On the role of the strengthened CBS inequality in the theory of algebraic multilevel iteration (AMLI) methods

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Introduction

We are interested in the efficient solution large-scale FEM linear systems

$$A\mathbf{u} = \mathbf{f}.$$

The construction of robust Preconditioned Conjugate Gradient (PCG) solution methods is addressed to some special properties of the stiffness matrix A, among which are that:

- A **is symmetric and positive definite (SPD);**
- A **is large and even very large but sparse.**

Let A be SPSD with SPD diagonal block A_{11}. Then

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & \\ A_{21} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ & I_2 \end{bmatrix},$$

where S states for the Schur complement

$$S = A_{22} - A_{21} A_{11}^{-1} A_{12}.$$

Unfortunately, starting with a sparse matrix A, the same does not hold in general for the Schur complement S.

Lemma 1.1.

Let A be SPSD with A_{11}-SPD, and let $\mathbf{x}_2$ is fixed where $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T]^T$. Then

$$\mathbf{x}_2^T S \mathbf{x}_2 = \min_{\mathbf{x}_1} \ \mathbf{x}^T A \mathbf{x}.$$

Proof.

$$\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= [\mathbf{x_1}^T \ \mathbf{x_2}^T] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\
&= \mathbf{x}_1^T A_{11} \mathbf{x}_1 + \mathbf{x}_1^T A_{12} \mathbf{x}_2 + \mathbf{x}_2^T A_{21} \mathbf{x}_1 + \mathbf{x}_2^T A_{22} \mathbf{x}_2 \\
&\quad + \mathbf{x}_2^T A_{21} A_{11}^{-1} A_{12} \mathbf{x}_2 - \mathbf{x}_2^T A_{21} A_{11}^{-1} A_{12} \mathbf{x}_2 \\
&= \mathbf{x}_2^T S \mathbf{x}_2 + \left(\mathbf{x}_1 + A_{11}^{-1} A_{12} \mathbf{x}_2^T\right)^T A_{11} \left(\mathbf{x}_1 + A_{11}^{-1} A_{12} \mathbf{x}_2^T\right) \\
&\geq \mathbf{x}_2^T S \mathbf{x}_2
\end{aligned}$$

The CBS constant

Let V_1, V_2 be finite dimensional spaces, $W = V_1 \times V_2$, and let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a partitioning consistently with V_1, V_2. Let us assume that A is SPSD and A_{11} is PD, and let $W_1 = \{\mathbf{v} = [\mathbf{v}_1^T, 0^T]^T\}$, $W_2 = \{\mathbf{v} = [0^T, \mathbf{v}_1^T,]^T\}$. Consider the smallest possible constant γ such that

$$|\mathbf{w}_1^T A \mathbf{w}_2| \leq \gamma \{\mathbf{w}_1^T A \mathbf{w}_1 \;\; \mathbf{w}_2^T A \mathbf{w}_2\}^{1/2} \quad \forall \mathbf{w}_1 \in W_1, \; \mathbf{w}_2 \in W_2.$$

- This inequality is referred to as the strengthened CBS inequality.
- The constant γ is associated with the cosine of the angle between W_1 and W_2, with energy inner product defined by A, and is of a crucial importance for analysis of certain preconditioners of A.

The strengthened CBS inequality is equivalent to:

$$|\mathbf{v}_1^T A_{12}\mathbf{v}_2| \le \gamma\{\mathbf{v}_1^T A_{11}\mathbf{v}_1 \ \ \mathbf{v}_2^T A_{22}\mathbf{v}_2\}^{1/2} \ \forall \mathbf{v}_i \in W_i, \ i = 1, 2$$

Lemma 1.2.

(a) $\gamma \le 1$.

(b) $\gamma = 1$ **if there exists** $\mathbf{w} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathcal{N}(A)$ **for which** $\mathbf{v}_2 \notin \mathcal{N}(A_{22})$.

(c) $\gamma < 1$ **if** $\forall \mathbf{w} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathcal{N}(A)$ **it holds that** $\mathbf{v}_2 \in \mathcal{N}(A_{22})$.

(d) **Under the assumption of (c),**

$$\gamma = \sup_{\mathbf{v}_i \in V_i \setminus \mathcal{N}(A_{ii}), \ \ i=1,2} \frac{\mathbf{v}_1{}^t A_{12}\mathbf{v}_2}{\{\mathbf{v}_1{}^T A_{11}\mathbf{v}_1 \ \ \mathbf{v}_2{}^T A_{22}\mathbf{v}_2\}^{1/2}}.$$

[By assumption $\mathcal{N}(A_{11}) = \{0\}$**.]**

Proof.

It is obviously enough to consider $\mathbf{v}_i \neq 0, i = 1, 2$. The following simple relations will be also used:

(i) $\mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 + 2\mathbf{v}_1^T A_{12} \mathbf{v}_2 \geq 0 \quad \forall(\mathbf{v}_1, \mathbf{v}_2)$ **(A is SPSD)**

(ii) $\gamma(\mathbf{v}_1, \mathbf{v}_2) = \gamma(\alpha \mathbf{v}_1, \beta \mathbf{v}_2) \, \forall(\alpha \neq 0, \beta \neq 0)$**, i.e., in analyzing γ, we could assume that the vectors are properly scaled**

(a) **If** $\mathbf{v}_2^T A_{22} \mathbf{v}_2 \neq 0 \Rightarrow^{(ii)} \mathbf{v}_1 : \mathbf{v}_1^T A_{11} \mathbf{v}_1 = \mathbf{v}_1^T A_{22} \mathbf{v}_2 \Rightarrow^{(i)}$
$|\mathbf{v}_1^T A_{12} \mathbf{v}_2| \leq \mathbf{v}_1^T A_{12} \mathbf{v}_1 \Rightarrow \gamma \leq 1$**.**

If $\mathbf{v}_2^T A_{22} \mathbf{v}_2 = 0 \Rightarrow$ **let** $\mathbf{v}_1 = \tau \hat{\mathbf{v}}_1$ **for some** $\hat{\mathbf{v}}_1$ **and some** $\tau > 0 \Rightarrow^{(i)}$
$\tau \hat{\mathbf{v}}_1^T A_{11} \hat{\mathbf{v}}_1 + 2\hat{\mathbf{v}}_1 A_{12} \mathbf{v}_2 \geq 0 \Rightarrow \hat{\mathbf{v}}_1 A_{12} \mathbf{v}_2 \geq 0$**.**
If $A_{12}\mathbf{v}_2 \neq 0 \Rightarrow$ **let** $\hat{\mathbf{v}}_1 = -A_{12}\mathbf{v}_2 \Rightarrow ||A_{12}\mathbf{v}_2|| = 0 \Rightarrow A_{12}\mathbf{v}_2 = 0 \Rightarrow$ **the CBS inequality holds for any γ.**

(b) **If** $A\mathbf{w} = 0$ **and** $\mathbf{v}_2^T(A_{22})\mathbf{v}_2 \neq 0 \Rightarrow^{(ii)} \mathbf{v}_1 : \mathbf{v}_1^T A_{11}\mathbf{v}_1 = \mathbf{v}_1^T A_{22}\mathbf{v}_2 \Rightarrow^{(i)}$ $\mathbf{v}_1^T A_{12}\mathbf{v}_2 = -\mathbf{v}_1^T A_{11}\mathbf{v}_1 \Rightarrow \gamma = 1$

(c) **If** $A\mathbf{w} \neq 0 \Rightarrow \mathbf{v}_1^T A_{11}\mathbf{v}_1 + \mathbf{v}_2^T A_{22}\mathbf{v}_2 + 2\mathbf{v}_1^T A_{12}\mathbf{v}_2 > 0 \Rightarrow \gamma < 1$

If $A\mathbf{w} = 0 \Rightarrow A_{22}\mathbf{v}_2 = 0 \Rightarrow$ **let** $\mathbf{v}_1 = \tau\hat{\mathbf{v}}_1$ **for some** $\hat{\mathbf{v}}_1$ **and some** $\tau > 0 \Rightarrow^{(i)} \tau\hat{\mathbf{v}}_1^T A_{11}\hat{\mathbf{v}}_1 + 2\hat{\mathbf{v}}_1 A_{12}\mathbf{v}_2 = 0 \Rightarrow \hat{\mathbf{v}}_1 A_{12}\mathbf{v}_2 \Rightarrow \hat{\mathbf{v}}_1^T A_{11}\hat{\mathbf{v}}_1 = 0 \Rightarrow A_{11}\mathbf{v}_1 = 0 \Rightarrow \mathbf{v}_1 = 0 \Rightarrow$ **the CBS inequality holds for any** γ**.**

(d) **Under the assumption of (c), the equality**

$$\gamma = \sup_{\mathbf{v}_1 \neq 0, \mathbf{v}_2 \in V_2 \setminus \mathcal{N}(A_{22})} \frac{\mathbf{v}_1{}^T A_{12}\mathbf{v}_2}{\{\mathbf{v}_1{}^T A_{11}\mathbf{v}_1 \;\; \mathbf{v}_2{}^T A_{22}\mathbf{v}_2\}^{1/2}}.$$

follows directly from the CBS inequality.

Lemma 1.3.

Let A be SPSD, with A_{11} - PD, such that $A\mathbf{w}=0 \Rightarrow A_{22}\mathbf{v}_2 = 0$. Then

(a) $\gamma^2 = \sup_{\mathbf{v}_2\in V_2\backslash\mathcal{N}(A_{22})} \dfrac{\mathbf{v_2}^T A_{21}A_{11}^{-1}A_{12}\mathbf{v}_2}{\mathbf{v}_2^T A_{22}\mathbf{v}_2}.$

(b) $1-\gamma^2 \le \dfrac{\mathbf{v_2}^T S\mathbf{v}_2}{\mathbf{v}_2^T A_{22}\mathbf{v}_2} \le 1 \;\; \forall \mathbf{v}_2$**, where the left-hand side inequality is sharp and the right-hand side is sharp if $\mathcal{N}(A_{12})\neq\{0\}$.**

Proof.

$$\gamma = \sup_{\mathbf{v}_i\in V_i\backslash\mathcal{N}(A_{ii})} \frac{\mathbf{v}_1^T A_{12}\mathbf{v}_2}{\{\mathbf{v}_1^T A_{11}\mathbf{v}_1 \; \mathbf{v}_2^T A_{22}\mathbf{v}_2\}^{1/2}} = \sup_{\mathbf{v}_i\in V_i\backslash\mathcal{N}(A_{ii})} \frac{\mathbf{v}_1^T A_{11}^{-1/2}A_{12}\mathbf{v}_2}{\{\mathbf{v}_1^T\mathbf{v}_1 \; \mathbf{v}_2^T A_{22}\mathbf{v}_2\}^{1/2}},$$

where the supremum is taken for $\mathbf{v}_1 = A_{11}^{-1/2}A_{12}\mathbf{v}_2$, so

$$\gamma^2 = \sup_{\mathbf{v}_2\in V_2\backslash\mathcal{N}(A_{22})} \frac{\mathbf{v}_2^T A_{21}A_{11}^{-1}A_{12}\mathbf{v}_2}{\mathbf{v}_2^T A_{22}\mathbf{v}_2}.$$

This proves (a), and (b) follows from the definition of the Schur complement.

Algebraic two-level methods

A lot of the recent multilevel methods can be viewed as a recursive generalization of the additive (C_A) and multiplicative (C_M) two-level preconditioners:

$$C_A = \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix}, \quad C_M = \begin{bmatrix} A_{11} & \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ & I \end{bmatrix}.$$

Theorem 1.1. **(additive preconditioning)**
Let A be SPSD with A_{11} - PD. Then, for any $\mathbf{w} = [\mathbf{v}_1^T, \mathbf{v}_2^T]^T$

$$(1-\gamma)\left(\mathbf{v}_1^T A_{11}\mathbf{v}_1 + \mathbf{v}_2^T A_{22}\mathbf{v}_2\right) \leq \mathbf{w}^T A\mathbf{w} \leq (1+\gamma)\left(\mathbf{v}_1^T A_{11}\mathbf{v}_1 + \mathbf{v}_2^T A_{22}\mathbf{v}_2\right)$$

and therefore

$$(1-\gamma)\mathbf{w}^T C_A \mathbf{w} \leq \mathbf{w}^T A\mathbf{w} \leq (1+\gamma)\mathbf{w}^T C_A \mathbf{w}.$$

Proof.

Combining the CBS inequality with $2ab \leq a^2 + b^2$ we obtain

$$\begin{aligned}
\mathbf{w}^T A \mathbf{w} &= \mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 + \mathbf{v}_2^T A_{21} \mathbf{v}_1 + \mathbf{v}_1^T A_{12} \mathbf{v}_2 \\
&\leq \mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 + 2\gamma \sqrt{\mathbf{v}_1^T A_{11} \mathbf{v}_1} \sqrt{\mathbf{v}_2^T A_{22} \mathbf{v}_2} \\
&\leq (1+\gamma) \left(\mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 \right) .
\end{aligned}$$

We apply in a similar way $-2ab \geq -a^2 - b^2$ and get

$$\begin{aligned}
\mathbf{w}^T A \mathbf{w} &= \mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 + \mathbf{v}_2^T A_{21} \mathbf{v}_1 + \mathbf{v}_1^T A_{12} \mathbf{v}_2 \\
&\geq \mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 - 2\gamma \sqrt{\mathbf{v}_1^T A_{11} \mathbf{v}_1} \sqrt{\mathbf{v}_2^T A_{22} \mathbf{v}_2} \\
&\geq (1-\gamma) \left(\mathbf{v}_1^T A_{11} \mathbf{v}_1 + \mathbf{v}_2^T A_{22} \mathbf{v}_2 \right) ,
\end{aligned}$$

which completes the proof.

Theorem 1.2. **(multiplicative preconditioning)**

Let A be SPSD with A_{11} - PD. Then, for any $\mathbf{w} = [\mathbf{v}_1^T, \mathbf{v}_2^T]^T$

$$(1-\gamma^2)\mathbf{w}^t C_M \mathbf{w} \leq \mathbf{w}^t A \mathbf{w} \leq \mathbf{w}^t C_M \mathbf{w}$$

Proof.

The estimates follow directly if Lemma 1.3. (b) is applied to the factorized matrices A and C_M, namely

$$A = \begin{bmatrix} I_1 & \\ A_{21}A_{11}^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A_{11} & \\ & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ & I_2 \end{bmatrix},$$

$$C_M = \begin{bmatrix} I_1 & \\ A_{21}A_{11}^{-1} & I_2 \end{bmatrix} \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ & I_2 \end{bmatrix}.$$

In a detailed form we have

$$\begin{aligned}
\mathbf{w}^T A \mathbf{w} &= \hat{\mathbf{w}}^T \begin{bmatrix} A_{11} & \\ & S \end{bmatrix} \hat{\mathbf{w}} = \hat{\mathbf{v}}_1^T A_{11} \hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2^T S \hat{\mathbf{v}}_2 \\
&\geq \hat{\mathbf{v}}_1^T A_{11} \hat{\mathbf{v}}_1 + (1-\gamma^2) \hat{\mathbf{v}}_2^T A_{22} \hat{\mathbf{v}}_2 \\
&\geq (1-\gamma^2) \hat{\mathbf{w}}^T \begin{bmatrix} A_{11} & \\ & A_{22} \end{bmatrix} \hat{\mathbf{w}} = (1-\gamma^2) \mathbf{w}^T C_M \mathbf{w},
\end{aligned}$$

where $\mathbf{w} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}$ **and** $\hat{\mathbf{w}} = \begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} I_1 & A_{11}^{-1} A_{12} \\ & I_2 \end{bmatrix} \mathbf{w}$**.**

The right-hand side of the statement follows in a completely similar way which completes the proof.

Corollary 1.1.

Let A be SP. Then,

$$\kappa\left(C_A^{-1}A\right) \leq \frac{1+\gamma}{1-\gamma}$$

$$\kappa\left(C_M^{-1}A\right) \leq \frac{1}{1-\gamma^2}$$

The derived estimates clearly state the importance of such splittings where the CBS constant γ is uniformly bounded.

Hierarchical two-level splitting

We consider the elliptic problem associated with the bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\Omega, \quad u,v \in H_0^1(\Omega),$$

where the coefficient matrix $[a_{ij}]$ is SPD.

FEM $\Rightarrow$ find $u_h \in V \subset H_0^1(\Omega)$:

$$a(u_h, v_h) = (f, v_h) \quad \forall \; v_h \in V \subset H_0^1(\Omega)$$

Let $\mathcal{T}_1$ be an initial triangulation describing the geometry of Ω, and let the coefficients $[a_{ij}]$ be piece-wise constants with respect to the triangles from $\mathcal{T}_1$, i.e.,

$$a(u_h, v_h) = \sum_{e \in T} a_e(u_h, v_h) = \sum_{e \in T} \int_e \sum_{i,j=1}^{2} a_{ij}(e) \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} de.$$

Let us consider the sequence of nested triangulations $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \ldots \subset \mathcal{T}_\ell$, corresponding FE spaces $V_1 \subset V_2 \subset \ldots V_\ell$ and related stiffness matrices $A^{(1)}, A^{(2)}, \ldots, A^{(\ell)}$. The goal is to solve the finest discretization FEM system

$$A^{(\ell)}\mathbf{u}^{(\ell)} = \mathbf{f}^{(\ell)}.$$

Consider the 2×2 block presentation of $A^{(k+1)}$ corresponding to the splitting of the nodes $\mathcal{N}^{(k+1)}$ from $\mathcal{T}_{k+1}$ into the subsets $\mathcal{N}^{(k+1)} \setminus \mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$.

$$A^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix}$$

Let us denote by $\Phi^{(k+1)} = \{\phi_i^{(k+1)}\}_{i=1}^{N^{(k+1)}}$ the standard Lagrangian nodal basis of the P1 conforming FE space V_{k+1}.

Then, the hierarchical two-level basis is defined as follows:

$$\tilde{\Phi}^{(k+1)} = \{\tilde{\phi}_i^{(k+1)}\}_{i=1}^{N^{(k+1)}} = \left\{ \begin{array}{c} \phi_i^{(k+1)} \\ \phi_i^{(k)} \end{array} \right\} \quad \begin{array}{l} - \; i \in \mathcal{N}^{(k+1)} \setminus \mathcal{N}^{(k)} \\ - \; i \in \mathcal{N}^{(k)} \end{array}$$

The hierarchical basis is simply computable by

$$\tilde{\Phi}^{(k+1)} = J^{(k+1)} \Phi^{(k+1)}, \qquad J^{(k+1)} = \begin{bmatrix} I & \\ J_{21}^{(k+1)} & I \end{bmatrix}$$

where $J_{21}^{(k+1)}$ is very sparse.

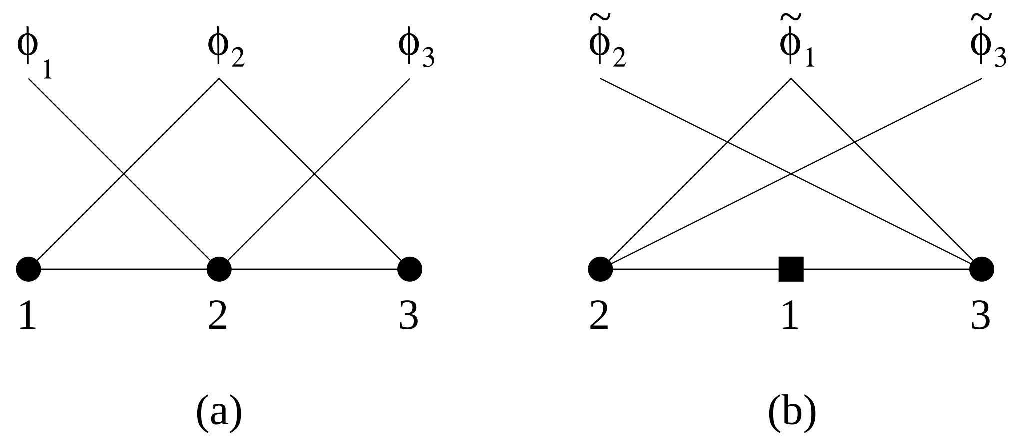


Figure 1: (a) standard nodal basis, and (b) hierarchical basis

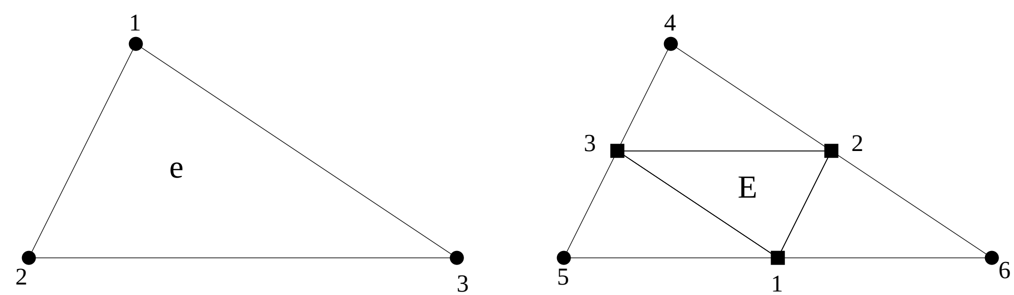


Figure 2: Uniform mesh refinement

Theorem 1.3.

Consider the 2×2 block presentation of the stiffness matrix $\tilde{A}^{(k+1)}$ computed using the hierarchical basis:

$$\tilde{A}^{(k+1)} = \begin{bmatrix} \tilde{A}_{11}^{(k+1)} & \tilde{A}_{12}^{(k+1)} \\ \tilde{A}_{21}^{(k+1)} & \tilde{A}_{22}^{(k+1)} \end{bmatrix}.$$

Then,

(a) $\tilde{A}_{11}^{(k+1)} = A_{11}^{(k+1)}, \qquad \tilde{A}_{22}^{(k+1)} = A^{(k)};$

(b) $\tilde{S}^{(k+1)} = S^{(k+1)}.$

Proof.

The equalities (a) follow straightforwardly from the definitions. To prove (b) we use the transformation formula $\tilde{A}^{(k+1)} = J^{(k+1)} A^{(k+1)} {J^{(k+1)}}^T$, i.e.,

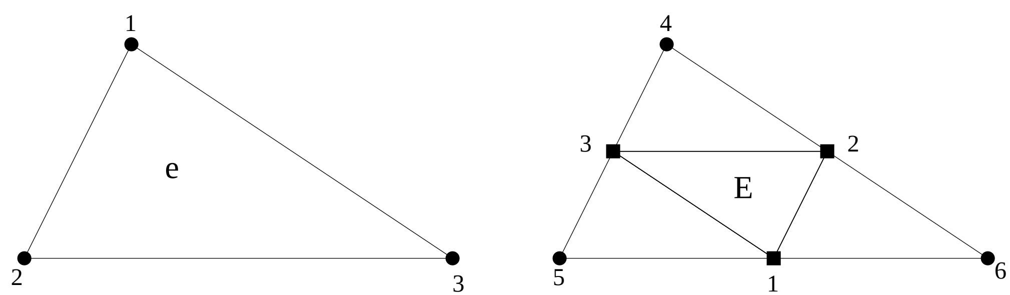
$$\tilde{A}^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{11}^{(k+1)} J_{12}^{(k+1)} + A_{12}^{(k+1)} \\ J_{21}^{(k+1)} A_{11}^{(k+1)} + A_{21}^{(k+1)} & \begin{array}{l} J_{21}^{(k+1)} A_{11}^{(k+1)} J_{12}^{(k+1)} + J_{21}^{(k+1)} A_{12}^{(k+1)} \\ + A_{21}^{(k+1)} J_{12}^{(k+1)} + A_{22}^{(k+1)} \end{array} \end{bmatrix}$$

and finally get

$$\begin{aligned} \tilde{S}^{(k+1)} &= J_{21}^{(k+1)} A_{11}^{(k+1)} J_{12}^{(k+1)} + J_{21}^{(k+1)} A_{12}^{(k+1)} + A_{21}^{(k+1)} J_{12}^{(k+1)} + A_{22}^{(k+1)} \\ &\quad -(J_{21}^{(k+1)} A_{11}^{(k+1)} + A_{21}^{(k+1)}) {A_{11}^{(k+1)}}^{-1} (A_{11}^{(k+1)} J_{12}^{(k+1)} + A_{12}^{(k+1)}) \\ &= A_{22}^{(k+1)} - A_{21}^{(k+1)} {A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)} \\ &= S^{(k+1)}. \end{aligned}$$

We can assemble the global stiffness matrices $A^{(k+1)}$ and $\tilde{A}^{(k+1)}$ by related macroelement matrices

$$A_E^{(k+1)} = \begin{bmatrix} A_{11;E}^{(k+1)} & A_{12;E}^{(k+1)} \\ A_{21;E}^{(k+1)} & A_{22;E}^{(k+1)} \end{bmatrix}, \quad \tilde{A}_E^{(k+1)} = \begin{bmatrix} A_{11;E}^{(k+1)} & \tilde{A}_{12;E}^{(k+1)} \\ \tilde{A}_{21;E}^{(k+1)} & A_e^{(k)} \end{bmatrix}.$$



Here $E \in \mathcal{T}_{k+1}$, $e \in \mathcal{T}_k$, and the 3×3 blocks $A_{ij;E}^{(k+1)}$, $\tilde{A}_{ij;E}^{(k+1)}$ enable for efficient local analysis.

Local estimate of γ

Let the spaces W_1 and W_2 correspond to the block 2×2 hierarchical basis stiffness matrix, and let W_1^h and W_2^h be the spaces of the related FE functions. The stiffness matrix is SPD and the related CBS constant reads as

$$\gamma^2 = \sup_{v_i^h \in W_i^h,\ \ i=1,2} \frac{[a(v_1^h, v_2^h)]^2}{a(v_1^h, v_1^h)\, a(v_2^h, v_2^h)}.$$

We introduce the macroelement CBS constant

$$\gamma_E^2 = \sup_{v_i^h \in W_i^h,\ v_2^h{}_{|E} \neq const} \frac{[a_E(v_1^h, v_2^h)]^2}{a_E(v_1^h, v_1^h)\, a_E(v_2^h, v_2^h)},$$

where $a_E(u, v) = \int_E \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dE$.

Note that, if $\begin{bmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{bmatrix} \in \mathcal{N}(\tilde{A}_E^{(k+1)}) \Rightarrow \mathbf{v_2} \in \mathcal{N}(\tilde{A}_{E;22}^{(k+1)}) = \mathcal{N}(A_E^{(k)}) = const$.

Theorem 1.4.

The following estimate holds for the CBS constant of the hierarchical 2×2 block presentation of the stiffness matrix computed with respect to the hierarchical basis:

$$\gamma \le \max_{E \in \mathcal{T}_{k+1}} \gamma_E .$$

Proof.

$$|a(v_1, v_2)| \le \sum_{E \in \mathcal{T}_{k+1}} |a_E(v_1, v_2)| \le \sum_{E \in \mathcal{T}_{k+1}} \gamma_E \sqrt{a_E(v_1, v_1)} \sqrt{a_E(v_2, v_2)}$$

$$\le \max_{E \in \mathcal{T}_{k+1}} \gamma_E \sum_{E \in \mathcal{T}_{k+1}} \sqrt{a_E(v_1, v_1)} \sqrt{a_E(v_2, v_2)}$$

$$\le \max_{E \in \mathcal{T}_{k+1}} \gamma_E \sqrt{\sum_{E \in \mathcal{T}_{k+1}} a_E(v_1, v_1)} \sqrt{\sum_{E \in \mathcal{T}_{k+1}} a_E(v_2, v_2)}$$

$$= \max_{E \in \mathcal{T}_{k+1}} \gamma_E \sqrt{a(v_1, v_1)} \sqrt{a(v_2, v_2)}$$

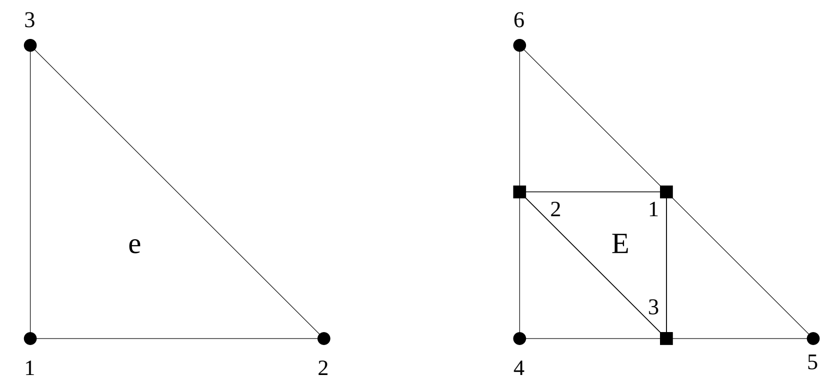
It is important to note, that the local CBS constants γ_E do not depend on the refinement level, i.e. we have got an estimate for γ which is uniform with respect to the problem size N.

The local analysis is based on the fact that $(1-\gamma_E^2)$ is the maximal constant satisfying the inequality

$$1-\gamma_E^2 \leq \frac{\mathbf{v_2^t}\tilde{S}_E\mathbf{v_2}}{\mathbf{v_2}^t A_e \mathbf{v_2}},$$

i.e., $(1-\gamma_E^2)$ is the minimal eigenvalue of the generalized eigenproblem

$$\tilde{S}_E\mathbf{v_2} = \mu A_e \mathbf{v_2}, \qquad \mathbf{v_2^T} \neq (1,1,1).$$



Model problem on uniform rectangle mesh

Example

$$a(u,v) = \int_\Omega a(e)\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) d\Omega.$$

$$A_e = \frac{a(e)}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A_E = \frac{a(e)}{2}\left[\begin{array}{rrr|rrr} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & -1 \\ -2 & 0 & 4 & -1 & -1 & 0 \\ \hline 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left|\begin{array}{cc} 5-8\mu & -1 \\ -1 & 5-8\mu \end{array}\right| = 0$$

$$1-\gamma_E^2 = \mu_1 = 1/2 \Rightarrow \gamma_E^2 = 1/2 \Rightarrow \gamma^2 \leq \frac{1}{2}$$

Remark

We will show later, that in a very general setting about $\mathcal{T}_1$ and $[a_{ij}]$

$$\gamma^2 \leq \frac{3}{4}.$$

AMLI methods

Let us consider the recursive multilevel generalization of the multiplicative two-level method introduced originally by Axelsson and Vassilevski.

$C^{(1)} = A^{(1)}$;

for $k = 1, 2, \ldots, \ell - 1$

$$C^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & 0 \\ A_{21}^{(k+1)} & \tilde{A}^{(k)} \end{bmatrix} \begin{bmatrix} I & {A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)} \\ 0 & I \end{bmatrix},$$

where the Schur complement approximation is stabilized by

$$ {\tilde{A}^{(k)}}^{-1} = \left[I - p_\beta \left({C^{(k)}}^{-1} A^{(k)} \right) \right] {A^{(k)}}^{-1}.$$

The acceleration polynomial is explicitly defined by

$$p_\beta(t) = \frac{1 + T_\beta\left(\dfrac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\beta\left(\dfrac{1+\alpha}{1-\alpha}\right)},$$

were $\alpha \in (0,1)$ is a properly chosen parameter, and T_β stands for the Chebyshev polynomial of degree β.

Theorem 1.5.

There exists $\alpha \in (0,1)$, such that the AMLI preconditioner $C = C^{(\ell)}$ has optimal condition number $\kappa\left(C^{-1}A\right) = O(1)$, and the total computational complexity is $O(N)$, if β satisfies the condition

$$4 > \beta > \frac{1}{\sqrt{1-\gamma^2}}.$$

For the model problem:

- $\gamma^2 < 3/4 \Rightarrow \beta \in \{2, 3\}$
- $\kappa(A_{11}^{(k+1)}) = O(1) \Rightarrow \mathcal{N}_{AMLI} = O(N)$

Construction of Robust Algebraic Multilevel Preconditioners:

- **Uniform estimates of the CBS constant with respect to anisotropy and/or possible small parameters.**
- **Optimal order preconditioning (approximation) of the first pivot block $A_{11}^{(k+1)}$ with respect to anisotropy and/or possible small parameters.**