Algebraic multilevel preconditioning using local Schur complements

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A class of two-level preconditioners:

Let A be an SPD matrix. Replace the exact block factorization

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & \\ A_{21}{A_{11}}^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{12} \\ & S \end{pmatrix}$$

with approximate factorization

$$B = \begin{pmatrix} I & \\ A_{21}P^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} P & A_{12} \\ & Q \end{pmatrix} = \begin{pmatrix} P & A_{12} \\ A_{21} & Q + A_{21}P^{-1}A_{12} \end{pmatrix}$$

i.e., approximate pivot matrix $A_{11} \approx P$ and Schur complement $S \approx Q$

Don't use hierarchical (two-level) basis representation of A, i.e., don't modify off-diagonal blocks A_{12} and A_{21}!

Theorem 1 ([Notay 98]) Let A and B, be symmetric nonnegative definite matrices such that A_{11} and P are invertible. Moreover, let

$$\underline{\alpha}\mathbf{v}_1^T A_{11}\mathbf{v}_1 \le \mathbf{v}_1^T P\mathbf{v}_1 \le \overline{\alpha}\mathbf{v}_1^T A_{11}\mathbf{v}_1 \quad \forall \mathbf{v}_1,$$

$$\underline{\beta}\mathbf{v}_2^T S\mathbf{v}_2 \le \mathbf{v}_2^T Q\mathbf{v}_2 \le \overline{\beta}\mathbf{v}_2^T S\mathbf{v}_2 \quad \forall \mathbf{v}_2, \qquad \text{and}$$

$$\overline{\alpha}\mathbf{v}_2^T A_{21}P^{-1}A_{12}\mathbf{v}_2 \le (1-\xi)\mathbf{v}_2^T A_{22}\mathbf{v}_2 + \xi\mathbf{v}_2^T A_{21}A_{11}^{-1}A_{12}\mathbf{v}_2 \quad \forall \mathbf{v}_2 \qquad \text{(A1)}$$

for some $\xi \le 1$. Then

$$\underline{\gamma}\mathbf{v}^T A\mathbf{v} \le \mathbf{v}^T B\mathbf{v} \le \overline{\gamma}\mathbf{v}^T A\mathbf{v} \quad \forall \mathbf{v}$$

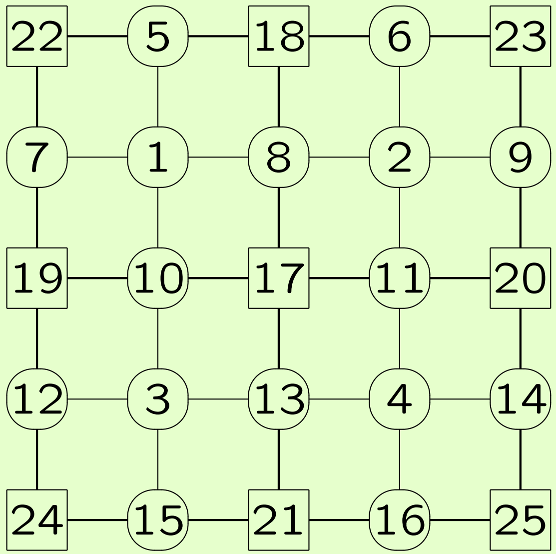
where $\underline{\gamma}$ is the smallest root of

$$\gamma^2 - \gamma(\underline{\beta} + \overline{\alpha} - \xi(\overline{\alpha} - \underline{\alpha})) + \underline{\alpha}\underline{\beta}$$

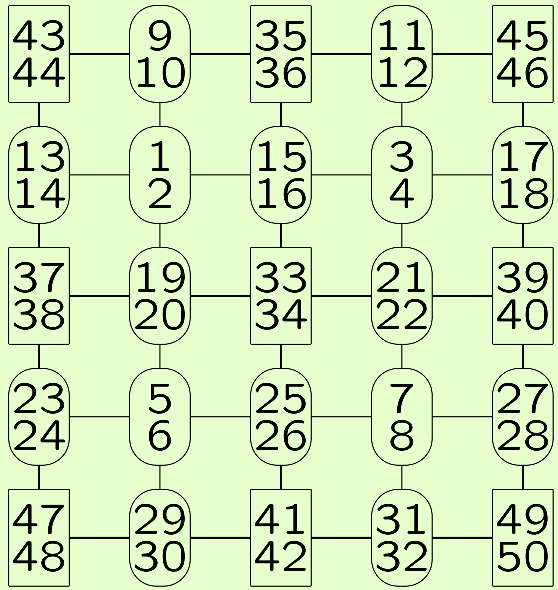
and $\overline{\gamma}$ is the largest root of

$$\gamma^2 - \gamma(\overline{\beta} + \overline{\alpha} - \xi(\overline{\alpha} - \underline{\alpha})) + \underline{\alpha}\overline{\beta}$$

Partitioning the degrees of freedom:



(a) one scalar PDE



(b) system of two PDEs

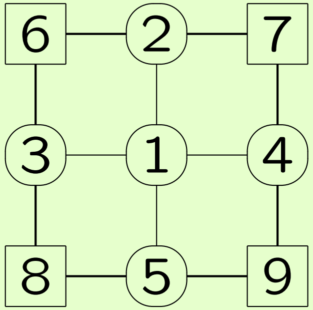
This yields the structure for the multilevel block-factorization:

$$B^{(k)} = \begin{pmatrix} I^{(k)} & \\ A_{21}^{(k)}(P^{(k)})^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} P^{(k)} & A_{12}^{(k)} \\ & Q^{(k)} \end{pmatrix},$$

where $A^{(k+1)} := Q^{(k)}$ for $0 \leq k < l$.

Schur complement approximation:

Local labeling within each agglomerate a (on the fine grid)



yields the following block form of the agglomerate matrices A_a:

$$A_a = \begin{pmatrix} A_{a,11} & A_{a,12} \\ A_{a,21} & A_{a,22} \end{pmatrix}$$

Local Schur complements serve as element matrices on level $k+1$:

$$S_a = A_{a,22} - A_{a,21}(A_{a,11})^{-1}A_{a,12} \quad \forall a$$

$$Q^{(k)} := \sum_e A_e^{(k+1)} := \sum_a S_a^{(k)} \quad 0 \le k < l.$$

Incomplete factorization of the pivot matrix:

Instead of using an incomplete factorization of the (global) pivot matrix $A_{11} = \sum_a A_{a,11}$, we compute the complete LU factorization of all local pivot blocks

$$A_{a,11} = L_a U_a \quad \forall a,$$

where $\mathrm{diag}(L_a) = I_a$ (assuming that $A_{a,11}$ is non-singular $\forall a$).

Then

$$U := \sum_a U_a$$

yields an approximate upper triangular factor of A_{11} and a preconditioner P can be defined by

$$P := LU, \quad U := \sum_a U_a, \quad L := U^T(\mathrm{diag}(U))^{-1}.$$

Local analysis using macro elements:

According to Theorem 1 we have to find the constants

$$\begin{aligned} 0 < \; & \underline{\alpha}, \underline{\beta} \; \leq 1, \\ 1 \leq \; & \overline{\alpha}, \overline{\beta} \; < \infty, \end{aligned}$$

in the relations

$$\underline{\alpha}\mathbf{v}_1^T A_{11}\mathbf{v}_1 \leq \mathbf{v}_1^T P\mathbf{v}_1 \leq \overline{\alpha}\mathbf{v}_1^T A_{11}\mathbf{v}_1 \quad \forall \mathbf{v}_1, \tag{1}$$

$$\underline{\beta}\mathbf{v}_2^T S\mathbf{v}_2 \leq \mathbf{v}_2^T Q\mathbf{v}_2 \leq \overline{\beta}\mathbf{v}_2^T S\mathbf{v}_2 \quad \forall \mathbf{v}_2. \tag{2}$$

Remark 2 *If the preconditioner P arises from incomplete factorization of A_{11} satisfying a row-sum criterion then the assumption (A1) can be verified under quite general conditions (see, e.g., [Notay 98]).*

Let us now prove that the right-hand side inequalities in (1) and (2) hold true for $\overline{\alpha} = \overline{\beta} = 1$.

Lemma 3 *For* $i = 1, 2, \ldots, N$ *let* X_i *be real* $n \times k$ *and* Y_i *real* $n \times m$ *matrices. Then, if the* $m \times m$ *matrix* $Z_{11} := \sum_{i=1}^N Y_i^T Y_i$ *is invertible the following inequality holds:*

$$\sum_{i=1}^N X_i^T X_i - \left(\sum_{i=1}^N X_i^T Y_i\right)\left(\sum_{i=1}^N Y_i^T Y_i\right)^{-1}\left(\sum_{i=1}^N Y_i^T X_i\right) \geq 0$$

Proof.

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}^T \begin{pmatrix} Y_i^T Y_i & Y_i^T X_i \\ X_i^T Y_i & X_i^T X_i \end{pmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} &= \langle (Y_i \mathbf{v}_1 + X_i \mathbf{v}_2), (Y_i \mathbf{v}_1 + X_i \mathbf{v}_2) \rangle \\ &= \| Y_i \mathbf{v}_1 + X_i \mathbf{v}_2 \|^2 \geq 0 \quad \forall \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \quad \forall i. \end{aligned}$$

Hence,

$$Z := \begin{pmatrix} \sum_i Y_i^T Y_i & \sum_i Y_i^T X_i \\ \sum_i X_i^T Y_i & \sum_i X_i^T X_i \end{pmatrix} \geq 0, \tag{3}$$

and since $Z_{11} = \sum_{i=1}^N Y_i^T Y_i$ is SPD, the last inequality (3) holds if and only if

$$S_Z := \sum_{i=1}^N X_i^T X_i - \left(\sum_{i=1}^N X_i^T Y_i\right)\left(\sum_{i=1}^N Y_i^T Y_i\right)^{-1}\left(\sum_{i=1}^N Y_i^T X_i\right) \geq 0.$$

□

Now, by choosing

$$X_a := (\text{diag}(U_a))^{-1/2} U_a \quad \text{and} \quad Y_a := (\text{diag}(U_a))^{1/2}$$

it follows from Lemma 3 that

$$\mathbf{v}_1^T P \mathbf{v}_1 \leq \mathbf{v}_1^T A_{11} \mathbf{v}_1 \quad \forall \mathbf{v}_1,$$

i.e., the right-hand side inequality in (1) holds for $\overline{\alpha} = 1$:

$$\begin{aligned} A_{11} - P &= \sum_a A_{a,11} - \left(\sum_a U_a^T\right) (\text{diag}(U))^{-1} \left(\sum_a U_a\right) \\ &= \sum_a U_a^T \text{diag}(U_a)^{-1} U_a - \left(\sum_a U_a^T\right) \left(\sum_a \text{diag}(U_a)\right)^{-1} \left(\sum_a U_a\right) \\ &= \sum_a X_a^T X_a - \left(\sum_a X_a^T Y_a\right) \left(\sum_a Y_a^T Y_a\right)^{-1} \left(\sum_a Y_a^T X_a\right) \geq 0 \end{aligned}$$

Similarly, by choosing $X_a := A_{a,11}^{-1/2} A_{a,12}$ and $Y_a := A_{a,11}^{1/2}$ we find that the right-hand side inequality in (2) holds for $\overline{\beta} = 1$.

Let $\mathcal{N}_a$ denote the set of adjacent macro elements of a (sharing at least one node with a) and $|\mathcal{N}_a|$ its cardinality. Let the (local) macro-element matrix $B_{a,11}$ be defined by $B_{a,11} := U_a^T D_a^{-1} U_a$ where $D_a := D|_a := \mathrm{diag}(U)|_a$ and let

$$\hat{\lambda} := \lambda_{\max}\left((B_{a,11})^{-1} A_{a,11}\right).$$

Assume that the relations below hold for some constant $c \geq 1$:

$$(c-1) \cdot \left(\frac{1}{|\mathcal{N}_a|} U_a^T D_a^{-1} U_a + \frac{1}{|\mathcal{N}_{a'}|} U_{a'}^T D_{a'}^{-1} U_{a'}\right)$$
$$+ c \cdot \left(U_a^T D_{(a \cup a')}^{-1} U_{a'} + U_{a'}^T D_{(a \cup a')}^{-1} U_a\right) \;\geq\; 0 \quad \forall a \; \forall a' \in \mathcal{N}_a.$$

Then, by choosing $1/\underline{\alpha} := c \cdot \hat{\lambda}$ we get the desired condition number bound:

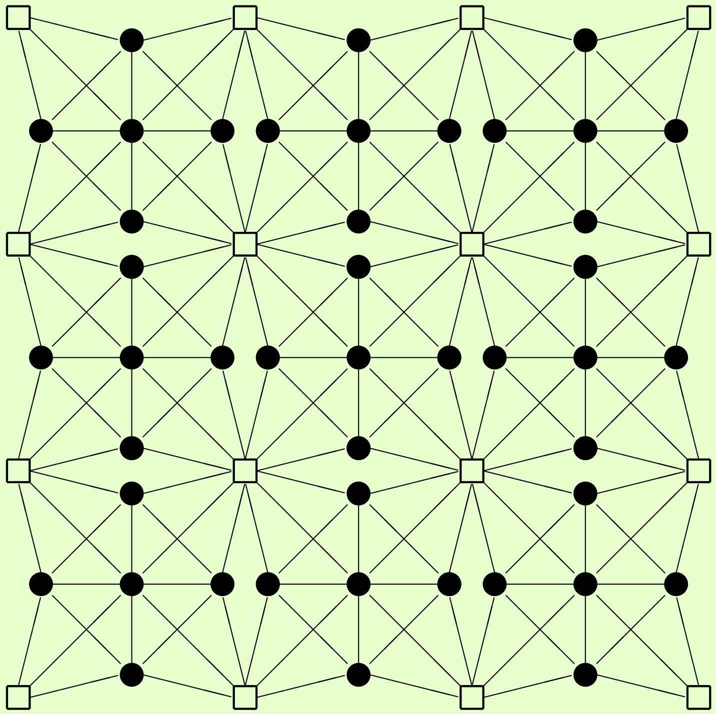
$$
\begin{aligned}
\frac{1}{\underline{\alpha}} \cdot \mathbf{v}_1^T P \mathbf{v}_1 &= c \cdot \hat{\lambda} \cdot \mathbf{v}_1^T \left(\sum_a U_a^T D^{-1} \sum_a U_a \right) \mathbf{v}_1 \\
&\geq \hat{\lambda} \cdot \mathbf{v}_1^T \left(\sum_a U_a^T D_a^{-1} U_a \right) \mathbf{v}_1 \\
&= \hat{\lambda} \cdot \mathbf{v}_1^T \left(\sum_a B_{a,11} \right) \mathbf{v}_1 \\
&\geq \mathbf{v}_1^T \left(\sum_a A_{a,11} \right) \mathbf{v}_1 = \mathbf{v}_1^T A_{11} \mathbf{v}_1 \quad \forall \mathbf{v}_1 .
\end{aligned}
$$

In many cases (e.g., for the scalar model problems considered later) the condition number of $P^{-1}A_{11}$ can be estimated by:

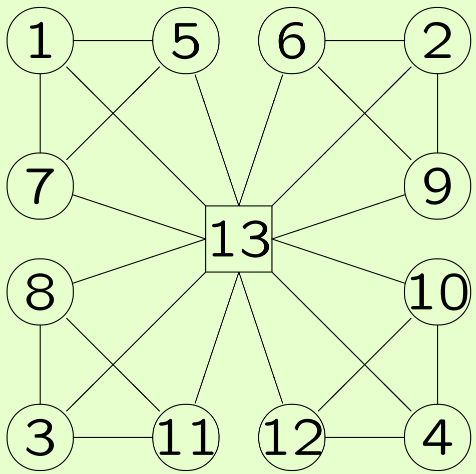
$$
\kappa(P^{-1}A_{11}) \lessapprox \hat{\lambda} := \lambda_{\text{max}} \left((B_{a,11})^{-1} A_{a,11} \right)
$$

Let us now focus our attention on the Schur complement approximation Q.

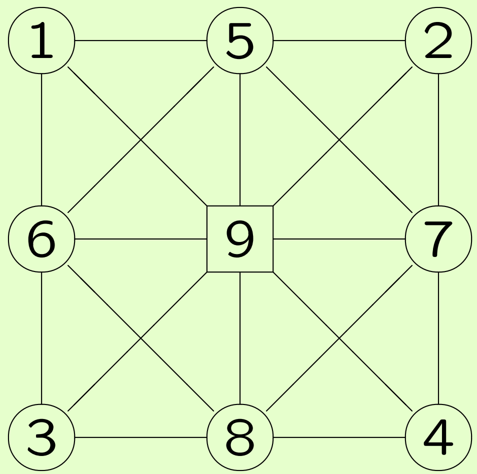
In order to compute $\underline{\beta}$ locally (based on analyzing certain macro elements) we tear the mesh:



Macro-element used for conditioning analysis of Q:



(a) $\widehat{g}$ on torn open mesh



(b) g on original mesh

$$B_{\widehat{g}} = R_{\widehat{g}}^T A_g R_{\widehat{g}}, \qquad R_{\widehat{g}} = \begin{pmatrix} I & & \\ & T & \\ & & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Theorem 4 *Let the (local) macro-element matrix $B_{\widehat{g}}$ be defined by $B_{\widehat{g}} := R_{\widehat{g}}^T A_g R_{\widehat{g}}$ where $R_{\widehat{g}}$ is a (constant) submatrix of a global transformation matrix R associated with an arbitrary macro element $\widehat{g}$, i.e., $R_{\widehat{g}}$ fulfills the compatibility condition*

$$\left(R_{\widehat{g}_1}\mathbf{v}_{\widehat{g}_1}\right)(d) = \left(R_{\widehat{g}_2}\mathbf{v}_{\widehat{g}_2}\right)(d) \quad \forall \mathbf{v},\ \forall d \in g_1 \cap g_2,\ \forall g_1, g_2.$$

Further let

$$\underline{\beta}\mathbf{v}_{\widehat{g}}^T B_{\widehat{g}} \mathbf{v}_{\widehat{g}} \leq \mathbf{v}_{\widehat{g}}^T A_{\widehat{g}} \mathbf{v}_{\widehat{g}} \quad \forall \mathbf{v}_{\widehat{g}}.$$

Then the following bound on the condition number of $Q^{-1}S$ holds:

$$\kappa(Q^{-1}S) \leq 1/\underline{\beta}.$$

Remark 5 *The specific choice of $R_{\widehat{g}}$ on the last slide is a simple example of a local transformation matrix satifying the compatibility assumption. However, other choices are possible and may improve the condition number bound, e.g., for (strongly) anisotropic problems.*

Proof.

$$
\begin{aligned}
\mathbf{v}_2^T Q \mathbf{v}_2 &= \min_{\widehat{\mathbf{v}}_1} \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ \mathbf{v}_2 \end{bmatrix}^T \widehat{A} \begin{bmatrix} \widehat{\mathbf{v}}_1 \\ \mathbf{v}_2 \end{bmatrix} \\
&= \min_{\widehat{\mathbf{v}}_1} \sum_{\widehat{g}} \left(\begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix}^T A_{\widehat{g}} \begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix} \right) \\
&\geq \underline{\beta} \cdot \min_{\widehat{\mathbf{v}}_1} \sum_{\widehat{g}} \left(\begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix}^T B_{\widehat{g}} \begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix} \right) \\
&= \underline{\beta} \cdot \min_{\widehat{\mathbf{v}}_1} \sum_{\widehat{g}} \left(\begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix}^T R_{\widehat{g}}^T A_g R_{\widehat{g}} \begin{bmatrix} \widehat{\mathbf{v}}_{\widehat{g},1} \\ \mathbf{v}_{\widehat{g},2} \end{bmatrix} \right) \\
&\geq \underline{\beta} \cdot \min_{\mathbf{v}_1} \sum_{g} \left(\begin{bmatrix} \mathbf{v}_{g,1} \\ \mathbf{v}_{g,2} \end{bmatrix}^T A_g \begin{bmatrix} \mathbf{v}_{g,1} \\ \mathbf{v}_{g,2} \end{bmatrix} \right) = \underline{\beta} \cdot \mathbf{v}_2^T S \mathbf{v}_2 \quad \forall \mathbf{v}_2
\end{aligned}
$$

The last inequality requires the compatibility assumption. □

Numerical results:

Model problems:

$$-\nabla\cdot[C\nabla u] = f \quad \text{in} \quad \Omega = [0,1]\times[0,1]$$
$$u = g \quad \text{on} \quad \Gamma_D \subset \partial\Omega$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_N = \partial\Omega \setminus \Gamma_D$$

Problem 1

$$C = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix} \textit{in } \Omega, \quad \Gamma_D = \partial\Omega,\ 0 \le \delta < 1$$

Problem 2

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \textit{in } \Omega, \quad \Gamma_D = \partial\Omega,\ 0 < \varepsilon \le 1$$

Model problems:

Problem 3

$$C = \begin{cases} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} & \textit{for all } (x,y) \in (0,1) \times (0,\frac{1}{2}) \\ \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} & \textit{for all } (x,y) \in (0,1) \times (\frac{1}{2},1) \end{cases}, \quad \Gamma_D = \partial\Omega,\ 0 < \varepsilon \leq 1$$

Problem 4

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2} \cdot \frac{\partial^2 u}{\partial y^2} + \frac{1+\mu}{2} \cdot \frac{\partial^2 v}{\partial x \partial y} &= f \quad \textit{on} \quad \Omega \\ \frac{1+\mu}{2} \cdot \frac{\partial^2 u}{\partial x \partial y} + \frac{1-\mu}{2} \cdot \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= g \quad \textit{on} \quad \Omega \\ u = v &= 0 \quad \textit{on} \quad \Gamma = \partial\Omega \end{aligned}$$

$\mu = \nu/(1-\nu) \in (0,1)$ is the modified Poisson ratio $(\nu \in (0,\frac{1}{2}))$

Approximation Q; computed κ and local bound $1/\underline{\beta}$: Problem 2

ϵ	16 elts	64 elts	256 elts	loc. bnd.
1.0	1.23	1.47	1.56	1.67
0.5625	1.32	1.69	1.86	1.93
0.25	1.41	2.03	2.36	2.67
0.0625	1.31	2.12	2.90	6.67
0.01	1.08	1.42	2.22	(34.67)

Preconditioner P; computed κ and local estimate $\hat{\lambda}$: Problem 2

ϵ	16 elts	64 elts	256 elts	estimate
1.0	1.20	1.27	1.29	2.29
0.5625	1.20	1.27	1.29	2.31
0.25	1.24	1.30	1.32	2.38
0.0625	1.44	1.65	1.70	3.24
0.01	1.82	2.95	4.11	10.18

If we require $\text{diag}(\tilde{P}) = \text{diag}(A_{11})$ we get $\tilde{P} = \tilde{U}^T(\text{diag}(\tilde{U}))^{-1}\tilde{U}$:

$$\tilde{u}_{ii} := a_{ii} - \sum_{j=1}^{i-1} \frac{u_{ji}^2}{u_{jj}} \quad \forall i.$$

Preconditioner $\tilde{P}$; computed κ: Problem 1

δ	16 elts	64 elts	256 elts
0	1.07	1.08	1.08
0.5	1.07	1.07	1.07
0.9	1.11	1.11	1.11
0.99	1.12	1.12	1.12

Preconditioner $\tilde{P}$; computed κ: Problem 2

ϵ	16 elts	64 elts	256 elts
0.25	1.17	1.20	1.21
0.0625	1.10	1.14	1.15
0.01	1.03	1.03	1.04
0.0001	1.00	1.00	1.00

Preconditioner $\tilde{P}$; computed κ: Problem 4

μ	16 elts	64 elts	256 elts
0.1	1.43	1.47	1.48
0.25	1.53	1.55	1.56
$0.\dot{3}$	1.60	1.61	1.61
0.5	1.76	1.75	1.75

If we require the preconditioner $P = LU$ to satisfy

$$A_{11} \cdot (1, 1, \ldots, 1)^T = P \cdot (1, 1, \ldots, 1)^T \tag{4}$$

the diagonal of U needs a recalculation:
Let $C = (c_{ij}) := A_{11}$ and $P = U^T(\text{diag}(U))^{-1}U$, where $U = (u_{ij})$ is an upper triangular matrix. Then the criterion (4) is equivalent to

$$u_{kk} = \sum_{j=1}^{n} c_{kj} - \sum_{j=k+1}^{n} u_{kj} - \sum_{i=1}^{k-1} \frac{u_{ik}}{u_{ii}} \sum_{j=i}^{n} u_{ij} \quad \forall k = 1, 2, \ldots, n$$

This a-posteriori modification of the diagonal of U can be done efficiently using the following algorithm:

Algorithm 6 [A-posteriori diagonal modification of U]

for $k = 1$ **to** n

$(a):\quad s_C(k) := \sum_{j=1}^{n} c_{kj},\; s_U(k) := \sum_{j=k+1}^{n} u_{kj},\; u_{kk} := s_C(k) - s_U(k) - \sum_{i=1}^{k-1} \frac{u_{ik} s_U(i)}{u_{ii}},$

$(b):\quad s_U(k) := s_U(k) + u_{kk}$

end

Nonlinear AMLI (cf., [Axelsson, Vassilevski 94], [Kraus 02], and [Kraus 05]): 2 inner GCG iterations at every other level.

We compare two methods

I: $A_{11} \approx P_{\mathsf{MILU}}$; $S \approx Q$

II: $A_{11} \approx P_{\mathsf{MILUE}}$; $S \approx Q$

using diagonal modification of U to satisfy criterion (4);

$1/h = 8, 16, \ldots, 512$ corresponds to $3, 4, \ldots, 9$ levels

Number of outer GCG iterations for Problem 1

	$1/h$	8	16	32	64	128	256	512
$\delta = 0$	(I)	9	11	11	11	11	11	11
	(II)	4	5	5	6	6	6	6
$\delta = 0.5$	(I)	11	14	14	14	14	14	14
	(II)	4	5	5	5	5	5	5
$\delta = 0.75$	(I)	13	18	19	19	19	19	19
	(II)	5	5	5	6	6	6	6
$\delta = 0.9$	(I)	14	22	24	24	24	24	24
	(II)	5	5	6	6	6	6	7

Number of outer GCG iterations for Problem 2

	$1/h$	8	16	32	64	128	256	512
$\epsilon = 1$	(I)	5	6	6	7	7	7	7
	(II)	5	6	7	7	7	7	7
$\epsilon = 0.5$	(I)	5	7	7	8	8	8	8
	(II)	5	7	8	8	8	8	8
$\epsilon = 0.25$	(I)	7	9	9	9	9	9	9
	(II)	5	7	9	9	9	9	9
$\epsilon = 0.1$	(I)	10	16	18	18	18	18	18
	(II)	5	8	10	10	11	11	11

Number of outer GCG iterations for Problem 3

	$1/h$	8	16	32	64	128	256	512
$\epsilon = 0.5$	(I)	6	7	7	8	8	8	8
	(II)	5	7	8	8	8	8	8
$\epsilon = 0.25$	(I)	8	9	9	9	9	10	10
	(II)	5	8	9	9	10	9	10
$\epsilon = 0.1$	(I)	12	19	20	21	21	21	21
	(II)	6	8	10	11	11	11	11
$\epsilon = 0.05$	(I)	16	47	56	59	60	60	61
	(II)	6	8	11	11	11	11	11

Number of outer GCG iterations for Problem 4

	$1/h$	8	16	32	64	128	256	512
$\mu = 0.1$	(I)	6	8	9	10	10	10	11
	(II)	6	8	9	10	10	11	14
$\mu = 0.33$	(I)	7	9	9	10	10	10	10
	(II)	6	8	9	9	10	11	16
$\mu = 0.5$	(I)	8	12	13	13	13	13	14
	(II)	6	8	9	9	11	13	21
$\mu = 0.67$	(I)	11	19	27	32	33	33	33
	(II)	7	9	10	11	13	17	29
$\mu = 0.75$	(I)	13	26	46	$*$	$*$	$*$	$*$
	(II)	7	9	11	12	14	20	36

The asterix indicates an iteration count greater than 100 for method (I).

The use of conforming bilinear elements for the displacements in the plane-stress elasticity problem is reasonable for moderate values of the Poisson ratio only. Otherwise, in order to overcome locking effects, one can use non-conforming Crouzeix-Raviart elements in the setting of [Blaheta, Margenov, Neytcheva 05], or, alternatively, extend the presented technique to non-conforming elements.

Conclusions:

- the presented multilevel preconditioner is based on element agglomeration
- by exact local elimination an efficient preconditioner for the pivot block can be constructed
- assembling local Schur complements one obtains a sparse and adequate approximation of the global Schur complement
- an analysis on macro-element level allows to derive spectral bounds for the Schur complement (and the pivot matrix) approximation
- a hierarchical basis representation of the stiffness matrix is avoided, which yields less densely populated off-diagonal blocks in the multilevel setting
- the presented techniques can be applied to nonsymmetric indefinite linear algebraic systems, see, e.g., [Bängtsson, Neytcheva 05]
- unstructured FE-meshes can be tackled (provided an agglomeration technique is available)

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