Robust AMLI preconditioning of FEM elasticity problems

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CONTENTS

1. Introduction

The target problem in this lecture is the Lamé system of elasticity:

$$\begin{aligned} \sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad & \mathbf{x} \in \Omega, \qquad i = 1,2 \\ \mathbf{u} = 0, \qquad & \mathbf{x} \in \Gamma_D \\ \sum_{j=1}^{2} \frac{\partial \sigma_{ij}}{\partial x_j} n_j = g_i \qquad & \mathbf{x} \in \Gamma_N, \qquad i = 1,2, \end{aligned}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ and $\partial\Omega = \Gamma_D + \Gamma_N$ is the boundary of Ω. The stresses σ_{ij} and the strains ε_{ij} are defined by the classical Hooke's law, i.e.

$$\sigma_{ij}(\mathbf{u}) = \lambda \left(\sum_{k=1}^{2} \varepsilon_{kk}(\mathbf{u}) \right) \delta_{ij} + 2\mu \varepsilon_{ij}(\mathbf{u}), \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

- We assume that the Lamé coefficients are piecewise constant in the domain Ω. The unknowns of the problem are the displacements $\mathbf{u}^t = (u_1, u_2)$.
- The Lamé coefficients are given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)},$$

where E stands for the elasticity modulus, and $\nu \in (0, \frac{1}{2})$ is the Poisson ratio.

- We use the notion **almost incompressible** for the case $\nu = \frac{1}{2} - \delta$ ($\delta > 0$ is a small parameter). Note that the boundary value problem becomes ill-posed when $\nu = \frac{1}{2}$ (the material is **incompressible**).

The weak formulation of the boundary value problem reads:

find $\mathbf{u} \in \mathcal{V} = \{\mathbf{v} \in (H^1(\Omega))^2, \mathbf{v} \mid_{\Gamma_D} = 0\}$ **such that**

$$a(\mathbf{u}, \mathbf{v}) = -\int_\Omega \mathbf{f}^T \mathbf{v} dx + \int_{\Gamma_N} \mathbf{g}^T \mathbf{v}\, ds, \qquad \forall \mathbf{v} \in \mathcal{V}.$$

The bilinear form $a(\mathbf{u}, \mathbf{v})$ **is of the form**

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega [\lambda div(\mathbf{u}) div(\mathbf{v}) + 2\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})] dx = \int_\Omega \langle C d(\mathbf{u}), d(\mathbf{v}) \rangle dx,$$

where

$$C = \begin{bmatrix} \lambda + 2\mu & 0 & 0 & \lambda \\ 0 & \mu & \mu & 0 \\ 0 & \mu & \mu & 0 \\ \lambda & 0 & 0 & \lambda + 2\mu \end{bmatrix}, \quad d(\mathbf{u}) = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2} \right)^T.$$

The variational formulation with the bilinear form a is next discretized using FEM, i.e., the continuous space $\mathcal{V}$ is replaced by a finite dimensional subspace $\mathcal{V}^h$. This results in the system

$$A\mathbf{u}^h = \mathbf{f}^h,$$

where A stands for the stiffness matrix associated with the bilinear form a, and the pair of triangulation and related FE space, namely, $\{\mathcal{T}, \mathcal{V}^h\}$.

If the separable displacements ordering of the unknowns is used, then

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \begin{array}{l} \} \text{ displacements } \mathbf{u}_1^h \\ \} \text{ displacements } \mathbf{u}_2^h \end{array}$$

2. Displacement Decomposition

The block-diagonal Displacement Decomposition preconditioner C_{DD} has the form

$$A_{DD} = \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}, \quad C_{DD} = \begin{bmatrix} \widetilde{A}_{11} & \\ & \widetilde{A}_{22} \end{bmatrix},$$

where $\widetilde{A}_{ii}$ is a proper preconditioner of the scalar FEM elliptic block A_{ii}. The efficient application of DD preconditioner is motivated by the second Corn's inequality.

Lemma 4.1.

The condition number estimate

$$\varkappa(A_{DD}^{-1}A) = O\left(\frac{1}{1-2\nu}\right)$$

holds uniformly with respect to mesh size and mesh anisotropy, $\nu \in [0, 1/2)$.

- **The considered DD approach is robust for problems with strong mesh anisotropy. One good candidate for preconditioning of the scalar blocks A_{ii} is shown to be the AMLI method. In this case**

$$C_{DD}^{AMLI} = \begin{bmatrix} AMLI(A_{11}) & \\ & AMLI(A_{22}) \end{bmatrix},$$

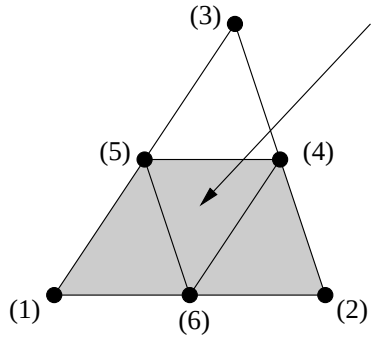
$$\varkappa((C_{DD}^{AMLI})^{-1}A) = O\left(\frac{1}{1-2\nu}\right).$$

- **Unfortunately, the DD preconditioner deteriorates if $\nu \to 1/2$.**
  A better preconditioning of the coupled matrix is required in the case of almost incompressible material.

3. Conforming FEM

Let us assume that the standard multilevel setting is applied to the coupled FEM elasticity system.

Case 1. **Linear FEs.**



Support for the piece-wise linear basis function φ_6

Lemma 4.2. **Margenov (1994), Achchab, Maitre (1996)**

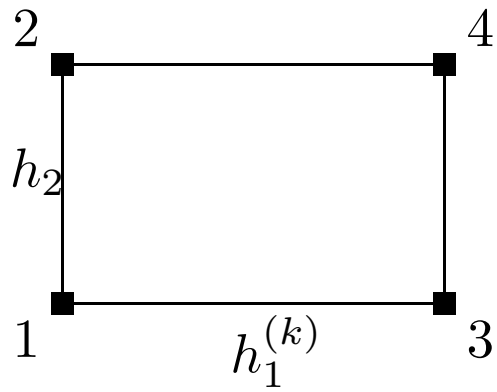
The CBS constant is uniformly bounded with respect to mesh anisotropy and Poisson's ratio $\nu \in [0, 1/2)$, namely,

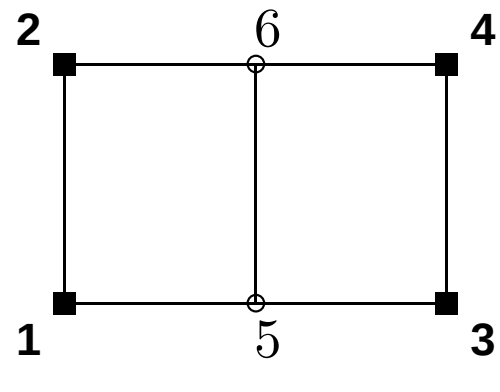
$$\gamma^2 < \frac{3}{4}.$$

Remark 4.1.

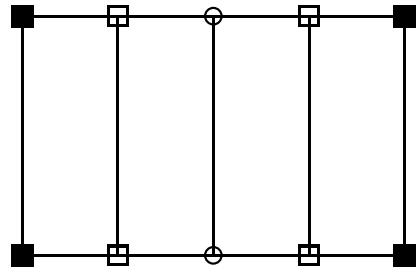
Unfortunately, the developed robust preconditioners for the block $A_{11}^{(k)}$ are not directly applicable to the case of almost incompressible materials.

Pure semicoarsening





■–level (k) ○–level $(k+1)$



□–level $(k+2)$

Lemma 4.2.

Consider the pure semi-coarsening refinement algorithm with $\rho = 2$, and let $\epsilon^{(k+1)} = \frac{h_2^{(k+1)}}{h_1^{(k+1)}} = 1$. Then the constant γ_e in the strengthened C.B.S. inequality is bounded uniformly on the Poisson ratio, and

$$\gamma_e^2 \leq \frac{3}{4}.$$

Remark 4.2.

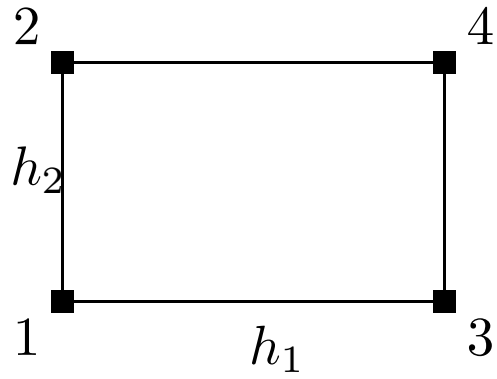
A more detailed analysis shows that

$$\frac{1}{\sqrt{1-\gamma^2}} \rightarrow \rho \text{ when } \nu \rightarrow \frac{1}{2} \text{ and } \epsilon^{(k+1)} \rightarrow \infty.$$

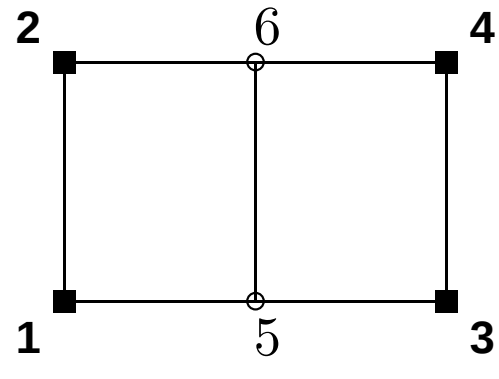
Tis is a negative in the sense of the theory of AMLI optimality conditions, i.e. no integer β satisfying the condition

$$\frac{1}{\sqrt{1-\gamma^2}} < \beta < \rho.$$

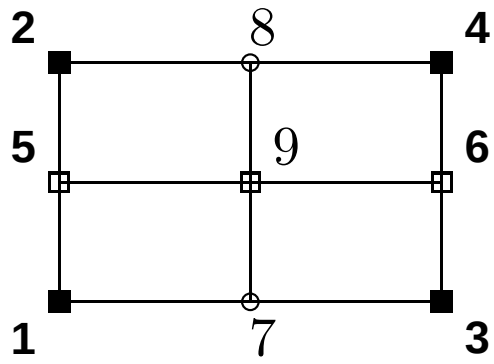
Balanced semicoarsening



■–level (k)



○–level $(k+1)$



□–level $(k+2)$.

Theorem 4.1. **Margenov (1998)**

Consider the balanced semi-coarsening AMLI with $\rho = 2$, and square initial mesh τ_1. Then the constant γ in the strengthened CBS inequality is bounded uniformly on the Poisson ratio, and

$$\gamma^2 \leq \frac{6}{7}.$$

Remark 4.3.

- **The hybrid V-cycle balanced semi-coarsening AMLI algorithm, defined on a square initial mesh τ_1, and with parameters $\rho = 2$, $k_0 = 2$ and $\beta_{2j} = 3$, is of optimal convergence rate, uniformly with respect to the Poisson ratio $\nu \in (0, 1/2)$.**
- **The blocks $A_{11}^{(\tilde{k}+1)} = A_{11}^{(k+1)}$ are $2(2\rho+1)$-diagonal matrices, providing $O(N)$ optimal complexity of the PCG algorithm.**

Locking phenomenon

There were presented here some robust results concerning the efficient solution of elasticity FEM systems as they arise applying standard conforming finite elements. The following theorem well illustrates why this is not enough.

Theorem 4.2. **(see, e.g., Brenner, Sung (1992))**

Let us introduce the relative error indicator

$$L_{\lambda,h} := \sup \left\{ \frac{|u - u_h|_{[H^1(\Omega)]^2}}{\|\nabla \cdot T_\sigma(u)\|_{[L^2(\Omega)]^2}} : \; u \in [H^2(\Omega)]^2 \cap [H_0^1(\Omega)]^2, \; u \neq 0 \right\}.$$

Then, for lower order conforming FEM, there exists a positive constant C independent of h, such that

$$\liminf_{\lambda \to \infty} L_{\lambda,h} \geq C$$

The complete solution of the so-called locking phenomenon, requires the development of robust solution methods for locking-free discretization methods.

4. Locking-free algorithms

Let us consider the pure-displacement ($\partial\Omega = \Gamma_D$) elasticity problem associated with the modified bilinear form

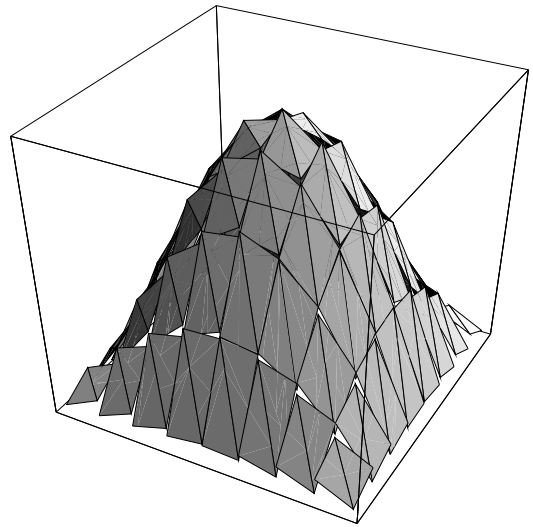
$$a^s(\mathbf{u},\mathbf{v}) = \int_\Omega \langle C^s d(\mathbf{u}), d(\mathbf{v})\rangle dx,$$

where

$$C^s = \begin{bmatrix} \lambda+2\mu & 0 & 0 & \lambda+\mu \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ \lambda+\mu & 0 & 0 & \lambda+2\mu \end{bmatrix}, \quad d(\mathbf{u}) = \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}\right)^T.$$

The modified weak formulation holds due to homogeneous pure displacement boundary conditions as for $u, v \in \mathcal{V}$ we have $\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}dx = \int_\Omega \frac{\partial u_i}{\partial x_i}\frac{\partial v_j}{\partial x_j}dx$.

It turns out that locking can be overcome if the C.-R. FEs are used.

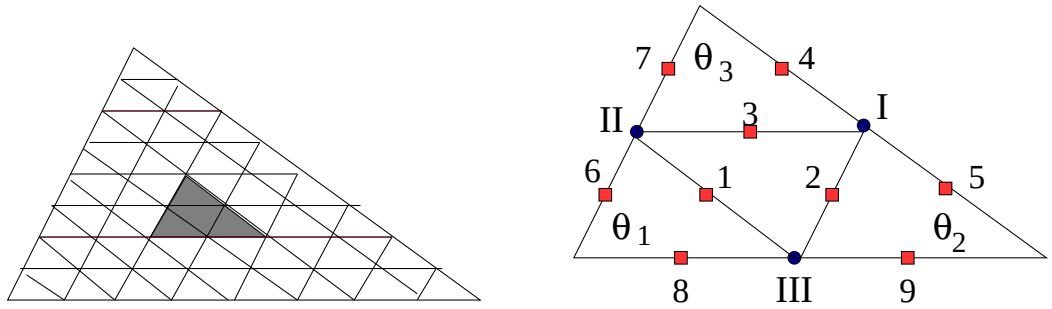


Function from $\mathcal{V}^0_{cr,h}$

Now, if $\mathbf{u}_h$ is the solution of the discrete problem: find $\mathbf{u}_h \in \mathcal{V}_h$ such that $a^s_h(\mathbf{u}_h, \mathbf{v}_h) = F(\mathbf{v}_h), \forall \mathbf{v}_h \in \mathcal{V}_h$, then the following locking-free error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C_{\Omega,\theta}\, h\, \|\mathbf{f}\|_{[L_2(\Omega)]^2},$$

where $\|\cdot\|_h := \sqrt{a^s_h(\cdot,\cdot)}$ and $C_{\Omega,\theta}$ is independent of h,ν.



Regular refinement (a), and macroelement local node numbering (b)

Let us consider two consecutive mesh refinements $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$.

- **As already noted, for Crouzeix-Raviart non-conforming linear elements, the FE spaces associated with two consecutive mesh refinements are not nested.**
- **To enable the use of the general multilevel scheme, we will apply the so called ”first reduce” (FR) and ”differences and aggregates” (DA) approaches to construct hierarchical two-level decomposition of the FEM systems.**

CBS constant estimates

Theorem 4.3. **Blaheta, Margenov, Neytcheva (2005)**

Consider the problem where the Lamé coefficients are constant on the coarse triangles $E \in \mathcal{T}_k$, discretization by the Crouzeix-Raviart FE and the DA decomposition of the stiffness matrix. Then for any element size and shape and for any Poisson ratio $\nu \in (0, \frac{1}{2})$, there holds

$$\gamma_{DA} < \sqrt{\frac{3}{4}}.$$

Remark 4.4.

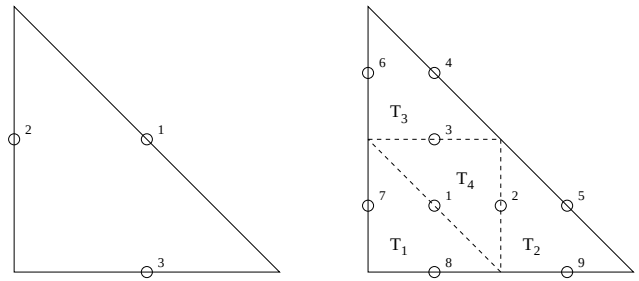
This new result is a generalization of the earlier estimate by Kolev, Margenov (1999), namely

$$\gamma_{FR} < \frac{\sqrt{8+\sqrt{8}}}{4} \approx 0.822,$$

which is derived in the case of a regular triangulation $\mathcal{T}_0$ obtained by a diagonal subdivision of a square mesh.

Proof.

The constant γ_{DA} will be estimated by the maximum of the constants over the macroelements. Let us first consider the case of a right angled reference macroelement.



The reference coarse grid triangle and the macroelement $\widehat{E}$.

Let $\widetilde{\mathcal{V}}_1(\widehat{E})$, $\widetilde{\mathcal{V}}_2(\widehat{E})$, be the two-level splitting for the reference macroelement $\widehat{E}$. For $u \in \widetilde{\mathcal{V}}_1(\widehat{E})$ and $v \in \widetilde{\mathcal{V}}_2(\widehat{E})$ denote $d^{(r)} = d^{(r)}(\mathbf{u})\,|_{T_r}$, $\delta^{(r)} = d^{(r)}(\mathbf{v})\,|_{T_r}$, $r = 1 \cdots, 4$. Then it is easy to show that

$$
\begin{aligned}
d^{(1)} + d^{(2)} + d^{(3)} + d^{(4)} &= \mathbf{0}, \\
\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = -\delta^{(4)} &= \delta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
a_{\widehat{E}}(u,v) &= \sum_{r=1}^{4} \int_{T_r} \langle C^s d(u),\, d(v)\rangle \, dx = \sum_{r=1}^{4} \triangle \langle C^s d^{(r)},\, \delta^{(r)}\rangle \\
&= \triangle \langle C^s \delta,\, d^{(1)} + d^{(2)} + d^{(3)} - d^{(4)}\rangle \\
&= -2\triangle \langle C^s \delta,\, d^{(4)}\rangle \leq 2\triangle \mid \delta \mid_{C^s} \mid d^{(4)} \mid_{C^s}
\end{aligned}
$$

where $\triangle =$ **area**(T_k), $\langle x, y\rangle = x^T y$ **denotes the inner product in** $\mathbb{R}^4$**, and** $\mid d \mid_{C^s} = \sqrt{\langle C^s d, d\rangle}$ **is the norm induced by the coefficient matrix** C^s**.**

Here, it is important to remind that the modified coefficient matrix C^s **is SPD.**

Further,

$$| d^{(4)} |^2_{C^s} = | d^{(1)} + d^{(2)} + d^{(3)} |^2_{C^s} \leq 3 \sum_{k=1}^{3} | d^{(k)} |^2_{C^s}$$

leads to

$$a_{\widehat{E}}(u,u) = \sum_{k=1}^{4} | d^{(k)} |^2_{C^s} \triangle \geq \left(1 + \frac{1}{3}\right) \triangle | d^{(4)} |^2_{C^s}$$

and

$$a_{\widehat{E}}(v,v) = 4\triangle | \delta |^2_{C^s} \ .$$

Thus,

$$a_{\widehat{E}}(u,v) \leq \sqrt{\tfrac{3}{4}} \sqrt{a_{\widehat{E}}(u,u)} \sqrt{a_{\widehat{E}}(v,v)} \, .$$

In the case of an arbitrary shaped macroelement E we can use the affine mapping $F : \widehat{E} \to E$ for transformation of the problem to the reference macroelement, which changes the coefficient matrix C^s, but the estimate $\gamma_E \leq \sqrt{\frac{3}{4}}$ still holds.

The standard multiplicative two-level preconditioner, can be written in the form

$$\widetilde{C}^{(k+1)} = \begin{bmatrix} \widetilde{A}_{11}^{(k+1)} & 0 \\ \widetilde{A}_{21}^{(k+1)} & \widetilde{A}_{22}^{(k+1)} \end{bmatrix} \begin{bmatrix} I_1 & (\widetilde{A}_{11}^{(k+1)})^{-1} \widetilde{A}_{12}^{(k+1)} \\ 0 & I_2 \end{bmatrix}.$$

The following theorem is useful for extending the two-level to multilevel preconditioners.

Theorem 4.4.

Let $\widetilde{A}_{22}^{(k+1)}$ be the stiffness matrix corresponding to the space $\widetilde{\mathcal{V}}_2^{(k+1)}$ from the introduced DA splitting, and let $A^{(k)}$ be the stiffness matrix, corresponding to the coarser triangulation $\mathcal{T}_k$, equipped with the standard nodal finite element basis. Then

$$\widetilde{A}_{22}^{(k+1)} = 4\, A^{(k)}.$$

Preconditioning of $\widetilde{A}_{11}^{(k+1)}$

- We consider now the construction of optimal preconditioners for the coarse grid complement systems $\widetilde{C}_{11}^{(k+1)}$ for the blocks $\widetilde{A}_{11}^{(k+1)}$.
- We search for optimal preconditioners in the sense that they are spectrally equivalent to the top-left matrix block independently on mesh size, element shape and Poisson ratio.
- Moreover the cost of applying the preconditioner is aimed to be proportional to the number of degrees of freedom.
- Similarly to the scalar case, we construct preconditioners on macroelement level and assemble the local contributions to obtain $\widetilde{C}_{11}^{(k+1)}$.

One possible approach is first to impose a displacement decomposition ordering, then use a block-diagonal approximation of $\widetilde{A}_{11}^{(k+1)}$, and then precondition the diagonal blocks which are elliptic. Let us assume that the **multiplicative preconditioner** is applied to the diagonal blocks of $\widetilde{A}_{11}^{(k+1)}$. Then, for homogeneous isotropic materials, the following simplified estimate holds

$$\kappa\left((\widetilde{B}_{11}^{(k+1)})^{-1}\widetilde{A}_{11}^{(k+1)}\right) \leq \frac{1-\nu}{1-2\nu}\frac{15}{8}.$$

- The presented construction is optimal with respect to mesh size and mesh anisotropy but is applicable for moderate values of $\nu \in (0, \frac{1}{2})$ only.
- When the material is **almost incompressible**, it is better to apply a macroelement level static condensation of $\widetilde{A}_{11}^{(k+1)}$ first, which is equivalent to the elimination of all unknowns corresponding to the interior nodes of the macroelements.

Let us assume that the triangulations $\mathcal{T}_0$ is constructed by diagonal subdivision of a square mesh. Let the Schur compliment, corresponding to the static condensation, be approximated by its diagonal $S_d^{(k+1)}$.

$$\widetilde{B}_{11}^{(k+1)} = \begin{bmatrix} I & 0 \\ \widetilde{A}_{11,SI}^{(k+1)} (\widetilde{A}_{11,II}^{(k+1)})^{-1} & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_{11,II}^{(k+1)} & \widetilde{A}_{11,IS}^{(k+1)} \\ 0 & S_d^{(k+1)} \end{bmatrix}.$$

Then the following estimate holds (Kolev, Margenov (99))

$$\kappa \left((\widetilde{B}_{11}^{(k+1)})^{-1} \widetilde{A}_{11}^{(k+1)} \right) \leq const \approx 8.301 \cdots ,$$

which is uniform with respect to the Poisson ratio ν.

5. Numerical tests

- The presented numerical tests illustrate the behavior of the FEM error as well as the optimal convergence rate of the AMLI algorithm when the size of the discrete problem is varied and $\nu \in [0, 1/2)$ tends to the incompressible limit.
- We consider the model test problem in the unit square $\Omega = (0,1)^2$ with $E = 1$. The right hand side corresponds to the exact solution $\mathbf{u}(x,y) = [\sin(\pi x)\sin(\pi y), y(y-1)x(x-1)]\,.$
- The relative stopping criterion for the PCG iterations is

$$\frac{(C_{AMLI}^{-1} r^{N_{it}}, r^{N_{it}})}{(C_{AMLI}^{-1} r^{0}, r^{0})} < \varepsilon^2,$$

where r^i stands for the residual at the i-th iteration step.

Table 1 well illustrates the locking-free C.R. FEM approximation. The number of refinement steps is $\ell = 4$, $N = 1472$, and $\varepsilon = 10^{-9}$.

Table 1. Relative error stability for $\nu \to 1/2$

ν	$\|u - u_h\|_{[L_2]^2} / \|f\|_{[L_2]^2}$	ν	$\|u - u_h\|_{[L_2]^2} / \|f\|_{[L_2]^2}$
0.4	.3108249106503572	0.4999	.3771889077038727
0.49	.3695943747405575	0.49999	.3772591195613628
0.499	.3764879643773666	0.499999	.3772661419401481

In Table 2, the number of iterations are presented as a measure of the robustness of the multilevel preconditioner. The optimal order *locking-free* convergence rate of the AMLI algorithm is well expressed. Here β stands for the degree of the stabilization polynomial.

Table 2. Iterations: $\varepsilon = 10^{-3}$, $\beta = 2$

ℓ	N	0.3	0.49	0.4999	0.499999
4	**1472**	**13**	**12**	**13**	**13**
5	**6016**	**12**	**12**	**13**	**13**
6	**24320**	**12**	**12**	**13**	**13**
7	**97792**	**11**	**11**	**13**	**13**
8	**196096**	**11**	**11**	**12**	**13**

In the last Table, a modification of the AMLI algorithm is used where two inner PCG iterations have been applied to stabilize the multilevel algorithm instead of the acceleration polynomial $[I - p_\beta(.)]$.

Table 3. Modified AMLI: two inner PCG iterations

ℓ	N	0.3	0.49	0.4999	0.499999
4	**1472**	**10**	**11**	**10**	**10**
5	**6016**	**10**	**11**	**11**	**11**
6	**24320**	**10**	**11**	**11**	**11**
7	**97792**	**11**	**11**	**11**	**12**
8	**196096**	**10**	**11**	**11**	**12**

6. Concluding remarks

- The presented results well illustrate the motivation of the expanding interest to applications of non-conforming FEs for solving problems, where the standard conforming elements may suffer from locking effects.
- The success of the Crouzeix-Raviart and other non-conforming FEs can be explained by the fact that they produce algebraic systems that are equivalent to the Schur complements for the Lagrange multipliers arising from the mixed FEM for Raviart-Thomas elements.
- There are also other advantages of the non-conforming Crouzeix-Raviart finite elements, such as less density of the stiffness matrix.
- The presented multilevel algorithms have some well expressed inherently parallel features. The key point here is that the considered approximations of the coarse grid complement blocks $\widetilde{A}_{11}$ are of either diagonal or generalized tridiagonal form.

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