Optimal multilevel preconditioning of strongly anisotropic problems. Part II: non-conforming FEM.

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CONTENTS

1. Introduction

- We study strategies to construct hierarchical basis functions (HB) multilevel preconditioners to solve algebraic systems arising from second order elliptic problems, discretized by the non-conforming **Crouzeix-Raviart finite elements.**
- To this end we follow the known framework for constructing HB preconditioners for conforming FEM.
- However, applying the latter framework and the existing theoretical results to non-conforming FEM is not straightforward as the classical construction of hierarchical preconditioner relies on a nested sequence of finite element spaces, in most cases related to nested grids, while: **The non-conforming FEM on nested grids produces non-nested FE spaces.**

2. Crouzeix-Raviart FEs

Consider the selfadjoint elliptic boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv -\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x})) &= f(\mathbf{x}) \quad \textbf{in} \quad \Omega, \\ u(x) &= 0 \quad \textbf{on} \quad \Gamma_D, \\ (a(\mathbf{x})\nabla u(\mathbf{x})) \cdot \mathbf{n} &= 0 \quad \textbf{on} \quad \Gamma_N, \end{aligned}$$

where Ω is a polygonal domain in $\mathbf{R}^2$, $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, $a(\mathbf{x}) = [a_{ij}(\mathbf{x})]_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω, $\mathbf{n}$ is the unit vector of outward normal to the boundary $\Gamma = \partial\Omega$, and $\Gamma = \Gamma_D \cup \Gamma_N$. We assume that the entries $a_{ij}(\mathbf{x})$ are piece-wise smooth functions on $\bar{\Omega} = \Omega \cup \partial\Omega$.

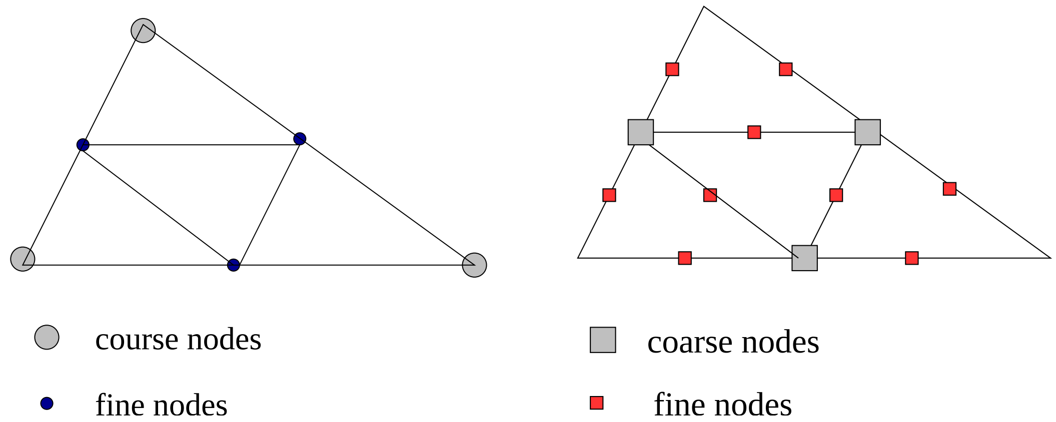
The weak formulation of the above problem reads as follows: for some given f, find $u \in \mathcal{V} \equiv H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0$ on $\Gamma_D\}$, which satisfies

$$\mathcal{A}(u, v) = (f, v) \quad \forall v \in H^1_D(\Omega),$$

where

$$\mathcal{A}(u, v) = \int_\Omega a(\mathbf{x}) \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}.$$

- **Consider two partitionings of Ω: a coarse triangulation $\mathcal{T}_H$ and a fine one $\mathcal{T}_h$, which is obtained by a regular refinement of $\mathcal{T}_H$.**
- **The partitioning $\mathcal{T}_H$ is assumed to be aligned with the discontinuities of the coefficient $a(\mathbf{x})$ so that over each element $E \in \mathcal{T}_H$ the function $a(\mathbf{x})$ is smooth.**



Crouzeix-Raviart non-conforming FEs

We discretize the variational problem using the Crouzeix-Raviart FEs, i.e., we seek the solution in the finite dimensional space

$$\mathcal{V}_h = \{v \in L_2(\Omega) : v\,|_e \text{ is linear} \forall e \in \mathcal{T}_h\,,\ v \text{ is continuous at the midpoints of the edges of} e \in \mathcal{T}_h \text{ and } v \text{ is zero at the midpoints on } \Gamma_D\}.$$

The nodal FE basis $\phi_i^h\ (i = 1, \ldots, n_h)$ **in** $\mathcal{V}_h$ **is naturally defined as** ϕ_i^h **being equal to unity at one midpoint** m_k**in** e **and zero at the other two midpoints.**

The discrete formulation becomes: find $u_h \in \mathcal{V}_h$, which satisfies

$$\mathcal{A} : 0_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in \mathcal{V}_h,$$

where

$$\mathcal{A}_h(u_h, v_h) = \sum_{e \in \mathcal{T}_h} \int_e a(e) \nabla u_h \cdot \nabla v_h d\mathbf{x}.$$

Here $a(e)$ is a piece-wise constant coefficient matrix, defined by the integral averaged values of $a(\mathbf{x})$ over each triangle from the coarser triangulation $\mathcal{T}_H$. Let us note that in this way arbitrary large coefficient jumps across the boundaries between adjacent finite elements from $\mathcal{T}_H$ are allowed.

Using the nodal basis, the FEM problem reads as

$$A_h \mathbf{u}_h \; = \; \mathbf{b}_h, \; \mathbf{u}_h, \qquad \mathbf{b}_h \in R^{n_h}$$

where $A_h = (a_{ij})$, is the global stiffness matrix with entries $a_{ij} = \mathcal{A}_h(\phi_i^h, \phi_j^h)$.

We pose no restrictions on the mesh and/or coefficient anisotropy.

As it is known, to derive estimates for the CBS constant γ, it suffices to consider an isotropic problem in an arbitrary shaped triangle, T.

Let us denote the angles in T by θ_1, θ_2 and $\theta_3 = \pi - (\theta_1 + \theta_2)$, where $a = \cot\theta_1$, $b = \cot\theta_2$ and $c = \cot\theta_3$. Without loss of generality, for each triangle T, we assume that $\theta_1 \geq \theta_2 \geq \theta_3$, and then denote $\alpha = a/c$ and $\beta = b/c$.

A simple computation shows that the standard nodal basis element stiffness matrix for Crouzeix-Raviart non-conforming linear elements A_e^{CR} coincides with that for the conforming linear elements A_e^{cl}, up to a scalar factor 4:

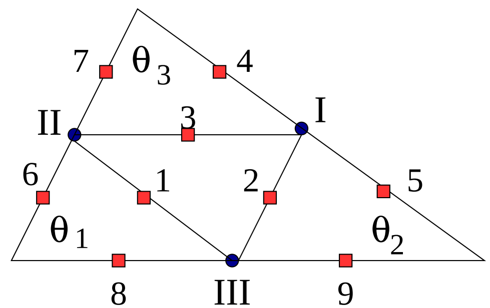
$$A_e^{CR} = 2\begin{bmatrix} b+c & -c & -b \\ -c & a+c & -a \\ -b & -a & a+b \end{bmatrix} = 2c\begin{bmatrix} 1+\beta & -1 & -\beta \\ -1 & 1+\alpha & -\alpha \\ -\beta & -\alpha & \alpha+\beta \end{bmatrix}.$$

3. Two-level decompositions

Consider now the two consecutive mesh refinements, $\mathcal{T}_H$ and $\mathcal{T}_h$. To build a hierarchical preconditioner, we need a suitable decomposition of $\mathcal{V}_h$, $\mathcal{V}_h = \mathcal{V}_1 \oplus \mathcal{V}_2$.
In the case of conforming FEM, one of the spaces ($\mathcal{V}_2$) is naturally induced by the coarse mesh. For non-conforming FEM, $\mathcal{V}_H \not\subseteq \mathcal{V}_h$ and the direct construction with, say, $\mathcal{V}_2 \equiv \mathcal{V}_H$ is impossible. To overcome the latter difficulty, we consider two constructions of suitable hierarchical decompositions of the Crouzeix-Raviart spaces were introduced.
Following the introduced terminology, Blaheta, Margenov, Neytcheva (2004), we refer them as:

- Two-level *first reduce* decomposition **(FR)**;
- Two-level decomposition with *differences and aggregates* **(DA)**.

3.1. Two-level FR decomposition



Crouzeix-Raviart macro-element

On macroelement level, we have the space $\mathcal{V}(E) =$ span $\{\phi_1, \ldots, \phi_9\}$. Let the basis functions ϕ_i correspond to the midpoints m_i, ordered as shown in the Figure.

The FR splitting $\mathcal{V}(E) = \widehat{\mathcal{V}}_1(E) \oplus \widehat{\mathcal{V}}_2(E)$ is defined as follows

$$
\begin{aligned}
\widehat{\mathcal{V}}_1(E) &= \textbf{span}\ \{\phi_1, \phi_2, \phi_3, \phi_4 - \phi_5, \phi_6 - \phi_7, \phi_8 - \phi_9\}, \\
\widehat{\mathcal{V}}_2(E) &= \textbf{span}\ \{\phi_4 + \phi_5, \phi_6 + \phi_7, \phi_8 + \phi_9\}.
\end{aligned}
$$

Using the transformation matrix J_{FR},

$$J_{FR} = \frac{1}{2}\begin{bmatrix} 2I & & \\ & J_1 & \\ & & J_2 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 1 & -1 & & & & \\ & & 1 & -1 & & \\ & & & & 1 & -1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & & & 1 & 1 \end{bmatrix},$$

we introduce a corresponding hierarchical basis

$$\widehat{\varphi}_E = \{\widehat{\phi}_i\}_{i=1}^{9} = J_{FR}\,\varphi_E.$$

The hierarchical macroelement stiffness matrix is then computed as

$$\widehat{A}_E = J_{FR} A_E J_{FR}^T,$$

and the related global stiffness matrix is obtained as $\widehat{A}_h = \sum_{E\in\mathcal{T}} \widehat{A}_E$.

The hierarchical stiffness matrix $\widehat{A}_h$ admits the 2×2 block structure

$$\widehat{A}_h = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix} \begin{matrix} \} \, \widehat{\mathcal{V}}_1 \\ \} \, \widehat{\mathcal{V}}_2 \end{matrix} ,$$

where $\widehat{\mathcal{V}}_1, \widehat{\mathcal{V}}_2$ are associated with the locally introduced red FR splitting.

The matrix $\widehat{A}_h$ can be also seen as having a block 3×3 structure:

$$\widehat{A}_h = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12}^{(0)} & \bar{A}_{13}^{(0)} \\ \bar{A}_{21}^{(0)} & \bar{A}_{22}^{(0)} & \bar{A}_{23}^{(0)} \\ \bar{A}_{31}^{(0)} & \bar{A}_{32}^{(0)} & \bar{A}_{33}^{(0)} \end{bmatrix} \begin{matrix} \} \text{ interior basis functions} \\ \} \text{ half-difference basis functions} \\ \} \text{ half-sum basis functions } (\widehat{\mathcal{V}}_2) \end{matrix}$$

For our purposes, however, $\widehat{A}_h$ is first decomposed into a 2×2 block form

$$\widehat{A}_h = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{array}{l} \} \text{ interior basis functions,} \\ \} \text{ rest.} \end{array} .$$

As a first step of the FR algorithm, the interior unknowns are eliminated and $\widehat{A}_h$ is reduced to its Schur complement $B = \bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12}$. Next we consider a two-level splitting of the matrix B, again in a block 2×2 form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where the first block corresponds to the half-difference and the second block corresponds to the half-sum basis functions.

The 2×2 block decomposition of B can now be used to construct two-level preconditioners, since the matrix B_{22} is associated with the coarse grid.

3.2. CBS constant of FR algorithm

Let us consider the local eigenvalue problem

$$S_E \mathbf{v} = \lambda B_{E,22} \mathbf{v}, \qquad \mathbf{v} \neq const = (c, c, c)^T, c \neq 0,$$

where

$$S_E = B_{E,22} - B_{E,21} B_{E,11}^{-1} B_{E,12}.$$

The minimum eigenvalue of $B_{E,22}^{-1} S_E$ is found to be of the form

$$\lambda_{min}(B_{E,22}^{-1} S_E) = \frac{5\sigma - \sqrt{\sigma(\sigma - 8\alpha\beta)}}{8\sigma}, \qquad \sigma = (\alpha+1)(\beta+1)(\alpha+\beta).$$

The local problem is associated with the angles of the current triangle T, namely θ_1, θ_2 and $\theta_3 = \pi - (\theta_1 + \theta_2)$, where $a = \cot\theta_1$, $b = \cot\theta_2$ and $c = \cot\theta_3$, assuming that $\theta_1 \geq \theta_2 \geq \theta_3$, and then $\alpha = a/c$ and $\beta = b/c$.

To estimate γ_E^2, we substitute σ, α, β by $c_i = \cos\theta_i$, using meanwhile $s_i = \sin\theta_i$, $i = \in \{1, 2, 3\}$. Applying $a = (1 - bc)/(b + c)$ in the form of $ab + bc + ca = 1$ we get $(a + c)(b + c) = 1 + c^2$ and therefore $\sigma c^3 = (1 + c^2)(a + b)$.

By definition, $a + b = \frac{c_1}{s_1} + \frac{c_2}{s_2} = \frac{c_1 s_1 + c_2 s_1}{s_1 s_2} = \frac{s_3}{s_1 s_2} \Rightarrow 1 + c^2 = 1 + \frac{c_3^2}{s_3^2}$.

Then $\sigma c^3 = \frac{1}{s_1 s_2 s_3} \Rightarrow \frac{\alpha\beta}{\sigma} = \frac{ab}{c^2\sigma} = \frac{abc}{c^3\sigma} = \frac{c_1 c_2 c_3}{s_1 s_2 s_3 c^3 \sigma} = c_1 c_2 c_3$, that is

$$\gamma_E^2 = \frac{3}{8} + \frac{1}{8}\sqrt{1 - 8c_1 c_2 c_3}.$$

Since $c_1 c_2 c_3 > -1$, $1 - 8c_1 c_2 c_3 < 9$ which simply leads to $\gamma_E^2 < \frac{3}{4}$.

Theorem 3.1. **(Blaheta, Margenov, Neytcheva (2004))**

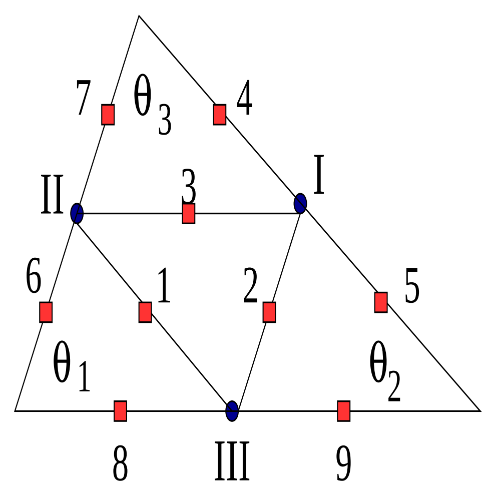
The related FR constant in the strengthened CBS inequality is uniformly bounded with respect to both coefficient and mesh anisotropy, i.e.,

$$\gamma^2 < \frac{3}{4}.$$

3.2. Two-level DA decomposition

This decomposition is referred to as ”(D)ifferentiation and (A)ggregation splitting”. If $\phi_1, \ldots, \phi_9$ are the standard nodal non-conforming linear finite element basis functions on the macroelement, then we define

$$\begin{aligned}
\mathcal{V}(E) &= \textbf{span}\,\{\phi_1, \ldots, \phi_9\} = \mathcal{V}_1(E) \oplus \mathcal{V}_2(E)\,, \\
\mathcal{V}_1(E) &= \textbf{span}\,\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4 - \phi_5,\ \phi_6 - \phi_7,\ \phi_8 - \phi_9\}, \\
\mathcal{V}_2(E) &= \textbf{span}\,\{\phi_1 + \phi_4 + \phi_5,\ \phi_2 + \phi_6 + \phi_7,\ \phi_3 + \phi_8 + \phi_9\}\,.
\end{aligned}$$



The related matrix J transforms the macroelement stiffness matrix into a hierarchical form

$$\tilde{A}_E = J_E A_E J_E^T = \begin{bmatrix} \tilde{A}_{E,11} & \tilde{A}_{E,12} \\ \tilde{A}_{E,21} & \tilde{A}_{E,22} \end{bmatrix} \begin{matrix} \tilde{\phi}_i \in \mathcal{V}_1(E) \\ \tilde{\phi}_i \in \mathcal{V}_2(E) \end{matrix} .$$

For the whole finite element space $\mathcal{V}_h$ with the standard nodal finite element basis $\varphi = \{\phi_h^{(i)} : i = 1, \ldots, N_h\}$, we can similarly construct a new hierarchical basis $\tilde{\varphi} = \tilde{\varphi}_1 \cup \tilde{\varphi}_2 \cup \tilde{\varphi}_3$ and a corresponding splitting

$$\mathcal{V}_h = \mathcal{V}_1 \oplus \mathcal{V}_2 ,$$

$$\mathcal{V}_1 = \textbf{span}\{\tilde{\phi}_h^{(i)} \in \tilde{\varphi}_1 \cup \tilde{\varphi}_2\}, \quad \mathcal{V}_2 = \textbf{span}\{\tilde{\phi}_h^{(i)} \in \tilde{\varphi}_3\},$$

and respectively

$$\tilde{A}_h = J A_h J^T = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{matrix} \tilde{\phi}_i \in \mathcal{V}_1 \\ \tilde{\phi}_i \in \mathcal{V}_2 \end{matrix} .$$

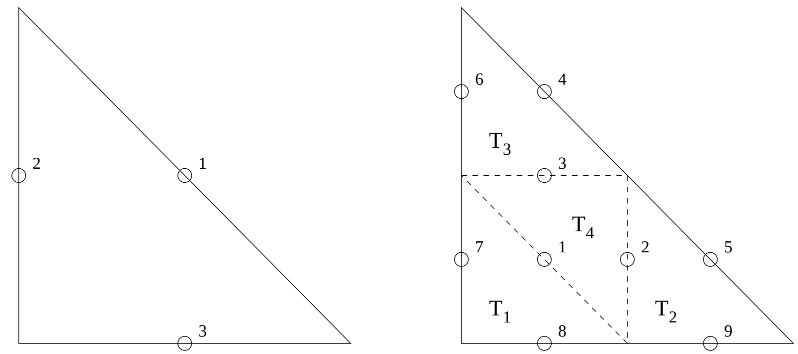
Again, the analysis of the related two-level method is performed locally, by considering the corresponding problems on macroelements. The obtained result is summarized in the following theorem, which is analogous to the estimate in the case of FR decomposition.

Theorem 3.2. (Blaheta, Margenov, Neytcheva (2004))
Let us consider the two–level DA splitting. Then the CBS constant is uniformly bounded with respect to both coefficients and mesh anisotropy,

$$\gamma^2 \leq 3/4 \,.$$

The latter estimate is independent on the discretization (mesh) parameter h and possible coefficient jumps aligned with the finite element partitioning $\mathcal{T}_H$.

Proof.



The reference coarse grid triangle and the macroelement $\hat{E}$.

Let $\mathcal{V}_1(\hat{E})$, $\mathcal{V}_2(\hat{E})$ be the two-level splitting for the reference macroelement and for $u \in \mathcal{V}_1(\hat{E})$, $v \in \mathcal{V}_2(\hat{E})$ denote $d^{(k)} = \nabla u|_{T_k}$, $\delta^{(k)} = \nabla v|_{T_k}$.

Then the relations between the function values in some nodal points, namely

$u(P_4) = -u(P_5)$, $u(P_6) = -u(P_7)$, $u(P_8) = -u(P_9)$ **and**
$v(P_1) = v(P_4) = v(P_5)$, $v(P_2) = v(P_6) = v(P_7)$, $v(P_3) = v(P_8) = v(P_9)$,

imply that

$$d^{(1)} + d^{(2)} + d^{(3)} + d^{(4)} = 0, \qquad \delta^{(1)} = \delta^{(2)} = \delta^{(3)} = -\delta^{(4)} = \delta.$$

Hence,

$$\begin{aligned} \mathcal{A}_{h,\hat{E}}(u,v) &= \sum_{k=1}^{4} \int_{T_k} a\nabla u \cdot \nabla v dx = \sum_{k=1}^{4} \triangle \langle a\delta^{(k)},\, d^{(k)} \rangle \\ &= \triangle \langle a\delta,\, d^{(1)} + d^{(2)} + d^{(3)} - d^{(4)} \rangle \\ &= -2\triangle \langle a\delta,\, d^{(4)} \rangle \leq 2\triangle \parallel \delta \parallel_a \parallel d^{(4)} \parallel_a \end{aligned}$$

where $\triangle =$ **area**(T_k), $\langle x, y \rangle = x^T y$**, and** $\parallel x \parallel_a = \sqrt{\langle ax, x \rangle}$**. Further,**

$$\parallel d^{(4)} \parallel_a^2 = \parallel d^{(1)} + d^{(2)} + d^{(3)} \parallel_a^2 \leq 3 \sum_{k=1}^{3} \parallel d^{(k)} \parallel_a^2$$

leads to

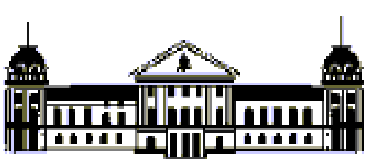
$$\mathcal{A}_{h,\hat{E}}(u,u) = \sum_{k=1}^{4} \parallel d^{(k)} \parallel_a^2 \triangle \geq \left(1 + \frac{1}{3}\right) \triangle \parallel d^{(4)} \parallel_a^2$$

and

$$\mathcal{A}_{h,\hat{E}}(v,v) = 4\triangle \parallel \delta \parallel_a^2 .$$

Thus,

$$\mathcal{A}_{h,\hat{E}}(u,v) \leq 2\Delta \sqrt{\frac{3}{4\Delta} \mathcal{A}_{h,\hat{E}}(u,u)} \sqrt{\frac{1}{4\Delta} \mathcal{A}_{h,\hat{E}}(v,v)} = \sqrt{\frac{3}{4}} \sqrt{\mathcal{A}_{h,\hat{E}}(u,u)} \sqrt{\mathcal{A}_{h,\hat{E}}(v,v)} .$$



The following theorem is useful for extending the two-level to multilevel case.

Theorem 3.3. (Blaheta, Margenov, Neytcheva (2004))

Let $\tilde{A}_{22}$ be the stiffness matrix corresponding to the space $\mathcal{V}_2$ with the basis $\tilde{\varphi}_3$ from the splitting DA and let A_H be the stiffness matrix corresponding to the coarse discretization $\mathcal{T}_H$ FE space $\mathcal{V}_H$, equipped with the standard nodal finite element basis $\{\phi_H^{(k)} : k = 1, \ldots, N_H\}$. Then

$$\tilde{A}_{22} = 4\, A_H \, .$$

Proof.

Consider the nodal basis function $\phi_H^{(i)} \in \mathcal{V}_H$ and DA basis function $\tilde{\phi}_h^{(i)} \in \tilde{\phi}_3$. Let both basis functions be equal to unity in the nodes belonging to the C-edges. Then for any macroelement $E = \cup_{k=1}^{4} T_k$ we get

$$d_i^{(1)} = d_i^{(2)} = d_i^{(3)} = -d_i^{(4)} = 2\nabla\phi_H^{(i)}, \quad \text{where} \quad d_i^{(k)} = \nabla\tilde{\phi}_h^{(i)}|_{T_k} \, .$$

4. Preconditioning of B_{11} (FR)

Recall that the top-left block of $\bar{A}_{11}$ in FR algorithm is block-diagonal. After the elimination, one obtains a (6×6) element matrix B_E, which constitutes the macroelement contribution to the matrix B. Next we split B_E as:

$$B_E = \begin{bmatrix} B_{E,11} & B_{E,12} \\ B_{E,21} & B_{E,22} \end{bmatrix} \begin{matrix} \}\text{two-level half-difference basis functions} \\ \}\text{two-level half-sum basis functions} \end{matrix}$$

The matrix block $B_{E,11}$ in is found explicitly, namely,

$$B_{E,11} = \frac{2p}{q} \begin{bmatrix} 3\,q + 2(1+\beta+\beta^2) & q+2 & -q-2\,\beta^2 \\ q+2 & 3\,q+2(1+\alpha+\ \alpha^2) & -q-2\,\alpha^2 \\ -q-2\,\beta^2 & -q-2\,\alpha^2 & 3\,q+2\,(\alpha^2+\alpha\beta+\beta^2) \end{bmatrix},$$

$q = \alpha + \alpha\beta + \beta$**, and**

$p = 3(\alpha+\alpha\beta+\beta) + 3(\alpha^2+\alpha\beta+\beta^2) + \alpha\beta(3\alpha+3\beta+1)$.

The following relations are known to hold or can be derived from Lemma 2.1.

(r1) $\alpha + \alpha\beta + \beta = \alpha(\beta + 1) + \beta = \dfrac{1}{c^2} \geq 0,\ c = \cot\theta_3$, i.e., $-\alpha\beta \leq \alpha + \beta$;

(r2) $\alpha^2 + \alpha\beta + \beta^2 = \alpha^2 + \beta(\alpha + \beta) \geq 0$.

Consider the off-diagonal elements of $B_{E,11}$. It is easy to observe that

1. $B_{E,11}(1,2) = \alpha + \alpha\beta + \beta + 2 > 0$ for all valid values of α and β
2. $B_{E,11}(1,3) = -\alpha - \alpha\beta - \beta - 2\,\beta^2 < 0$
3. $B_{E,11}(2,3) = -\alpha - \alpha\beta - \beta - 2\,\alpha^2 < 0$ and
4. $|B_{E,11}(2,3)| \leq |B_{E,11}(1,3)| \leq B_{E,11}(1,2)$, and
   $|B_{E,11}(1,3)| = B_{E,11}(1,2)$ only for $\beta = 1$.

Remark 3.1.

Note that the direction of the strongest off-diagonal coupling of the macroelement matrix $B_{E,11}$ is the same as of the related conforming FE.

4.1. Additive preconditioner

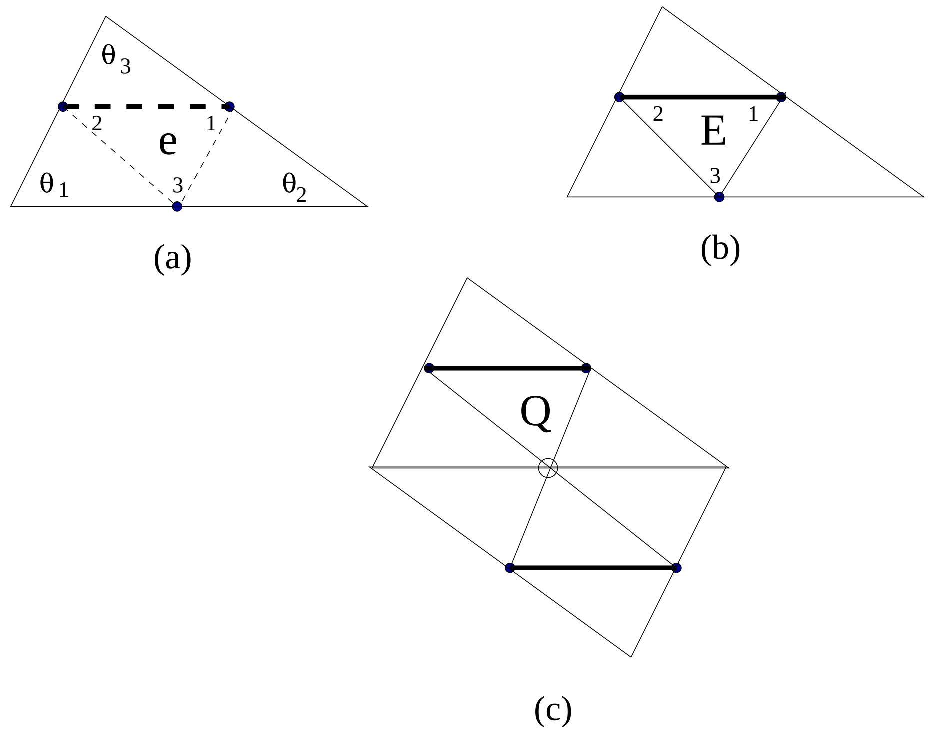


Figure: Dominating off-diagonal couplings for (a) the element stiffness matrix corresponding to $e \in \mathcal{T}_h$, (b) the macroelement matrix $B_{E,11}$,and (c) the macroelement matrix $B_{Q,11}$.

We construct the additive preconditioner $C_{E,11}$ to $B_{E,11}$ by deleting the weakest off-diagonal couplings in $B_{E,11}$, i.e., we let

$$C_{E,11} = \frac{2p}{q} \begin{bmatrix} 3\,q + 2(1+\beta+\beta^2) & q+2 & 0 \\ q+2 & 3\,q + 2(1+\alpha+\ \alpha^2) & 0 \\ 0 & 0 & 3\,q + 2\,(\alpha^2+\alpha\beta+\beta^2) \end{bmatrix}.$$

Then C_{11} is obtained by assembling the modified element matrices $C_{E,11}$.

Lemma 3.1.

For any element size and shape and for any anisotropy $a(\mathbf{x})$, there holds that

$$\left(1-\sqrt{7/15}\right) C_{E,11} \quad \leq \quad B_{E,11} \leq \left(1+\sqrt{7/15}\right) C_{E,11}.$$

Proof.

Consider the generalized eigenvalue problem $B_{E,11}\mathbf{v} = \lambda C_{E,11}\mathbf{v}$ **and the corresponding characteristic equation for** λ**,** $det\,(B_{E,11} - \lambda C_{E,11}) = 0$**.**

The determinant is found to be

$$\begin{vmatrix} (1-\lambda)(3\,q+2(1+\beta+\beta^2)) & (1-\lambda)(q+2) & -q-2\,\beta^2 \\ (1-\lambda)(q+2) & (1-\lambda)(3\,q+2(1+\alpha+\ \alpha^2)) & -q-2\,\alpha^2 \\ -q-2\,\beta^2 & -q-2\,\alpha^2 & (1-\lambda)(3\,q+2\,(\alpha^2+\alpha\beta+\beta^2)) \end{vmatrix}.$$

Straightforward computation shows that $\mu_i = 1-\lambda_i, i = 1, 2, 3$ **satisfy**

$$\mu_1 = 0, \quad \textbf{i.e.}, \quad \lambda_1 = 1$$

$$\mu_{2,3}^2 = \frac{(\alpha+\beta)(\alpha+\alpha\beta+\beta+2(2\alpha^2-\alpha\beta)+\beta^2)}{(\alpha+\beta+2)[2(\alpha^2+\alpha\beta+\beta^2)+3(\alpha+\alpha\beta+\beta)]}.$$

We show below that

$$\mu_{2,3}^2 \le \frac{7}{15},$$

which expanded form becomes

$$\mathcal{E}(\alpha,\beta) \equiv 16\alpha^3+16\beta^3-34\alpha^2-34\beta^2-34\alpha^2\beta-34\alpha\beta^2-82\alpha\beta-42\alpha-42\beta \le 0$$

$$\forall(\alpha,\beta)\in D = \left\{(\alpha,\beta) : -\tfrac{1}{2} < \alpha \le 1,\ 0 < \beta \le 1,\ \alpha+\beta > 0,\ |\alpha| \le \beta\right\}.$$

- **Case 1:** Let $\alpha = 0$. In this case $\mathcal{E}(0,\beta) \le 16\beta^2 - 34\beta^2 - 42\beta = -18\beta^2 - 42\beta \le 0$.
- **Case 2:** Let $\alpha > 0$. Then $16\alpha^3 + 16\beta^3 - 34\beta^2 \le 16\beta^2 + 16\beta^2 - 34\beta^2 \le 0$ and the remaining terms in $\mathcal{E}$ are negative, so $\mathcal{E} \le 0$.

- **Case 3: Let $-\frac{1}{2} < \alpha < 0$. We use the fact that $\beta \leq 1$, i.e., $\beta^2 \leq \beta$ and that $-\alpha\beta \leq \alpha + \beta$.**

$$
\begin{aligned}
\mathcal{E} = \quad & 16\alpha^3 + 16\beta^3 - 34\alpha^2 - 34\beta^2 - 34\alpha\beta(\alpha+\beta) - 82\alpha\beta - 42(\alpha+\beta) \\
\leq \quad & 16\alpha^3 + 16\beta^3 - 14\alpha\beta - 42(\alpha+\beta) \\
\leq \quad & 16(\alpha^3 + \beta^3) + 14(\alpha+\beta) - 42(\alpha+\beta) \\
= \quad & 16(\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2) - 28(\alpha+\beta)
\end{aligned}
$$

It remains to prove that $16(\alpha^2 - \alpha\beta + \beta^2) \leq 28$ or $\alpha^2 - \alpha\beta + \beta^2 \leq 7/4$.

The latter is true, since

$$
\sup_{\alpha,\beta} (\alpha^2 - \alpha\beta + \beta^2) = \sup_{\alpha \in (-1/2, 0)} (\alpha^2 - \alpha + 1) = 7/4.
$$

$\mathcal{E}$ achieves its maximum value 0 for (α, β) equal to $(0,0)$ and $(-1/2, 1)$.

The assembled global preconditioner C_{11} inherits the properties of $C_{E,11}$. We collect the results in the following theorem.

Theorem 3.1.

By deleting the two smallest off-diagonal elements in the local matrix $B_{E,11}$ and assembling the modified element matrix, we construct a preconditioner C_{11} to B_{11} such that

$$\left(1 - \sqrt{7/15}\right) C_{11} \leq B_{11} \leq \left(1 + \sqrt{7/15}\right) C_{11},$$

$$\varkappa(C_{11}^{-1} B_{11}) < \frac{1}{4}\left(11 + \sqrt{105}\right) \approx 5.31.$$

The spectral equivalence and the related condition number estimate hold independently of element size and shape, and problem anisotropy as well.

4.2. Multiplicative preconditioner

Following the construction first proposed by Margenov, Vassilevski (1994) (see Axelsson, Margenov (2003) for the complete analysis in the case of conforming FE), we partition the nodes in the block B_{11} into two groups, where the first one contains the centers of parallelogram superelements Q which are weakly connected in the sense that the off-diagonal couplings are relatively small.

With respect to this partitioning, B_{11} admits the two-by-two block-factored form

$$B_{11} = \begin{bmatrix} D_{11} & F_{11} \\ {F_{11}}^T & E_{11} \end{bmatrix} = \begin{bmatrix} D_{11} & 0 \\ {F_{11}}^T & S_{11} \end{bmatrix} \begin{bmatrix} I & {D_{11}}^{-1} F_{11} \\ 0 & I \end{bmatrix}.$$

Then, the multiplicative preconditioner C_{11} is defined as

$$C_{11} = \begin{bmatrix} D_{11} & 0 \\ {F_{11}}^T & E_{11} \end{bmatrix} \begin{bmatrix} I & {D_{11}}^{-1} F_{11} \\ 0 & I \end{bmatrix}.$$

Theorem 3.2.

The multiplicative preconditioner C_{11} of B_{11} has an optimal order convergence rate with a relative condition number uniformly bounded by

$$\kappa \left({C_{11}}^{-1} B_{11} \right) < \frac{15}{8} = 1.875.$$

Proof.

We again consider the generalized eigenvalue problem

$$S_{11:Q} \mathbf{v}_Q = \lambda_Q E_{11:Q} \mathbf{v}_Q .$$

As it is seen from the analysis in the case of conforming FEM,

$$\lambda_Q^{(2)} = \lambda_Q^{(3)} = \lambda_Q^{(4)} = 1, \quad \text{and} \quad \lambda_Q^{(1)} = 1 - \left(\mu^{(2,3)} \right)^2 .$$

Here $\mu^{(2,3)} = 1 - \lambda^{(2,3)}$ stand for the eigenvalues introduced in the analysis of the additive preconditioner to B_{11}. This immediately gives

$$\left(\mu^{(2,3)} \right)^2 < \frac{7}{13} \quad \Rightarrow \quad \lambda_Q^{(1)} > \frac{8}{15}$$

after what the proof of the theorem follows straightforwardly.

4.3. Solvers for C_{11}

- **The computational complexity of solving systems with C_{11} is determined by their connectivity pattern only. This means, that the optimal order direct solver considered in the case of conforming FEM are directly applicable here.**
- **For the additive algorithm, the matrix C_{11} has a generalized tridiagonal structure, that is, it is tridiagonal under a proper ordering of the elements.**
- **For the multiplicative preconditioner of B_{11}, the optimal order direct solver incorporates the** ***nested dissection*** **(ND) algorithm for the** ***reduced*** **system.**

5. Preconditioner of $\widetilde{A}_{11}$ (DA)

The block $\widetilde{A}_{11}$ can be further decomposed by splitting the unknowns into two groups - associated with the interior (I), and associated with the sides (S) of the macroelements,

$$\widetilde{A}_{11} = \begin{bmatrix} \widetilde{A}_{11,II} & \widetilde{A}_{11,IS} \\ \widetilde{A}_{11,SI} & \widetilde{A}_{11,SS} \end{bmatrix}.$$

Elimination of the block $\widetilde{A}_{11,II}$ corresponding to the inner nodes gives rise to the Schur complement

$$S = \widetilde{A}_{11,SS} - \widetilde{A}_{11,SI}\widetilde{A}^{-1}_{11,II}\widetilde{A}_{11,IS} \quad \Rightarrow S = B_{11},$$

where B_{11} is the block appearing in the FR decomposition. Therefore, the optimal preconditioners C_{11} of B_{11} are directly applicable to construct $\widetilde{C}_{11}$, namely

$$\widetilde{C}_{11} = \begin{bmatrix} I & 0 \\ \widetilde{A}_{11,SI}\widetilde{A}^{-1}_{11,II} & I \end{bmatrix} \begin{bmatrix} \widetilde{A}_{11,II} & \widetilde{A}_{11,IS} \\ 0 & C_{11} \end{bmatrix}.$$

Theorem 3.3.

For any element size and shape and any problem anisotropy it holds that

- $\kappa\left(\widetilde{C}_{11}^{-1}\widetilde{A}_{11}\right) < \frac{1}{4}(11+\sqrt{105})$, **if C_{11} is the preconditioner to B_{11}, based on a modified element matrix;**
- $\kappa\left(\widetilde{C}_{11}^{-1}\widetilde{A}_{11}\right) < \frac{15}{8}$, **if C_{11} is the preconditioner to B_{11} of factorized form;**
- **the cost of the application of preconditioners in both cases is proportional to the number of unknowns.**

Proof.

The spectral equivalence is readily seen from the following expressions

$$\widetilde{C}_{11}=\begin{bmatrix}\widetilde{A}_{11,II} & \widetilde{A}_{11,IS} \\ \widetilde{A}_{11,SI} & \widetilde{A}_{11,SI}\widetilde{A}_{11,II}^{-1}\widetilde{A}_{11,IS}+C_{11}\end{bmatrix},\ \widetilde{A}_{11}=\begin{bmatrix}\widetilde{A}_{11,II} & \widetilde{A}_{11,IS} \\ \widetilde{A}_{11,SI} & \widetilde{A}_{11,SI}\widetilde{A}_{11,II}^{-1}\widetilde{A}_{11,IS}+B_{11}\end{bmatrix}.$$

6. Concluding remarks

- To be able to construct multilevel hierarchical preconditioners for the non-conforming discretization we have to answer the question if the two-level splitting of the finite element spaces is recursively applicable, or in other words, how $\mathcal{V}_2^{(k)}$ relates to $\mathcal{V}^{(k-1)}$.
- In the FR case, it turns out that the block B_{22} approximates A_H. The spectral relations of B_{22} and A_H can be studied locally. For the reference macroelement, we find numerically that

$$c_1 \mathbf{v}^T B_{22} \mathbf{v} \;\leq\; \mathbf{v}^T A_H \mathbf{v} \;\leq\; c_2 \mathbf{v}^T B_{22} \mathbf{v} \quad \text{for all } \mathbf{v},$$

  where $c_1 = 0.5,\ c_2 = 0.75$ for isotropic problems, $0.25 \leq c_1$ and $c_2 \leq 1$ for all considered anisotropies.
- In the DA case, Theorem 3.2. shows that $\tilde{A}_{22} = 4A_H$, which enables the recursive extension of the two-level hierarchical construction to the multilevel version straightforwardly.

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