(-x)(1-x))^2}{x}\, dx$$
$$\leq \int_0^1 \varphi^2(-x)\frac{x^2}{x}\, dx \leq \|\varphi\|^2_{L^2(-1,0)}$$

and

$$I_2(\varphi) = \int_0^1 \frac{(\varphi(-x)(1-x))^2}{1-x}\, dx \leq \|\varphi\|^2_{L^2(-1,0)}$$

Gathering all inequalities complete the proof. □

Our Goal is to construct Schur-complement on the lines(sub-boundaries) of the interior of the given domain.

$\Lambda = \bigcup_{i=1}^n \partial\Omega_i = \bigcup_{i=1}^m \lambda_i$

For crossing points,

$$k_i = O(1/h), \quad i = 1, \ldots, m_1$$

and for usual lines,

$$k_i = O(1/h), \quad i = m_1 + 1, \ldots, m$$

Assume $\exists\ r$ independent of h such that $\forall p \in \Lambda$ there exists λ_i:

$$B(p,r) \bigcap \Lambda \subset \lambda_i$$

Let $H = H_h(\Lambda)$ and $H = H_1 + H_2 + \cdots + H_m$ with

$$H_i = H_h(\lambda_i) = \{\varphi^h \in H_h(\Lambda) | \varphi(x) = 0, \exists x \notin \lambda_i\}$$

By previous Lemma 9.2, we have $\forall \varphi^h \in H, \exists \varphi_i^h \in H_i$,

$$\|\varphi_1^h\|^2_{\overset{00}{H}{}^{1/2}(\lambda_1)} + \cdots + \|\varphi_m^h\|^2_{\overset{00}{H}{}^{1/2}(\lambda_m)} \leq C\|\varphi^h\|^2_{H^{1/2}(\Lambda)}$$

Define

$$\tilde{H}_1 = H_1 + \cdots + H_{m_1}, \quad \tilde{H}_2 = H_{m_1+1} + \cdots + H_m$$

Then $H = \tilde{H}_1 + \tilde{H}_2$. So far we have constructed the space satisfying the conditions (1) and (2) in the previous theorem 9.3. Now, we'll construct a preconditioner for the Schur-complement by an additive form of pseudo-inverses such as

$$\Sigma^{-1} = \Sigma_1^+ + \cdots + \Sigma_m^+$$

as followings: For $i = m_1 + 1, \ldots, m$,

$$\Sigma_i = R_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} R_i^t$$

where R_i is the permutation matrix and X is the matrix corresponding to the 1 dim'l Laplacian. i. e.

$$X = \begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}$$

Hence

$$\Sigma_i^+ = R_i \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{-1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix} R_i^t$$

For the cross point, we introduce the following lemma.

Lemma 5.3 *Symmetric and positive definite matrices*

$$\Sigma : R^m \longrightarrow R^m, \quad S : R^n \longrightarrow R^n$$

are given. Let $t : R^m \longrightarrow R^n$ *such that*

$$\alpha(\varphi, \varphi)_\Sigma \le (t\varphi, t\varphi)_S \le \beta(\varphi, \varphi)_\Sigma, \quad \forall \varphi \in R^m$$

$$(t^T u, \varphi)_{R^n} = (u, t\varphi)_{R^n}$$

where $(\cdot, \cdot)_{R^i}$ *is the Euclidian inner product. Denote* $C = t\Sigma^{-1}t^T$. *Then we have*

$$\alpha(C^+u, u) \le (u, u)_S \le \beta(C^+u, u), \quad \forall u \in Im(t)$$

Remark 5.1 $m \le n$ *should be hold and* t *can be interpreted a kind of extension operator.*

Proof.
$\exists (t^T t)^{-1}$ by assumption. We note that

$$C^+ = t(t^T t)^{-1}\Sigma(t^T t)^{-1}t^T$$

,which is easily verified from the following observation

$$C^+C = t(t^T t)^{-1}\Sigma(t^T t)^{-1}t^T(t\Sigma^{-1}t^T) = t(t^T t)^{-1}t^T$$

Now it's sufficient to check $t(t^T t)^{-1}t^T$ is a projection. If $u \in Im(t)$, then $u = t\varphi$ for some φ. Thus

$$C^+Cu = t(t^T t)^{-1}t^T u = t\varphi = u$$

Also, for all $v_0 \in (Im(t))^\perp$,

$$0 = (v_0, t\varphi) = (v_0, t(t^T t)^{-1} t^T t\varphi) = (t(tTt)^{-1} t^T v_0, t\varphi), \quad \forall \varphi$$

Hence we have

$$C^+ C v_0 = 0, \quad \forall v_0 \in (Im(t))^\perp$$

Now, $\forall u \in Im(t)$

$$\begin{aligned}(C^+ u, u) &= (C^+ t\varphi, t\varphi) = (t(t^T t)^{-1} \Sigma (t^T t)^{-1} t^T t\varphi, t\varphi) \\ &= (t^T t(t^T t)^{-1} \Sigma\varphi, t\varphi) \\ &= (\Sigma\varphi, \varphi)\end{aligned}$$

Hence the proof is completed. □

In the Additive Schwartz Method, we need to define B_i^+. Now we'll try to set

$$B_i^+ = (C_i^+)^+ = C_i = t\Sigma^{-1} t^T$$

where $Im(t) := H_i$ by using certain proper extension operator t.

6 Additive Schwarz Method on interfaces

Let z_0 be a fixed cross point. Let λ be the union of branches emerging from z_0. Let L_i, for $i = 1, \cdots, m$ be each branch. Let $L_{m+1} = L_1$. Define a trace norm on λ by

$$\|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} = \sum_{i=1}^{m} \|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(L_i \cup L_{i+1})}$$

Let $x_{i,j}$ be the point on the branch L_i which has distance jh from z_0. Let

$$H_h(\lambda) = H_0 + H_1 + \cdots + H_m$$

$$H_i = \{\phi^h \in H_h(\lambda) | \phi^h(x) = 0, x \notin L_i\}$$

$$H_0 = \{\phi^h \in H^h(\lambda) | \phi^h(x_{1,j}) = \cdots = \phi^h(x_{m,j}), j = 1, 2, \cdots, k\}$$

Here we assume that each L_i has the same number k of nodes.

Lemma 6.1 *There exists c independent of h such that for each $\phi^h \in H^h(\lambda)$, there exist $\phi_i^h \in H_i$ which satisfy*

$$\phi_0^h + \phi_1^h + \cdots + \phi_m^h = \phi^h$$

$$\|\phi_0^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} + \|\phi_1^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} + \cdots + + \|\phi_m^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} \le c \|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)}$$

Proof. Let $\phi \in H_h(\lambda)$. Define $\phi_0^h(x_{i,j}) = \phi^h(x_{1,j})$, for $j = 1, \cdots, k$ and $i = 1, \cdots, m$.(Fix first branch and rotate it). It is clear that $\phi_0^h \in H_0$. Let $\psi^h = \phi^h|_{L_1}$ Since

$$\|\phi_0^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} = m\|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(L_1 \cup L_2)} \simeq \|\psi^h\|^2_{\check{H}^{\frac{1}{2}}(L_1)} \simeq (\Sigma_{ND}\psi, \psi),$$

there exists c_1 independent of h such that

$$\|\phi_0^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} \leq c_1\|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)}.$$

Let $\xi^h = \phi^h - \phi_0^h$. Define $\phi_i^h(x_{i,j}) = \xi^h(x_{i,j})$. Then

$$\|\xi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} \leq c_2\|\phi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)}$$

$$\|\phi_i^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} \simeq \|\phi_i^h\|^2_{H_{00}^{\frac{1}{2}}(L_i)} \simeq (\Sigma_{DD}\phi_i, \phi_i).$$

Hence we obtain

$$\|\phi_i^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)} \leq \|\xi^h\|^2_{H_{00}^{\frac{1}{2}}(\lambda)}$$

Continuing the above processes , we can prove the lemma.

□

The lemma shows that

$$\frac{1}{c}a(\phi^h, \phi^h) \leq a((P_0 + \cdots + P_m)\phi^h, \phi^h) \leq (m+1)a(\phi^h, \phi^h)$$

Let t be the extension operator such that for each $\psi^h = \begin{bmatrix}\psi_0 & \psi_1 & \cdots & \psi_m\end{bmatrix}^T$,

$$t\phi^h = \begin{bmatrix}\psi_0 & \eta & \cdots & \eta\end{bmatrix}^T$$

where $\eta = \begin{bmatrix}\psi_1 & \cdots & \psi_m\end{bmatrix}^T$. $H_0 = t \cdot F$, $F = H_h(L_1)$. And we have

$$\|\psi^h\|_{\check{H}^{\frac{1}{2}}(Ł_1)} \leq \|t\psi^h\|_{H_{00}^{\frac{1}{2}}(\lambda)} \leq C\|\psi^h\|_{\check{H}^{\frac{1}{2}}(L_1)}$$

So

$$B_i^+ = t\Sigma_{ND}^{-1}t^T$$

Now decompose the whole interface space $H_h(\Lambda)$ into subspaces. Let

$$H_h(\Lambda) = H_1^{(N)} + \cdots + H_{m_1}^{(N)} + H_{m_1+1}^{(0)} + \cdots + H_m^{(0)}$$

where $H_i^{(N)}$, for $i = 1, \cdots, m_1$ are the subspace corresponding to cross points

$$H_i^N = \{\varphi^h \in H_h(\Lambda) \mid \varphi^h(x) = t_i\psi^h(x), x \in \lambda_i, \varphi^h(x) = 0, x \notin \lambda_i\}$$

and $H_i^{(0)}$, for $i = m_1 + 1, \cdots, m$ are the subspace corresponding to intervals between cross points

$$H_i^0 = \{\varphi^h \in H_h(\Lambda) \mid \varphi^h(x) = 0, x \notin \lambda_i\}$$

Let

$$B^{-1} = B^+_{N,1} + \cdots + B^+_{N,m_1} + B^+_{0,m_1+1} + \cdots + B^+_{0,m},$$

where

$$B^+_{0,i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^{-1/2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

X is one dimensional Laplacian operator and

$$B^+_{N,i} = t_i \Sigma^{-1}_{ND} t_i^T.$$

Then B is a preconditioner of $S = A_0 - \sum_{i=1}^n A_{0i} A_i^{-1} A_{i0}$. Here A_i^{-1} is expensive. With a local preconditioner on Ω_i how to construct a global preconditioner ? We can use ASM.

6.1 Non-exact Solvers

Let $\overline{\Omega} = \cup_{i=1}^n \overline{\Omega_i}, \ \ \Omega_i \cap \Omega_j = \emptyset$, if $i \neq j$. Let

$$H_h(\Omega) = H_0 + H_1$$

where

$$H_0 = H_{h,0}(\Omega_1) \oplus \cdots \oplus H_{h,0}(\Omega_n)$$
$$H_{h,0}(\Omega_i) = \{\phi^h \in H_h(\Omega) | \phi^h(x) = 0, x \notin \Omega_i\}.$$

Assume that we have the followings.

$A) \ \exists B_i, \ \ c_1\|u^h\|^2_{H^1(\Omega)} \leq (B_i u, u) \leq c_2 \|u^h\|^2_{H^1(\Omega)}, \forall u^h \in H_{h,0}(\Omega_i)$

$B) \ \exists t, \ \ t : H_h(\Gamma_i) \to H_h(\Omega_i), s.t.$

$\|t\phi^h\|_{H_1(\Omega_i)} \leq c_3 \|\phi^h\|_{H^{\frac{1}{2}}(\Gamma_i)}$

where $H_1 = tH_h(\lambda)$

$C) \ c_4\|\phi^h\|^2_{H^{\frac{1}{2}}(\lambda)} \leq (\Sigma\phi, \phi) \leq c_5 \|\phi^h\|^2_{H^{\frac{1}{2}}(\Lambda)}, \forall \phi^h \in H_h(\Lambda)$

Theorem 6.1 *Let*

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1^{-1} & & \\ & & \cdots & \\ & & & B_n^{-1} \end{bmatrix} + t\Sigma^{-1} t^T.$$

Then there exist $\alpha, \beta = \alpha, \beta(c_1, c_2, \cdots, c_5)$ *such that*

$$\alpha(Bu, u) \leq (Au, u) \leq \beta(Bu, u), \ \forall u.$$

Proof. *a*) Let $u^h \in H_h(\Omega^h)$ and $\phi^h \in H^h(\Lambda)$ such that $\phi^h(x) = u^h(x), x \in \Lambda$. Then there exists c_6 independent of h such that

$$\|\phi^h\|_{H^{\frac{1}{2}}(\Lambda)} \leq c_6 \|u^h\|^2_{H^1(\Omega)}.$$

Let $u_1^h = t\phi^h$. Then

$$\|u_1^h\|^2_{H^1(\Omega)} \leq c_3 \|\phi^h\|_{H^{\frac{1}{2}}(\Lambda)} \leq c_3 c_6 \|u^h\|^2_{H^1(\Omega)}$$

Let $u_0^h = u^h - u_1^h$. We have the similar inequalities using triangle inequality.
b) $a((P_1 + P_2)u, u) \leq 2a(u, u), \forall u$
c) $B_i \simeq A$.
$A), B), C)$ and the lemma completes the proof.
□

6.2 Explicit Extension Operators

One simple extension operator is harmonic extension. But consider another one: Let (s, n) be a near boundary coordinate system. Using this we will construct a extension operator t. Let ϕ be a given function defined on boundary Γ of domain Ω For continuous case, we can define $u = t\phi$ by

$$u(s, n) = \xi(n) \frac{1}{n} \int_s^{s+n} \phi(t) dt.$$

where $\xi(n) = 1 - \frac{n}{D}$. For discrete case, we can define u by the following three steps.

Step 1)

$$V(z_{ij}) = \sum_{l=0}^{j} \phi(i + l).$$

Step 2)

$$V(z_{ij}) = \frac{1 - \frac{j}{n}}{j + 1} V(z_{ij}).$$

Step 3)

$$u^h(z_l) = \begin{cases} V(z_{ij}) & \text{if } z_l \in D_{ij} \\ 0 & \text{if } z_l \notin \cup_{ij} D_{ij} \end{cases}$$

Then $u = t\phi = P_3 P_2 P_1 \phi$.

$$P_3 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ & I & \end{bmatrix}$$

where lower identity matrix corresponds to D. Then

$$\|u^h\|_{H_h^1(\Omega^h)} \leq c\|V\|_{H_h^1(D^h)}.$$

Let $P_2 =$ Diag$\{\cdots, \frac{1-\frac{j}{n}}{j+1}, \cdots\}$.
Let

$$P_1 = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ 1 & \cdot & \cdot & \cdot 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$V(z_{i,0} = \phi(i), i = 0, \cdots, N-1$$

$$P_1\phi = V(z_{i,j+1}) = V(z_{i,j}) + \phi(i+j+1), 0 \leq i \leq N-1, \quad 0 \leq j \leq M$$

Cost of P_2 is $O(h^{-2})$. Let

$$t^T = P_1^T P_2^T P_3^T$$

For any given function V, define W by

$$W(z_{i,M}) = V(z_{i,M}), i = 0, \cdots, M$$

and for $i = 0, \cdots, N$ and $j = M, M-1, \cdots, 1$,

$$W(z_{i,j-1}) = W(z_{i,j}) + V(z_{i,j-1})$$

If we let $\phi_i = \sum_{j=0}^{N} \sum_{l=i-j}^{i} V(z_{l,j})$, $W(z_{i,M}) = V(z_{i,M})$

$$W_1(t_{i,j-1}) = W_1(z_{i,j}) + V(z_{i,j-1})$$

then $\phi_i = \sum_{j=0}^{M} W(z_{i-j})$. Cost of t^T is again $O(h^{-2})$.

7 Domain Decomposition with Many Subdomains ($n >> 1$)

Let Ω be a domain of diameter $O(1)$ with boundary Γ, and set

$$\Omega_\varepsilon = \{(x,y) : x = \varepsilon s,\ y = \varepsilon t,\ (x,y) \in \Omega\}$$

with boundary Γ_ε.

Lemma 7.1 *There exists $c_1 \neq c_1(\varepsilon)$ such that for all $u \in H^1(\Omega_\varepsilon)$,*

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad |\varphi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \leq c_1 |u|_{H^1(\Omega_\varepsilon)}.$$

There exists $c_2 \neq c_2(\varepsilon)$ such that for every $\varphi \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$, there exists $u \in H^1(\Omega_\varepsilon)$ satisfying

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad |u|_{H^1(\Omega_\varepsilon)} \leq c_2 |\varphi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}.$$

Proof.

$$|u|^2_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$
$$= \int_{\Omega} \left(\frac{\partial \tilde{u}}{\partial s}\right)^2 + \left(\frac{\partial \tilde{u}}{\partial t}\right)^2 = |\tilde{u}|^2_{H^1(\Omega)}.$$

$$|\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} = \int_{\Gamma_\varepsilon}\int_{\Gamma_\varepsilon} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dxdy$$
$$= \int_{\Gamma}\int_{\Gamma} \frac{(\tilde{\varphi}(s)-\tilde{\varphi}(t))^2}{|s-t|^2}\,dsdt = |\tilde{\varphi}|^2_{H^{\frac{1}{2}}(\Gamma)}$$

□

Now we define

$$\|\varphi\|^2_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma_\varepsilon)} = \varepsilon\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}.$$

Lemma 7.2 *There exists $c_1 \neq c_1(\varepsilon)$ such that for all $u \in H^1(\Omega_\varepsilon)$,*

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad \|\varphi\|_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma_\varepsilon)} \leq c_1\|u\|_{H^1(\Omega_\varepsilon)}.$$

There exists $c_2 \neq c_2(\varepsilon)$ such that for every $\varphi \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$, there exists $u \in H^1(\Omega_\varepsilon)$ satisfying

$$\varphi(x) = u(x),\ x \in \Gamma_\varepsilon, \qquad \|u\|_{H^1(\Omega_\varepsilon)} \leq c_2\|\varphi\|_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma_\varepsilon)}.$$

Proof.

$$\|u\|^2_{H^1(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} u^2 + \int_{\Omega_\varepsilon} |\nabla u|^2$$
$$= \varepsilon^2\|\tilde{u}\|^2_{L_2(\Omega)} + |\tilde{u}|^2_{H^1(\Omega)} \approx \varepsilon^2\|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\tilde{u}|^2_{H^1(\Omega)},$$
$$\varepsilon\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} = \varepsilon^2\|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma)}$$

□

Lemma 7.3 *There exists $c_1 \neq c_1(\varepsilon)$ such that if $\int_{\Gamma'_\varepsilon} \varphi(x)\,dx = 0$, $\mathrm{meas}(\Gamma'_\varepsilon) \approx \varepsilon$, then*

$$\frac{1}{\varepsilon}\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \leq c_1|\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}.$$

Proof.

$$\frac{1}{\varepsilon}\|\varphi\|^2_{L_2(\Gamma_\varepsilon)} + |\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} = \|\tilde{\varphi}\|^2_{L_2(\Gamma)} + |\tilde{\varphi}|^2_{H^{\frac{1}{2}}(\Gamma)}$$
$$\leq c_2\|\tilde{u}\|^2_{H^1(\Omega)} \leq \text{ Sobolev}$$
$$\leq c_3|\tilde{u}|^2_{H^1(\Omega)}$$
$$\leq c_1|\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}$$

□

Let $t : H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ be given by

$$u = t\varphi = \xi v, \qquad \xi(n) = 1 - \frac{n}{D}.$$

Then

$$|u|^2_{H^1(\Omega_\varepsilon)} \cong |\xi'|^2 \|v\|^2_{L_2(\Omega_\varepsilon)} + |\xi|\, |v|^2_{H^1(\Omega_\varepsilon)}, \qquad |\xi'| = \frac{1}{\varepsilon}.$$

For $\varphi \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$, let

$$\begin{aligned}
&\varphi = \varphi_0 + \varphi_1, \qquad \varphi_0 \equiv \text{ const}, \qquad \int_{\Gamma_\varepsilon} \varphi_1(x)\, dx = 0,\\
&u_0 \equiv \text{ const} = \varphi_0,\\
&u_1 = t\varphi_1 = \xi v,\\
&\|u_0\|_{L_2(\Omega_\varepsilon)} \le c_2 \varepsilon \|\varphi_0\|_{L_2(\Gamma_\varepsilon)},\\
&\left(\frac{1}{\varepsilon}\right)^2 \|v\|^2_{L_2(\Omega_\varepsilon)} + |v|^2_{H^1(\Omega_\varepsilon)} \le c_3 |\varphi_1|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} = c_3 |\varphi|^2_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}
\end{aligned}$$

We have(Lemma 9.2)

Lemma 7.4 *Let*

$$\varphi(x) = \begin{cases} (1-x)\varphi(-x), & x \in [0,1] \\ 0, & x \in [1,2]. \end{cases}$$

Then there exists c such that $\|\varphi\|_{H^{-\frac{1}{2}}(-1,2)} \le c \|\varphi\|_{H^{\frac{1}{2}}(-1,0)}$.

Lemma 7.5 *If* $\varphi \in H^{\frac{1}{2}}(0, 3\varepsilon)$*, and we have* $(c \ne c(\varepsilon))$

$$\frac{1}{\varepsilon}\|\varphi\|^2_{L_2(0,3\varepsilon)} + |\varphi|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} \le c\|\varphi\|^2_{H^{\frac{1}{2}}(0,3\varepsilon)}.$$

Let $\varphi = \varphi_1 + \varphi_2$ *with*

$$\varphi(x) = \begin{cases} \varphi_1(x), & x \in (0,\varepsilon) \\ \varphi_2(x), & x \in (2\varepsilon, 3\varepsilon) \end{cases}$$

where φ_1, φ_2 *are defined on* $[0, 3\epsilon]$ *according to Lemma 11.4. Then*

$$\frac{1}{\varepsilon}\|\varphi_1\|^2_{L_2(0,3\varepsilon)} + |\varphi_1|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} + \frac{1}{\varepsilon}\|\varphi_2\|^2_{L_2(0,3\varepsilon)} + |\varphi_2|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} \le c_1 \|\varphi\| H^{\frac{1}{2}}(0,3\varepsilon)^2.$$

Proof.

$$\int_0^{3\varepsilon}\int_0^{3\varepsilon} \frac{(\varphi(x) - \varphi(y))^2}{|x-y|^2}\, dxdy = \int_0^3 \int_0^3 \frac{(\tilde\varphi(s) - \tilde\varphi(t))^2}{|s-t|^2}\, dsdt$$
$$\frac{1}{\varepsilon}\int_0^{3\varepsilon} \varphi^2(x)\, dx = \int_0^3 \tilde\varphi(s)^2\, ds$$

□

Lemma 7.6 *Let* $\varphi^\varepsilon \in H^{1/2}(0,3\epsilon)$ *be continuous piecewise linear with* $\varphi^\varepsilon(i\varepsilon) = \varphi_i$*,* $i = 0,\cdots,3$*. Then*

$$\|\varphi^\varepsilon\|^2_{H^{\frac{1}{2}}_\varepsilon(0,3\varepsilon)} \approx \sum_{i=0}^{3} \varepsilon^2\varphi_i^2 + \sum_{i=0}^{3}\sum_{j=0}^{3}(\varphi_i - \varphi_j)^2.$$

Proof.

$$\varepsilon\|\varphi^\varepsilon\|^2_{L_2(0,3\varepsilon)} \approx \sum_{i=0}^{3} \varepsilon^2\varphi_i^2,$$

$$|\varphi^\varepsilon|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} \approx \sum_{i=0}^{3}\sum_{j=0}^{3}(\varphi_i - \varphi_j)^2.$$

□

Lemma 7.7 *There exists* $c \neq c(h,\varepsilon)$ *such that for every* $\varphi^h \in H_h(0,3\varepsilon)$*, there are* $\varphi^\varepsilon, \varphi_1^h, \varphi_2^h$ *satisfying*

$$\begin{aligned} &\varphi^h = \varphi^\varepsilon + \varphi_1^h + \varphi_2^h, \\ &\varphi^\varepsilon - \text{ piecewise linear}, \\ &\varphi_1^h(x) = 0, \qquad x \in (2\varepsilon, 3\varepsilon), \\ &\varphi_2^h(x) = 0, \qquad x \in (0, \varepsilon), \end{aligned}$$

and

$$\|\varphi^\varepsilon\|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} + \|\varphi_1^h\|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} + \|\varphi_2^h\|^2_{H^{\frac{1}{2}}(0,3\varepsilon)} \leq c\|\varphi^h\|^2_{H^{\frac{1}{2}}(0,3\varepsilon)}.$$

Proof. Define φ^ε by the values

$$\varphi_0 = \varphi_1 = \frac{1}{\varepsilon}\int_0^\varepsilon \varphi^h(x)\,dx,$$

$$\varphi_2 = \varphi_3 = \frac{1}{\varepsilon}\int_{2\varepsilon}^{3\varepsilon} \varphi^h(x)\,dx.$$

Then

$$(\varphi_i)^2 = \left(\frac{1}{\varepsilon}\int_{x_i}^{x_{i+1}} \varphi(x)\,dx\right)^2 \leq \frac{1}{\varepsilon^2}\varepsilon\int_{x_i}^{x_{i+1}} \varphi^2(x)\,dx$$

$$\sum_{i=0}^{3} \varepsilon^2(\varphi_i)^2 \leq \varepsilon\|\varphi\|^2_{L_2(0,3\varepsilon)}$$

$$
\begin{aligned}
(\varphi_i - \varphi_j)^2 &= \left(\frac{1}{\varepsilon} \int_{x_i}^{x_{i+1}} \varphi^h(x)\,dx - \frac{1}{\varepsilon} \int_{x_j}^{x_{j+1}} \varphi^h(x)\,dx \right)^2 \\
&= \frac{1}{\varepsilon^2} \left(\frac{1}{\varepsilon} \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \varphi^h(x)\,dydx - \frac{1}{\varepsilon} \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \varphi^h(x)\,dxdy \right)^2 \\
&\leq \frac{4}{\varepsilon^2} \left(\int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \frac{\varphi^h(x) - \varphi^h(y)}{|x-y|}\,dxdy \right)^2 \\
&\leq 4 \int_{x_i}^{x_{i+1}} \int_{x_j}^{x_{j+1}} \frac{(\varphi^h(x) - \varphi^h(y))^2}{|x-y|^2}\,dxdy
\end{aligned}
$$

$$
\psi^h = \varphi^h - \varphi^\varepsilon,
$$

$$
\int_0^\varepsilon \psi^h(x)\,dx = \int_{2\varepsilon}^{3\varepsilon} \psi^h(x)\,dx = 0
$$

□

Lemma 7.8 *Let* $\overline{\Omega} = \bigcup_{i=1}^n \overline{\Omega_i}$*, where* Ω_i *is polygonal and* $\operatorname{diam} \Omega_i = O(H)$*, and let* $\Lambda = \bigcup_{i=1}^m \lambda_i$*. Then there exists* $c \neq c(h, H)$ *such that for every* $\varphi^h \in H_h(\Lambda)$*, there are* $\varphi^H, \varphi_1^h, \cdots, \varphi_m^h$ *satisfying (i)* φ^H *- piecewise linear on the coarse grid* $\bigcup_{i=1}^n \partial\Omega_i$*, (ii)* $\varphi_i^h(x) = 0,\ x \notin \lambda_i,\ i = 1, \cdots, m$*. Then*

$$
\|\varphi^H\|_{H^{\frac{1}{2}}(\Lambda)} \leq c_1 \|\varphi^h\|_{H^{\frac{1}{2}}(\Lambda)}
$$

$$
\sum_{i=1}^m \|\varphi_i^h\|^2_{H^{1/2}(\Lambda)} \leq C_1 \|\varphi^h\|^2_{H^{1/2}(\Lambda)}
$$

$$
\Sigma^{-1} = \Sigma_H^+ + \Sigma_1^+ \cdots + \Sigma_m^+, \qquad (\Sigma_i \varphi, \varphi) \simeq \|\varphi^h\|_{\overset{00}{H}{}^{1/2}(\lambda_i)}
$$

$$
(\Sigma_H \varphi, \varphi) = H^2 \sum \varphi_i^2 + \sum_i \sum_j (\varphi_i - \varphi_j)^2
$$

$$
(\Sigma\varphi, \varphi) \simeq \|\varphi^h\|_{H^{1/2}(\Lambda)}
$$

8 Additive Schwarz Method(ASM) and Multi-level Decomposition

Let Ω be a domain in R^2, $\Omega_i, i = 1, \ldots, n$ be a disjoint subdomain of Ω and $\Lambda = \cup_{i=1}^n \partial\Omega_i$. In addition,

$$
\begin{aligned}
B_i &\longleftrightarrow -\Delta_{\Omega_i} \\
\Sigma &\longleftrightarrow H^{1/2}(\Lambda).
\end{aligned}
$$

Let $t\colon H^{1/2}(\Lambda) \to H(\Omega)$ be an extension operator. Then

$$B^{-1} = \begin{bmatrix} 0 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_n \end{bmatrix} \begin{matrix} \Lambda \\ \Omega_1 \\ \vdots \\ \Omega_n \end{matrix} + t\Sigma^{-1}t^T.$$

Let us fix the subdomain Ω_i and omit the index i. First consider the case when Ω is polygonal. Let $\Omega_0^h, \Omega_1^h, \ldots, \Omega_J^h$ be a sequence of grids on Ω and $W_0 \subset W_1 \subset \ldots \subset W_J = W$ be a sequence of nested spaces respectively. We denote the set $\{\phi_i^{(k)}\}_{i=1,2,\ldots,n_k}$ be the nodal basis in W_k and $\Phi_i^{(k)} = \{\alpha \cdot \phi_i^{(k)} | \alpha \in R\}$ be the spanned vector space of this basis. Then we have

$$\begin{aligned} W_k &= \Phi_1^{(k)} + \cdots + \Phi_{n_k}^{(k)} \\ W &= \sum_{k=0}^{J} \sum_{i=1}^{n_k} \Phi_i^{(k)}. \end{aligned} \tag{11}$$

Let $P_i^{(k)} : W \to \Phi_i^{(k)}$ be an orthogonal projection with respect to $a(\cdot,\cdot)$.

Theorem 8.1 *We have the two followings with α, β independent of h:*

(1) For every $u^h \in W$, there exists $u_i^{(k)} \in \Phi_i^{(k)}$ such that

$$\sum_{k=0}^{J} \sum_{i=1}^{n_k} u_i^{(k)} = u^h \tag{12}$$

and

$$\sum_{k=0}^{J} \sum_{i=1}^{n_k} \|u_i^{(k)}\|_{H^1(\Omega)}^2 \le \frac{1}{\alpha} \|u^h\|_{H^1(\Omega)}$$

$$\Longrightarrow \alpha \|u^h\|_{H^1(\Omega}^2 \le a(\sum_{k=0}^{J} \sum_{i=1}^{n_k} P_i^{(k)} u^h, u^h) \le \beta(u^h, u^h)$$

(2)

$$\|u^h\|_{H^1(\Omega)}^2 \le \beta \inf_{\sum_{k=0}^{J} \sum_{i=1}^{n_k} v_i^{(k)} = u^h} \|v_i^{(k)}\|_{H^1(\Omega)}^2$$

We use the following fundamental result.

Lemma 8.1 *Let $Q_k : W \to W_k$ be an orthogonal projection in $L_2(\Omega)$. Then there exist constants C_1, C_2 (independent of h and J) such that*

$$\begin{aligned} C_1 \|u^h\|_{H^1(\Omega)}^2 \le |||h^h||| &:= \|Q_0 u^h\|_{L2(\Omega)}^2 + \sum_{k=1}^{J} h_k^{-2} \|(Q_k - Q_{k-1}) u^h\|_{L2(\Omega)}^2 \\ &\le C_2 \|u^h\|_{H^1(\Omega)}^2 \end{aligned}$$

and

$$C_1|||u^h||| \le \inf_{\substack{u^h=u_0^h+\cdots+u_J^h\\ u_k^h\in W_k}} \sum_{k=0}^{J} h_k^{-2}\|u_k^h\|^2_{L_2(\Omega)} \le C_2|||u^h|||.$$

Here $Q_k : W \to W_k$ is L_2-orthogonal projection.

Proof. (of (1)) Let $u^h \in W$, then

$$u^h = Q_0u^h + \sum_{k=1}^{J}(Q_k - Q_{k-1})u^h = v_0^h + v_1^h + \cdots + v_J^h,$$

where $v_k^h \in W_k$. Note that $Q_Ju^h = u^h$. On the other hand, since $v_k^h \in W_k$,

$$v_k^h = \sum_{i=1}^{n_k} \alpha_i^{(k)}\phi_i^{(k)} = \sum_{i=1}^{n_k} v_i^{(k)}$$

where $v_k^h \in \Phi_i^{(k)}$. Thus

$$u^h = \sum_{k=0}^{J}\sum_{i=1}^{n_k} v_i^{(k)}.$$

$$\begin{aligned}\sum_{k=0}^{J}\sum_{i=0}^{n_k}\|v_i^{(k)}\|^2_{H^1(\Omega)} &\approx \sum_{k=0}^{J}\sum_{i=0}^{n_k} h_k^{-2}\|v_i^{(k)}\|^2_{L_2(\Omega)}\\ &\approx \sum_{k=0}^{J} h_k^{-2}\|v_k^h\|^2_{L_2(\Omega)}\\ &\le \|u^h\|^2_{H^1(\Omega)}.\end{aligned}$$

Note that we used Lemma and the fact $v_k^h = (Q_k - Q_{k-1})u^h$. □

Proof. (of (2))

$$\|u^h\|^2_{H^1(\Omega)} \le \beta \inf \sum_{k=0}^{J}\sum_{i=1}^{n_k} h_k^{-2}\|v_i^{(k)}\|^2_{L_2(\Omega)}.$$

$$\begin{aligned}\inf_{\substack{v_i^{(k)}\in\Phi_i^{(k)}\\ \sum_{k=0}^{J}\sum_{i=1}^{n_k} v_i^{(k)}=u^h}} \sum_{k=0}^{J}\sum_{i=1}^{n_k} h_k^{-2}\|v_i^{(k)}\|^2_{L_2(\Omega)} &= \inf_{\alpha_i^{(k)}} \sum_{k=0}^{J}\sum_{i=1}^{n_k} h_k^{-2}\|\alpha_i^{(k)}\phi_i^{(k)}\|^2_{L_2(\Omega)}\\ &\ge C \inf_{\alpha_i^{(k)}} \sum_{k=0}^{J} h_k^{-2}\|v_k^h\|^2_{L_2(\Omega)}\\ &= C \inf_{v_k^h\in W_k} \sum_{k=0}^{J} \|v_k^h\|^2_{L_2(\Omega)}\\ &\ge C\cdot C_1\|u^h\|^2_{H^1(\Omega)}\end{aligned}$$

□ Let us give an example of the above theorem. Let $A_i^{(k)} : \Phi_i^{(k)} \to \Phi_i^{(k)}$. Let us define L_2 orthogonal projection $Q_i^{(k)} : W \to \Phi_i^{(k)}$ as follows:

$$Q_i^{(k)} u^h = \frac{(u^h, \phi_i^{(k)})_{L_2(\Omega)}}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}} \phi_i^{(k)}.$$

Define $P_i^{(k)} : W \to \Phi_i^{(k)}$ by setting

$$P_i^{(k)} = (A_i^{(k)})^{-1} Q_i^{(k)}.$$

and $a(\cdot, \cdot)$ by

$$a(u^h, v^h) = (Au, v).$$

Then

$$(A_i^{(k)} \phi_i^{(k)}, \phi_i^{(k)}) = (A\phi_i^{(k)}, \phi_i^{(k)}) = a(\phi_i^{(k)}, \phi_i^{(k)}) = (\alpha_i^{(k)} \phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)},$$

where

$$\alpha_i^{(k)} = \frac{a(\phi_i^{(k)}, \phi_i^{(k)})}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}}.$$

We have the following equalities:

$$\begin{aligned} A_i^{(k)} \phi_i^{(k)} &= \frac{a(\phi_i^{(k)}, \phi_i^{(k)})}{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}} \phi_i^{(k)}, \\ (A_i^{(k)})^{-1} \phi_i^{(k)} &= \frac{(\phi_i^{(k)}, \phi_i^{(k)})_{L_2(\Omega)}}{a(\phi_i^{(k)}, \phi_i^{(k)})} \phi_i^{(k)}. \end{aligned}$$

So we have the following equality for the preconditioner B

$$\begin{aligned} B^{-1} u^h &= \sum_{k=0}^{J} \sum_{i=0}^{n_k} (A_i^{(k)})^{-1} Q_i^{(k)} u^h \\ &= \sum_{k=0}^{J} \sum_{i=0}^{n_k} \frac{(u^h, \phi_i^{(k)})_{L_2(\Omega)}}{a(\phi_i^{(k)}, \phi_i^{(k)})} \phi_i^{(k)}. \end{aligned}$$

Remark 8.1

$$\begin{aligned} a(\phi^h, \phi^h) &= O(1) \\ B_{BPX}^{-1} u^h &= \sum_{k=0}^{J} \sum_{i=0}^{n_k} (u^h, \phi_i^{(k)})_{L_2(\Omega)} \phi_i^{(k)}. \end{aligned}$$

9 Fictitious Space Method

Theorem 9.1 *Let H_0 and H be Hilbert spaces with $(\cdot,\cdot)_{H_0}, (\cdot,\cdot)_H$. Let $A : H_0 \to H_0$ and $B : H \to H$ be self adjoint positive definite operators, i.e., $A^* = A > 0$ and $B^* = B > 0$. Assume that there exists $R : H \to H_0$ such that*

$$(ARv, Rv)_{H_0} \le C_R(Bv, v)_H, \qquad \forall v \in H,$$

and $T : H_0 \to H$ such that

$$RTu_0 = u_0, \qquad \forall u_0 \in H_0,$$

and

$$C_T(BTu_0, Tu_0)_H \le (Au_0, u_0)_{H_0}, \qquad \forall u_0 \in H_0.$$

Set $C^{-1} = RB^{-1}R^*$ where $R^* : H \to H_0$ and $(R^*u_0, v)_H = (u_0, Rv)_{H_0}$. Then we have

$$C_T(A^{-1}u_0, u_0)_{H_0} \le (C^{-1}u_0, u_0) \le C_R(A^{-1}u_0, u_0) \quad \forall u_0 \in H.$$

Lemma 9.1 *Let $A = A^* > 0$ in Hilbert space. Then*

$$(A^{-1}u, u)^{1/2} = \sup_{v\in H} \frac{(u, v)}{(Av, v)^{1/2}}$$

Proof.

$$\begin{aligned}(u, v) = (A^{-1/2}u, A^{1/2}v) \le \text{ C. B. } &\le \|A^{-1/2}u\|\|A^{1/2}v\| \\ &= (A^{-1}u, u)^{1/2}(Av, v)^{1/2}\end{aligned}$$

and

$$v = A^{-1}u \Rightarrow (A^{-1}u, u)^{1/2} = \sup_{v\in H} \frac{(u, v)}{(Av, v)^{1/2}}$$

□

Proof. (of Theorem) The first inequality follows from

$$\begin{aligned}(RB^{-1}R^*u_0, u_0)^{1/2})_{H_0} = (B^{-1}R^*u_0, R^*u_0)_H &= \sup_{v\in H} \frac{(R^*u_0, v)_H}{(Bv, v)_H^{1/2}} \\ &\ge \sup_{v_0\in H_0} \frac{(R^*u_0, Tv_0)_H}{(BTv_0, Tv_0)_H^{1/2}} \ge \sqrt{C_T} \sup_{v_0\in H_0} \frac{(R^*u_0, Tv_0)_H}{(Av_0, v_0)^{1/2}} \\ &= \sqrt{C_T} \sup_{v_0\in H_0} \frac{(u_0, v_0)_{H_0}}{(Av_0, v_0)^{1/2}} = \sqrt{C_T}(A^{-1}u_0, u_0)^{1/2}.\end{aligned}$$

For the second one,

$$\begin{aligned}(RB^{-1}R^*u_0, u_0)^{1/2})H_0 = \sup_{v\in H} \frac{(u_0, Rv)_{H_0}}{(Bv, v)_H^{1/2}} &= \sup_{v\in H} \frac{(A^{-1/2}u_0, A^{1/2}Rv)_{H_0}}{(Bv, v)_H^{1/2}} \\ &\le \text{ C. B. } \le (A^{-1}u_0, u_0)_{H_0}^{1/2} \sup_{v\in H} \frac{(ARu_0, Rv)^{1/2}}{(Bv, v)^{1/2}} \\ &\le \sqrt{C_R}(A^{-1}u_0, u_0)^{1/2}.\end{aligned}$$

□

10 Application to Fictitious Domain Method

10.1 Neumann Boundary Condition

Let us consider the following model problem:

$$\begin{cases} -\Delta u + u &= f \text{ in } \Omega \\ \dfrac{\partial u}{\partial n} &= 0 \text{ on } \Gamma. \end{cases}$$

where Ω is not regular(not polygonal) and Γ is its boundary.

Let $H_0 = H^1(\Omega)$ and $H = H_0^1(\Pi)$. Let A and B be the differential operators according to the domain Ω and Π, i.e.,

$$A \longleftrightarrow -\Delta_\Omega + I$$
$$B \longleftrightarrow -\Delta_\Pi$$

Let $R : H_0^1(\Pi) \to H^1(\Omega)$ be a restriction operator. In this case, we define it by $R = I_\Omega$. Then we have

$$(Ru, Ru)_{H^1(\Omega)} \le C_R(\nabla u, \nabla u)_{L_2(\Pi)}.$$

Let $T : H^1(\Omega) \to H_0^1(\Pi)$ be an extension operator. Then

$$u \in H^1(\Omega) \to \|u\|_{H^1(\Omega)} \ge C_1\|\phi\|_{H^{1/2}(\Gamma)} \ge C_2\|Tu\|_{H^1(\Pi)}$$

and

$$RTu_0 = u_0 \qquad \forall u_0 \in H^1(\Omega).$$

We obtain preconditioner for the domain Ω by setting

$$C^{-1} = RB^{-1}R^*.$$

In matrix notation,

$$C^{-1} = \begin{bmatrix} I & 0 \end{bmatrix} (-\Delta_\Pi^{-1}) \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ and } Ru = u_{\bar{\Omega}}$$

where

$$R = \begin{bmatrix} I & 0 \end{bmatrix} \text{ and } u = \begin{bmatrix} u_{\bar{\Omega}} \\ u_{\Pi \setminus \Omega} \end{bmatrix}$$

and I is an identity block.

10.2 Dirichlet Boundary Condition(1-D case)

Let us first consider Dirichlet Boundary condition for 1-D problem:

$$\begin{cases} -\dfrac{d^2u}{dx^2} = f \text{ in } (a, b) \subset (0, 1) \\ u(a) = u(b) = 0. \end{cases}$$

Let $H_0 = H_0^1(a,b)$ and $H = H_0^1(0,1)$ with $\Pi = (0,1)$ and $\Omega = (a,b)$. Let $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. In order to extend u from Ω to u on Π, we define an extension operator $T : H_0^1(a,b) \to H_0^1(0,1)$ by

$$Tu = \begin{cases} u(x) & x \in (a,b) \\ 0 & x \in \Pi \backslash (a,b). \end{cases}$$

Then

$$(Tu_0, Tu_0)_{H^1(\Pi)} = (u_0, u_0)_{H^1(\Omega)} \Longrightarrow C_T = 1.$$

Next, we consider the restriction operator $R : H_{0,h}^1(\Pi) \to H_{0,h}^1(\Omega)$. There are many ways to define R. Here we consider two R and compare them.

1) The first one is defined as follows:

$$Ru^h = \begin{cases} u^h(x) & x_i \in (a,b) \\ 0 & x_i = a \text{ or } b. \end{cases}$$

$\|R\| \to \infty$ as $h \to 0$. So $C_T \to \infty$. This is not a good choice of restriction. 2) So we introduce another restriction operator. Let $I_\Omega : H_0^1(\Omega) \to H^1(\Omega)$ be natural restriction defined as follows:

$$(I_\Omega u)(x) = u(x), x \in \Omega, \quad \forall u \in H_0^1(\Omega)$$

and $I_\Gamma : H_0^1(\Pi) \to R^2$ be the trace operator defined by

$$I_\Gamma u = \begin{bmatrix} u(a) \\ u(b) \end{bmatrix}, \qquad \forall u \in H_0^1(\Pi).$$

Let $t : R^2 \to H^1(\Omega)$ be the extension operator defined by

$$t\left(\begin{bmatrix} u(a) \\ u(b) \end{bmatrix}\right) = u(a) + \frac{u(b) - u(a)}{b-a}(x-a).$$

Now we define the restriction operator $R : H_0^1(\Pi) \to H_0^1(\Omega)$ by

$$R = I_\Omega - tI_\Gamma.$$

Clearly, we have

$$|u(a)| \le C\|u\|_{H^1(\Pi)},$$
$$|u(b)| \le C\|u\|_{H^1(\Pi)},$$

and

$$|tI_\Gamma u|_{H^1}^2 = \int_a^b \frac{(u(b)-u(a))^2}{(b-a)^2} dx \le C\|u\|_{H^1(\Pi)}^2.$$

Thus

$$\|Ru\|_{H^1(\Omega)} \le \|I_\Omega u\|_{H^1(\Omega)} + \|tI_\Gamma u\|_{H^1(\Omega)} \le C_R\|u\|_{H^1(\Pi)}$$

since $\|I_\Omega u\|_{H^1(\Omega)} \le \|I_\Omega u\|_{H^1(\Pi)}$. Note that C_R is independent of h in this case(FEM). It is easy to see that $RTu_0 = u_0 - 0 = u_0, \quad \forall u_0 \in H_0^1(\Omega)$.

10.3 Dirichlet Boundary Condition(2-D case)

Let $H_0 = H_0^1(\Omega)$, $H = H_0^1(\Pi)$, $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. Let $T : H_0^1(\Omega) \to H_0^1(\Pi)$ be an extension operator defined by

$$Tu = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \Pi \backslash \Omega \end{cases}$$

as in the case of 1-D. Then $C_T = 1$. Let $R = I_\Omega - tI_\Gamma$, where t is the extension operator from subsection 10.2, then we obtain constant C_R(independent of h).

10.4 Mixed Boundary Condition(2-D case)

Let $\breve{H}^1(\Omega) = \{u \in H^1(\Omega) | u(x) = 0, x \in \Gamma_D\}$. Let $H = H_0^1(\Pi)$, $A = -\Delta_\Omega$ and $B = -\Delta_\Pi$. We see,

$$\overline{\Pi \backslash \Omega} = \overline{G}_N \cup \overline{G}_D.$$

Let $T_{ND} u_0 = T_N T_D u_0$. Define $T_D : \breve{H}^1_\Omega) \to H^1(\bar{\Omega} \cup G_D)$ for Dirichlet data by

$$T_D u_0 = \begin{cases} u_0(x) & x \in \Omega \\ 0 & x \in G_D. \end{cases}$$

Next, by trace theorem, there exists $T_N : H^1(\bar{\Omega} \cup G_D) \to H^1(\Pi)$. Now, we define a restriction operator R by $R = I_\Omega - t_\Gamma t_N \cdot I_D$ where $I_\Omega : H_0^1(\Pi) \to H^1(\Omega)$ and $I_D : H_0^1(\Pi) \to H^{1/2}(\Gamma_D)$. We define $t_N : H^{1/2}(\Gamma_D) \to H^{1/2}(\Gamma)$ by

$$(t_N \phi)(-s) = (1 - \frac{s}{D})\phi(s) \quad \text{for } \phi(s) \in H^{1/2}(\Gamma_D).$$

Here $(1 - s/D)$ is a linear cut-off function. Note that D is independent of h. We get the following estimation for t_N

$$\|t_N \phi\|_{H^{1/2}(\Gamma)} \le C_1 \|\phi\|_{H^{1/2}(\Gamma_D)}.$$

Let $t_\Gamma : H^{1/2}(\Gamma) \to H^1(\Omega)$ be the extension operator, then we have the following estimation for R

$$\begin{aligned} \|Ru\| &\le \|I_\Omega\| + \|t_\Gamma\| \cdot \|t_N\| \cdot \|I_D u\| \\ &\le C_R \|u\|_{H^1(\Pi)}. \end{aligned}$$

Here C_R is independent of h in FEM case.

10.5 Unstructured and nonuniform grid(2-D case)

Here we consider the case of unstructured and nonuniform grid of Ω. In the case of structured grid Ω, there is no problem. In other words, we can design a preconditioner for the differential operator on Ω from that on Π. In the case when Ω is not polygonal, though, we want to design a preconditioner from the uniform grid differential operator on Π. Let $Q^{h_\sharp}$ denote the uniform grid on Π and $h_\sharp$ be the mesh size of $Q^{h_\sharp}$ satisfying

$$h_\sharp < \frac{1}{\sqrt{2}} r_{\min}$$

where

$$r_{\min} = \min_{z_l \in \Omega^h} r_l.$$

Let $H_0 = H_h(\Pi^h)$ and $H = H_h(Q^{h_\sharp})$. Let $A = -\Delta_{\Pi^h}$ and $B \approx -\Delta_{Q^{h_\sharp}}$ be defined as previously. Now, we construct $R_Q : H_h(Q^{h_\sharp}) \to H_h(\Pi^h)$. Let $U^{h_\sharp} \in H_h(Q^{h_\sharp})$, then how can we define $u^h \in H_h(\Pi^h)$? Let z_l denote nodal point of Π^h and Z_{ij} is the node of some Q_{ij}. We set $u^h(z_l) = U^{h_\sharp}(Z_{i,j})$ i.e., $RU^{h_\sharp} = u^h$ is a simple restriction. Next, we define the extension operator $T : H_h(\Pi^h) \to H_h(Q^{h_\sharp})$ by the following way:(See Figure **??**.)

$$\begin{cases} U^{h_\sharp}(z_{ij}) = u^h(z_l) \text{ if } z_l \text{ belongs to some } Q_{ij} \\ U^{h_\sharp}(z_{ij}) = \frac{1}{3}(u^h(z_1) + u^h(z_2) + u^h(z_3)) \text{ otherwise} \end{cases}$$

With the condition on mesh size thatthere are only two cases.(See Figure **??**.) That is, there is a one-to-one correspondence between Π^h and some subset $\tilde{Q}^{h_\sharp}$ of $Q^{h_\sharp}$. Then we can see that

$$RTu^h = u^h \qquad \forall u^h \in H_h(\Pi^h).$$

Assume that $c_1 r_{\min} \leq h_\sharp$, i.e $h\sharp$ is of order h.

Lemma 10.1 *There exist two constants C_R^Q and C_T^Q (independent of h) such that*

$$\|R_Q U^{h_\sharp}\|_{H^1(\Pi)} \leq C_R^Q \|U^{h_\sharp}\|_{H^1(\Pi)}$$

and

$$\|T_Q u^h\|_{H^1(\Pi)} \leq C_T^Q \|u^h\|_{H^1(\Pi)}.$$

Proof. Let $u^h = RU^{h_\sharp}$, then

$$\begin{aligned} \|u^h\|_{H^1(\Pi)} &\approx \sum_{\tau_i \subset \Pi^h} (h^2\{(u^h(z_{i_1}))^2 + (u^h(z_{i_1}))^2 + (u^h(z_{i_1}))^2\} \\ &+ (u^h(z_{i_1}) - u^h(z_{i_2}))^2 + (u^h(z_{i_2}) - u^h(z_{i_3}))^2 + (u^h(z_{i_3}) - u^h(z_{i_1}))^2) \\ &= \sum_{\tau_i \subset \Pi^h} h^2((U^{h_\sharp}_{i_1,j_1})^2 + \cdots + (U^{h_\sharp}_{i_k,j_k})^2) \\ &+ \sum_{\tau_i \subset \Pi^h} ((U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 + (U^{h_\sharp}_{i_2,j_2} - U^{h_\sharp}_{i_3,j_3})^2 + (U^{h_\sharp}_{i_3,j_3} - U^{h_\sharp}_{i_1,j_1})^2). \end{aligned}$$

Clearly,

$$\sum_{\tau_i \subset \Pi^h} h^2((U^{h_\sharp}_{i_1,j_1})^2 + \cdots + (U^{h_\sharp}_{i_k,j_k})^2) \leq \|U^{h_\sharp}\|_{L_2,h}(Q^{h_\sharp}).$$

Next,

$$
\begin{aligned}
(U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 \le & \text{ some differences of neighbors} \\
& \le (U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i'_2,j'_2})^2 + \cdots + (U^{h_\sharp}_{i'_k,j'_k} - U^{h_\sharp}_{i_2,j_2})^2.
\end{aligned}
$$

So there exists a constant C_1 such that

$$
\begin{aligned}
\sum_{\tau_i \subset \Pi^h} ((U^{h_\sharp}_{i_1,j_1} - U^{h_\sharp}_{i_2,j_2})^2 + (U^{h_\sharp}_{i_2,j_2} - U^{h_\sharp}_{i_3,j_3})^2 + (U^{h_\sharp}_{i_3,j_3} - U^{h_\sharp}_{i_1,j_1})^2) \\
\le C_1 |U^{h_\sharp}|^2_{H^1(Q^{h_\sharp})}.
\end{aligned}
$$

This completes the proof of the existence of C_R. The existence of C_T is the same as the case of C_R. □

Theorem 10.1 *We have the preconditioner for the problem on Ω^h with $C_R(\neq C(h))$ and $C_T(\neq C(h))$ by letting $(-\Delta_{\Omega_h})^{-1} \approx R R_Q (-\Delta_{Q^{h_\sharp}})^{-1} R_Q^T R^T$.*

11 Fictitious Space Method and Multilevel ASM

We consider the following mixed boundary value problem:

$$
\begin{aligned}
\sum_{i,j=1} -\frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + a_0(x) u = f(x), \quad & x \in \Omega \\
u(x) = 0 \quad & \text{on } \Gamma_D \\
\frac{\partial u}{\partial n} + \sigma(x) u = 0 \quad & \text{on } \Gamma_N
\end{aligned}
$$

Then, we have

$$
a(u,v) = \int_\Omega \left(\sum \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x) uv \right) dxdy + \int_{\Gamma_N} \sigma(x) uv \, ds
$$

with

$$
\begin{aligned}
a(u,v) &= a(v,u) \\
a(u,u) &\approx \|u\|^2_{H^1_\Omega}
\end{aligned}
$$

We assume that $\Omega^h = \bigcup_{i=1}^M \tau_i$ is quasi-uniform and shape regular with $\partial\Omega^h = \Gamma^h_D \bigcup \Gamma^h_N$ and $\Gamma^h_D \subset \overline{\Omega}, \Gamma^h_N \subset \overline{(R^2 \setminus \Omega)}$. Also suppose that $h_\sharp \le r_{min}/2\sqrt{2}$. Define Q^h by the minimum collection of Q_{ij} enclosing Ω^h. Let $\partial Q^h = S^h$ with $S^h = S^h_D \bigcup S^h_N$ such that, if

$$
\overline{Q_{ij}} \bigcap \Gamma^h \neq \varnothing
$$

then

$$S^h \bigcap \overline{Q_{ij}} \in S^h_D \text{ and } S^h_N = S^h \setminus S^h_D$$

We have

$$(Au, v) = a(u^h, v^h)$$

$$(BU, U) \approx \|U^h\|^2_{H^1(Q^h)}, \forall U^h \in H_h(Q^h)$$

$$R_Q : H_h(Q^h) \to H_h(\Omega^h)$$

Theorem 11.1 *There exist C_1 and C_2, independent of h, such that*

$$C_1(A^{-1}u, v) \leq (RB^{-1}R^T u, v) \leq C_2(A^{-1}u, u), \quad \forall u$$

Proof. The proof is same as one in the previous theorem of the FSM. □

Remark 11.1 *The condition $h_\sharp \leq r_{min}/2\sqrt{2}$ instead of $h_\sharp < \frac{r_{\min}}{\sqrt{2}}$ is needed in the mixed boundary case.*

Assume $h_\sharp = l \cdot 2^{-J}, h_k = l \cdot 2^{-k}, k = 0, 1, \ldots, J$, and $h_J = h_\sharp$ and we have a sequences of triangulations and spaces

$$\Pi^h_0, \Pi^h_1, \ldots, \Pi^h_J$$

$$W^h_0 \subset W^h_1 \subset \cdots \subset W^h_J = \breve{H}_h(Q^h)$$

where $W^h_k = \{u^h_k = \sum_i \alpha^{(k)}_i \phi^{(k)}_i\}$ and $\phi^{(k)}_i$ is a nodal basis. Let $S^h = S^h_N$. Now,

$$B_N^{-1} U^h = \sum_{k=0}^{J} \sum_{\{supp\phi^{(k)}_i \cap Q^h \neq \phi\}} (\tilde{U}^h, \phi^{(k)}_i)_{L_2(Q^h)} \tilde{\phi}^{(k)}_i \quad \text{- part of BPX in } \Pi$$

where

$$\tilde{U}^h = \begin{cases} U^h(Z_{i,j}) & Z_{i,j} \in Q^h \\ 0 & \text{otherwise} \end{cases}$$

Theorem 11.2 *There exist C_1, C_2, independent of h, such that*

$$C_1 \|U^h\|^2_{H^1(Q^h)} \leq (B_N U, U) \leq C_2 \|U^h\|^2_{H^1(Q^h)}$$

Proof. Note that

$$B^{-1}_\Pi (U^h_\Pi) = \sum_{k=0}^{J} (U^h_\Pi, \Phi^{(k)}_i)_{L_2(\Pi)} \Phi^{(k)}_i \quad \text{- BPX in } \Pi,$$

$$R_N U^h_\Pi = \begin{cases} U^h_\Pi(Z_{i,j}) & Z_{i,j} \in Q^h \\ 0 & \text{otherwise} \end{cases}$$

and

$$R_N = \begin{bmatrix} I & 0 \end{bmatrix}$$

Then $R_N B_N^{-1} R_N^T = B_N^{-1}$ □

Drichlet Condition ($S^h = S_D^h$)

$$B^{-1}U^h = \sum_{k=0}^{J} \sum_{supp\Phi_i^{(k)} \subset Q^h} (U_\Pi^h, \Phi_i^{(k)})_{L_2(\Pi)} \Phi_i^{(k)}$$

Theorem 11.3 *There exists C_1, C_2 such that*

$$C_1 \|U^h\|^2_{H^1(Q^h)} \le (B_D U, U) \le C_2 \|U^h\|^2_{H^1(Q^h)} \forall U \in H_h(Q^h)$$

Proof. The proof is postponed and let us assume temporarily that the theorem was proved already. □

Mixed Condition ($S_D^h \neq \phi, S_N^h \neq \phi$)

$$B_M^{-1}U = \sum_{k=0}^{J} \sum (U^h, \Phi_i^{(k)})_{L_2(\Pi)} \Phi_i^{(k)}$$

where the second summation is taken on the set *supp*$(\Phi_i^{(k)} \cap Q^h) \neq \phi$ and *supp*$(\Phi_i^{(k)} \cap S_D^h) = \phi$.

Proof. Note that $\Pi^h \setminus Q^h = \bar{G}_D^h \bigcup \bar{G}_N^h$ and $G_D^h \bigcap G_N^h = \phi$. From the figure we observe that $\partial G_D^h \bigcap S^h = S_D^h$ and $\bar{G}^h = \bar{Q}^h \bigcup \bar{G}_N^h$. Now define

$$\dot{H}_h(G^h) = \{u^h | u^h(x) = 0, x \in \partial G^h\}$$

Then we have by previous case

$$B_{D,G}^{-1} U_G^h = \sum_{k=0}^{J} \sum_{supp\Phi_i^{(k)} \subset G^h} (U_G^h, \Phi_i^{(k)})_{L_2(G)} \Phi_i^{(k)}$$

and

$$C_1 \|G_G^h\|^2_{H^1(G)} \le (B_{D,G} U_G, U_G) \le C_2 \|U_G^h\|^2_{H^1(G)}$$

Also define $R_{N,G} : \dot{H}_h(G^h) \to \breve{H}_h(Q^h)$ by

$$R_{N,G} U_G^h(Z_{i,j}) = \begin{cases} U_G^h(Z_{i,j}) & Z_{i,j} \in Q^h \\ 0 & \text{otherwise} \end{cases}$$

Then, finally we have

$$R_{N,G} = \begin{bmatrix} I & 0 \end{bmatrix}$$

and

$$R_{N,G}B_{D,G}^{-1}R_{N,G}^{-1} = B_M^{-1}$$

□

Now, we'll show the Theorem for Dirichlet condition. Define $\dot{W}_k = W_k \bigcap \dot{H}_h(Q^h)$. Then, the proof of the theorem is completed if the following conditions are satisfied :

(a) For all $u^h \in \dot{W}_J, \exists u_i^{(k)} = \alpha_i^{(k)}\Phi_i^{(k)}$ such that

$$\sum_{k=0}^{J} \sum_{supp(u_i^{(k)})\subset Q^h} u_i^{(k)} = u^h$$

and

$$\alpha \sum_{supp(u_i^{(k)})\subset Q^h} \|u_i^{(k)}\|^2_{H^1(Q^h)} \leq \|u^h\|^2_{H^1(\Omega)}$$

(b) For all $u^h \in \dot{W}_J$,

$$\|u^h\|^2_{H^1(Q^h)} \leq \beta \inf \sum_{supp(u_i^{(k)})\subset Q^h} \|u_i^{(k)}\|^2_{H^1(Q^h)}$$

where the infimum is taken on $u_i^{(k)}$ decomposition satisfying

$$\sum_{k=0}^{J} \sum_{supp(u_i^{(k)})\subset Q^h} u_i^{(k)} = u^h$$

with α, β are independent of h.

Now, to prove the above condition (a) and (b), we need 3 lemmas. The proof of each lemma is easy, so omitted. The first and second lemmas imply the condition (b) and the last lemma with BPX in Π implied the condition (a). Now, we'll state the three lemmas.

Lemma 11.1 *There exists C, independent of h, such that*

$$(\nabla v^h, \nabla w^h)_{L_2(\tau_i)} \leq C(1/\sqrt{2})^{l-k}|v|_{H^1(\tau_i)}2^l\|w\|_{L_2(\tau_i)}$$

for all triangles τ_i of the triangluation $\Pi_k^h \cap$ Supp W_k , $v^h \in \dot{W}_k, w^h \in \dot{W}_l \quad (l > k)$.

Lemma 11.2 $\forall u^h = u_0^h + \sum_{k=1}^J u_k^h, \quad u_k^h \in \dot{W}_k$, *we have*

$$|u^h|^2 \leq C(|u_0^h|^2_{H^1(Q^k)} + \sum_{k=1}^{J} 4^k\|u_k^h\|^2_{L_2(Q^h)})$$

Lemma 11.3 *For given $u^h \in \dot{W}_J$, we define $\tilde{u}^h(x) = u^h(x)$ if $x \in Q^h$, otherwise, $\tilde{u}^h = 0$. Then, for a given decomposition*

$$\tilde{u}^h = \tilde{u}_0 + \sum_{k=1}^{J} \tilde{u}_k, \quad \tilde{u}_k \in W_k,$$

$\exists$ decomposition

$$u^h = u_0 + \sum_{k=1}^{J} u_k, \quad u_k \in \dot{W}_k$$

such that

$$4^k \|u_k\|^2_{L_2(Q^h)} \le C(|\tilde{u}_0| + \sum_{k=1}^{J} \|\tilde{u}_k\|^2_{L_2(\Pi)})$$

for some constant C, independent of h.