lecture notes on

Parameter Identification in Partial Differential Equations

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Barbara Kaltenbacher

Contents

1. Introduction

A large variety of natural, industrial, social and economical phenomena can be modeled by (systems of) partial differential equations (PDEs). When doing simulations based on a PDE model, it is assumed that all involved parameters — such as coefficients or source terms — are known. However, beforehand these parameters have to be determined. Since they are frequently hardly (or even not at all) accessible to direct measurements, they have to be fitted from indirect measurements. This amounts to the inverse problem of parameter identification. Inverse problems are often not well-posed in the sense of Hadamard, who postulated that for properly posed mathematical problems always

- a solution exists;
- the solution is unique;
- the solution depends continuously on the given data.

The task of uniqueness is crucial in parameter identification, since it is essential that the given data are sufficient for determining the searched for parameter uniquely. This question of identifiability is often a challenging mathematical problem and it is the main subject of this lecture to show some approaches concerning its answer in the context of several different PDEs.

Stability is even mostly violated in inverse problems, so especially also for parameter identification: Small perturbations in the data can lead to large deviations in the solution, which makes special numerical methods — so-called regularization methods — indispensable in solving such problems. Part of this lecture is devoted to such methods, however, for this topic we mainly refer to the lecture *Inverse Problems*.

By the way, the first of Hadamard's postulations is usually satisfied for parameter identification problems, provided the given model makes sense.

We mention in passing that parameter (especially coefficient) identification problems are often nonlinear even if the PDE itself is a linear one.

1.1. Some examples

Examplem 1 Heat equation:
Consider the heat equation

$$u_t - \nabla(c\nabla u) = f \text{ in } \Omega \times [0,T]\,, \tag{1.1}$$

which describes the development of the temperature u within a body Ω over the time interval $[0,T]$, starting from an initial temperature distribution u_0 according to

$$u(x,0) = u_0(x)\,, \quad x \in \Omega \tag{1.2}$$

Here f represents interior heat sources that may depend on location in space and/or time. The coefficient c is called the thermal conductivity. If, e.g., the boundary is isolated, this means that the boundary condition

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{1.3}$$

is prescribed, where n is the outer unit normal to $\partial\Omega$.

Assuming that u_0, c, and f are known functions, one can (in principle as well as numerically) compute the temperature u at every space point $x \in \Omega$ and every time instance $t \in [0,T]$ from the initial boundary value problem (1.1), (1.2), (1.3). This is the forward problem in this context.

An interesting inverse problem is to determine the distribution f of the sources from additional boundary measurements, e.g., of the temperature

$$u = g \text{ on } \partial\Omega\,. \tag{1.4}$$

This problem of source identification is linear.

Also the situation that the sources f are given but the thermal conductivity c — a parameter depending on the material of the body Ω — is unknown is of practical interest: Typically, when the temperature varies over a large range, c is not a constant but depends on the temperature u itself, i.e. (1.1) becomes

$$u_t - \nabla(c(u)\nabla u) = f \text{ in } \Omega \times [0,T]\,, \tag{1.5}$$

i.e., the forward problem involves a nonlinear PDE. Hence, the inverse problem of coefficient identification from additional measurements (1.4) is nonlinear as well.

Coefficient identification in PDEs is generally usually nonlinear, even if the PDEs under consideration is linear. As an example, consider the situation of a spatially varying c in (1.1), which occurs e.g., when operating only over a small temperature range, but with Ω consisting of regions with different materials having different thermal conductivity, i.e. $c = c(x)$. Therewith, (1.1) is obviously a linear PDE, but multiplying c by some factor λ does not lead to a multiplication of the boundary values (1.4) of u by λ so the parameter-to-measurement-map $c \mapsto g$ is nonlinear and so is its inverse, which we aim at evaluating when doing parameter identification.

Example 2 Groundwater filtration:
The groundwater level u (or, more precisely, the so-called piezometric head) in a domain Ω is in the static case basically governed by the elliptic PDE

$$\nabla(a\nabla u) = f \tag{1.6}$$

where f represents the sinks and sources in the domain, and a is the transmissivity of the ground, that depends on space $a = a(x)$. While in the previous example temperature measurements are typically available only at the boundary but not inside a body, inverse groundwater filtration is one of the (rare) examples, where it makes sense to assume that one has measurements in the interior of a region. The inverse groundwater filtration problem in this simple model (of course, there exist much more complicated ones!) consists of determining a from measurements of u in Ω.

In the one-dimensional case and assuming that $a(0)$ is known, we can solve this problem analytically: (1.6) becomes

$$(au_x)_x = f \tag{1.7}$$

and therewith

$$a(x) = \frac{a(0)u_x(0) + \int_0^x f(\xi)\, d\xi}{u_x(x)} \,.$$

Note that this formula involves differentiation of the data u, which is known to be an unstable problem. Moreover, in regions where u_x vanishes, a cannot be determined from formula (1.7). As a matter of fact, it is obvious from (1.7) that in regions where u_x is equal to zero, a may assume an arbitrary value without having any influence on the measurements. Thus, if such regions exist, the inverse problem is ill-posed in the sense of non-uniqueness.

Example 3 Nonlinear magnetics:
The well-known Maxwell's equations

$$\begin{aligned}
\nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{(Ampére's law)}\\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law)}\\
\nabla \vec{D} &= q \quad \text{(Gauss' law)}\\
\nabla \vec{B} &= 0 \quad \text{(solenoidal magnetic field)}
\end{aligned}$$

describe electromagnetic field phenomena. Here, $\vec{H}$ is the magnetic field intensity, $\vec{B}$ the magnetic induction (magnetic flux density) $\vec{E}$ the electric field, $\vec{D}$ the electric flux density (dielectric displacement), $\vec{J}$ the current density, and q the volume charges. Additionally, one has the constitutive equations

$$\vec{J} = J_i + \gamma(\vec{E} + \vec{v} \times \vec{B}) \tag{1.8}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H} \,, \tag{1.9}$$

where μ (magnetic permeability), ε (electric permittivity), γ (electric conductivity) are material parameters, v is the velocity of moving charges ($\rightarrow$ Lorentz forces), and J_i the impressed current density.

In the situation of large magnetic fields, the magnetic permeability is not constant but a function of the modulus $H = |\vec{H}|$ of the magnetic field intensity

$$\mu = \mu(H)\,.$$

For measuring this nonlinear relation, the usual experimental setup consists of a coil wound around a material probe, which is supplied with an impressed current I. The magnetic flux through the coil

$$\Phi = \int_{A_c} \vec{B} \cdot \vec{n}\, d\sigma$$

where A_c denotes the cross sectional area of the coil, is measured. A mathematical model of this setting is given by a combination of part of Maxwell's equations together with constitutive equations. Ampére's law together with the fact that in this context the change $\frac{\partial \vec{D}}{\partial t}$ of the electric flux density is negligible, as well as Faraday's law give the equation

$$\nabla \times \frac{1}{\gamma} \nabla \times \vec{H} = \nabla \times \frac{1}{\gamma} J_i - \vec{B}_t \text{ in } \Omega \times [0,T]\,,$$

where γ is the electric conductivity, and J_i the impressed current density according to

$$J_i(x,t) = \begin{cases} \frac{I(t)}{|A_c|} \vec{e}_J(x) & \text{for } x \text{ in the coil region } \Omega_c \subset \Omega \\ 0 & \text{else,} \end{cases}$$

(note that there is no movement, $v = 0$). The region $\Omega \subseteq \mathbb{R}^3$ includes the coil, the probe as well as the surrounding air and is supposed to be sufficiently large so that basically no magnetic field lines leave Ω

$$\vec{H} \cdot \vec{n} = 0 \text{ on } \partial\Omega \times [0,T]\,.$$

The inverse problem of interest here is to determine the so-called $B - H$ curve, (or equivalently, the curve $H \mapsto \mu(H)$) from measurements of the curve $I \mapsto \Phi(I)$ (see B.K&M.Kaltenbacher&Reitzinger)

Example 4 Population dynamics:
A basic model for the development of an age-structured population over a time interval $[0,T]$ is

$$\rho_t + \rho_a + \lambda\rho = 0 \quad a \in [0,L],\ t \in [0,T] \tag{1.10}$$

with initial condition

$$\rho(a,0) = \rho_0(a) \tag{1.11}$$

and boundary conditions

$$\rho(0,t) = \int_0^L \beta(a)\rho(a,t)\, da \tag{1.12}$$

$$\rho(L,t) = 0\,. \tag{1.13}$$

Here a is the age, t the time, $\rho(a,t)$ denotes the number of individuals of age a at time t, and L is the maximal life span. The functions $\beta = \beta(a)$ and $\lambda = \lambda(a)$ are the birth and death functions respectively, i.e., the probabilities with which an individual at age a gives birth to a new individual or dies, respectively. The initial population distribution $\rho_0(a)$ can be obtained from census data. Therewith, (1.10) and (1.12) can be seen as balance equations: For ages $a > 0$, the change of population with respect to time and age is determined by the death function and the current population structure. Individuals of age $a = 0$ can obviously only emerge from births. Since L is the maximal life span, no individual can be older than L, see (1.13).

Here the forward problem is to compute ρ from the initial boundary value problem (1.10), (1.11), (1.12), (1.13), (actually, since we deal with a transport equation, we expect the boundary condition at the right hand boundary to be redundant at least in case of constant λ), provided ρ_0, β, λ are given.

Often the parameters β and/or λ are unknown and have to be estimated from additional census data. Assuming, e.g., that we know ρ_0, β and additionally

$$\rho_T = \rho(a, T) \tag{1.14}$$

(from another census at time $t = T$), an interesting inverse problem is to identify λ as a function of the age in the initial boundary value problem (1.10), (1.11), (1.12), (1.13), from the over posed data (1.14), see Pilant&Rundell.

2. The electrical impedance tomography problem

Electric impedance tomography is a technique to recover spatially distributed properties in the inaccessible interior of a body from electrical measurements and has important applications in medical imaging and nondestructive testing. Consider the problem of determining a spatially varying electric conductivity $a = a(x)$ inside a body Ω by measuring currents corresponding to all possible distributions of boundary voltages that are impressed via electrodes attached to the body surface. This can be modeled in a somewhat simplified way by the boundary value problem

$$\nabla(a\nabla u) = 0 \quad \text{in } \Omega \tag{2.1}$$

$$a\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \tag{2.2}$$

where $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is a two or three dimensional domain, u the electric potential, and g is the measured current. Additionally, with f denoting the impressed voltage, we have

$$u = f \quad \text{on } \partial\Omega\,. \tag{2.3}$$

Obviously, the possible current patterns will show a dependency on the interior conductivity distribution. Of course for dimensionality reasons, it is not sufficient to measure one or finitely many voltage-current pairs (functions living on the $n-1$ dimensional boundary) to uniquely identify a as a function of the n components of x. Still one hopes to uniquely recover a from knowledge of the whole so-called Neumann-to-Dirichlet operator, i.e. the mapping of all possible currents to all corresponding voltages inducing them via (2.1), (2.3). This uniqueness question was originally posed and studied by Calderon (1980) and later on treated by several different authors (e.g., Sylvester, Uhlmann, Kohn, Vogelius, Nachmann, Isakov, Alessandrini, Päivärinta, Astala). We here give an identifiability proof for the three dimensional case following Isakov, (as well as the lecture notes by Scherzer). Note that the two dimensional case was considered as an open problem for several years and has been solved only recently (Nachman, Päivärinta, Astala).

Before defining the inverse problem, we state well-definedness of the forward problem and for this purpose assume from now on that

$$a \in C^2(\Omega,\mathbb{R})\,, \quad 0 < \gamma \leq a(x)\ x \in \Omega\,,$$

which implies ellipticity of (2.1).

Theorem 2.1. *Let $g \in C^2(\partial\Omega, \mathbb{R})$ and $\int_{\partial\Omega} g\, d\Gamma = 0$. Then the boundary value problem (2.1), (2.2) has a solution $u \in C^2(\Omega)$ and this solution is unique within the set*

$$\{u \in C^2(\Omega) \mid \int_\Omega u\, dx = 0\} \tag{2.4}$$

Proof. see Stampaccia.

$\diamondsuit$

2.1. The inverse problem

Given the Neumann-Dirichlet map

$$\begin{array}{rrcl} \tilde{\Lambda}_a : & C^2(\partial\Omega, \mathbb{R}) & \to & C^2(\partial\Omega, \mathbb{R}) \\ & g & \mapsto & u|_{\partial\Omega} \text{ where } u \text{ solves (2.1),(2.2) with } \int_\Omega u\, dx = 0 \end{array}$$

determine a.

Let Λ_a denote the extension of $\tilde{\Lambda}_a$ to complex valued functions

$$\begin{array}{rrcl} \Lambda_a : & C^2(\partial\Omega, \mathbb{C}) & \to & C^2(\partial\Omega, \mathbb{C}) \\ & g_1 + \imath g_2 & \mapsto & \tilde{\Lambda}_a g_1 + \imath \tilde{\Lambda}_a g_2 \, . \end{array}$$

Since a is a real-valued function, it can be identified from $\tilde{\Lambda}_a$ if and only if it can be identified from Λ_a.

2.2. Transformation to a Schrödinger equation

To prove that the mapping $a \mapsto \Lambda_a$ is injective (i.e., identifiability of the conductivity from the Neumann-Dirichlet map), we transform it to a Schrödinger (or Helmholtz) equation. Setting

$$v = a^{\frac{1}{2}} u \, , \tag{2.5}$$

we have $u = a^{-\frac{1}{2}} v$, $\nabla u = -\frac{1}{2} a^{-\frac{3}{2}} v \nabla a + a^{-\frac{1}{2}} \nabla v$ and therewith

$$\begin{aligned} \nabla(a \nabla u) &= \nabla\Big(-\frac{1}{2} a^{-\frac{1}{2}} v \nabla a + a^{\frac{1}{2}} \nabla v\Big) \\ &= \frac{1}{4} a^{-\frac{3}{2}} v \nabla a \nabla a - \frac{1}{2} a^{-\frac{1}{2}} \nabla v \nabla a - \frac{1}{2} a^{-\frac{1}{2}} v \Delta a \\ &\quad + \frac{1}{2} a^{-\frac{1}{2}} \nabla v \nabla a + a^{\frac{1}{2}} \Delta v \\ &= a^{\frac{1}{2}} \Delta v + \Big(\frac{1}{4} a^{-\frac{3}{2}} \nabla a \nabla a - \frac{1}{2} a^{-\frac{1}{2}} \Delta a\Big) v \, . \end{aligned}$$

Hence, if u solves (2.1) then $v = a^{\frac{1}{2}} u$ solves the Schrödinger equation

$$\Delta v + bv = 0, \tag{2.6}$$

where the coefficient b is given by

$$b = \frac{1}{4}\frac{|\nabla a|^2}{a^2} - \frac{1}{2}\frac{\Delta a}{a} \tag{2.7}$$

Especially, from the Neumann-Dirichlet map Λ_a for (2.1) one can compute the Neumann-Dirichlet map Λ_b^S for (2.6), provided a is known at the boundary and $\frac{\partial a}{\partial n}|_{\partial\Omega} = 0$, which we will assume in the following.

Moreover, using the map

$$\begin{array}{rrcl} G: & C^2(\Omega,\mathbb{R}) & \rightarrow & C(\Omega,\mathbb{R}) \\ & a & \mapsto & b \text{ according to (2.7)} \end{array}$$

we can show that b uniquely determines a:

Lemma 2.2. *Given $a_0 \in C^2(\partial\Omega,\mathbb{R})$, for all $b \in \mathcal{R}(G)$ there exists a unique $a \in C^2(\Omega,\mathbb{R})$ such that $a|_{\partial\Omega} = a_0$, $\frac{\partial a}{\partial n}|_{\partial\Omega} = 0$, and the relation (2.7) holds.*

Proof. Existence is obvious for $b \in \mathcal{R}(G)$.
We prove uniqueness, i.e., that $G(a_1) = G(a_2)$ implies $a_1 = a_2$.

$$\begin{aligned} 0 &= G(a_1) - G(a_2) \\ &= \frac{1}{4}\left(\frac{|\nabla a_1|^2}{a_1^2} - \frac{|\nabla a_2|^2}{a_2^2}\right) - \frac{1}{2}\left(\frac{\Delta a_1}{a_1} - \frac{\Delta a_2}{a_2}\right) \\ &= \frac{1}{4}\left(\frac{1}{a_1^2} - \frac{1}{a_2^2}\right)|\nabla a_1|^2 + \frac{1}{4}\frac{|\nabla a_1|^2 - |\nabla a_2|^2}{a_2^2} \\ &\quad -\frac{1}{2}\left(\frac{1}{a_1} - \frac{1}{a_2}\right)\Delta a_1 - \frac{1}{2}\frac{\Delta a_1 - \Delta a_2}{a_2} \\ &= -\frac{(a_1+a_2)(a_1-a_2)}{4a_1^2a_2^2}|\nabla a_1|^2 + \frac{1}{4a_2^2}(\nabla a_1 + \nabla a_2)(\nabla a_1 - \nabla a_2) \\ &\quad +\frac{a_1-a_2}{2a_1a_2}\Delta a_1 - \frac{1}{2a_2}(\Delta a_1 - \Delta a_2)\,. \end{aligned}$$

Therewith, $w := a_1 - a_2$ solves the homogeneous boundary value problem

$$\begin{array}{rcl} \Delta w - \frac{\nabla a_1 + \nabla a_2}{2a_2}\nabla w - \left(\frac{\Delta a_1}{a_1} - \frac{a_1+a_2}{2a_1^2a_2}|\nabla a_1|^2\right) w & = & 0 \text{ in } \Omega \\ w & = & 0 \text{ on } \partial\Omega \end{array} \tag{2.8}$$

where we have used the assumption that $a_1|_{\partial\Omega} = a_0 = a_2|_{\partial\Omega}$. A maximum principle (see e.g. Protter&Weinberger), applied to (2.8), yields $w = 0$ in all of Ω.
$\diamondsuit$

2.3. Completeness of products of harmonic functions

An essential ingredient of the uniqueness proof will be denseness of certain subsets of $L^2(\Omega, \mathbb{C})$. We start with proving the fact that the set of products of harmonic functions

$$\{u^1.u^2 \mid \Delta u^1 = \Delta u^2 = 0\} \tag{2.9}$$

is dense in $L^2(\Omega, \mathbb{C})$. For this purpose, we require some Lemmas.

Lemma 2.3. *Let, for some $\varepsilon \in \mathbb{C}^n$, $n \in \mathbb{N}$, the function u be defined by*

$$u(x) = e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}, \tag{2.10}$$

where $x \overset{\mathbb{C}^n}{\cdot} y := \sum_{j=1}^n x_k y_k$ (note that $\overset{\mathbb{C}^n}{\cdot}$ is <u>not</u> the Euclidean inner product in C^n, which would be given by $\langle x, y\rangle_{C^n} = \sum_{k=1}^n x_k \overline{y_k}$!). Then

$$\Delta u = 0 \;\Leftrightarrow\; \varepsilon \overset{\mathbb{C}^n}{\cdot} \varepsilon = 0\,.$$

Proof.

$$\Delta u = \sum_{k=1}^n \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^n \imath^2 (\varepsilon_k)^2\, e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}$$

Since $|e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}| = |e^{-\sum_{k=1}^n x_k \Im(\varepsilon_k)}|$ does not vanish, the assertion follows.
$\diamondsuit$

Lemma 2.4. *For arbitrary $\eta \in \mathbb{R}^n$, $n \geq 2$ there exist $\varepsilon^1, \varepsilon^2 \in \mathbb{C}^n$ with the properties*

$$\varepsilon^1 + \varepsilon^2 = \eta\,, \quad \varepsilon^j \overset{\mathbb{C}^n}{\cdot} \varepsilon^j = 0\,,\; j = 1, 2 \tag{2.11}$$

If $n \geq 3$ then for any $R > 0$ there exist $\varepsilon^1, \varepsilon^2$ such that additionally to (2.11) the property

$$|\Im(\varepsilon^j)|_{\mathbb{R}^n} \geq R\,,\; j = 1, 2 \tag{2.12}$$

holds.

Proof.
With the Ansatz

$$\varepsilon^1 = \alpha + \imath\beta\,, \quad \varepsilon^2 = \eta - \alpha - \imath\beta\,, \qquad \alpha, \beta \in \mathbb{R}^n\,, \tag{2.13}$$

satisfying the first identity in (2.11), the rest of (2.11) becomes equivalent to

$$\begin{aligned} 0 &= \Re(\varepsilon^1 \overset{\mathbb{C}^n}{\cdot} \varepsilon^1) = |\alpha|^2_{\mathbb{R}^n} - |\beta|^2_{\mathbb{R}^n}\,, \quad 0 = \Im(\varepsilon^1 \overset{\mathbb{C}^n}{\cdot} \varepsilon^1) = \langle \alpha, \beta\rangle_{\mathbb{R}^n}\,, \\ 0 &= \Re(\varepsilon^2 \overset{\mathbb{C}^n}{\cdot} \varepsilon^2) = |\eta - \alpha|^2_{\mathbb{R}^n} - |\beta|^2_{\mathbb{R}^n}\,, \quad 0 = \Im(\varepsilon^2 \overset{\mathbb{C}^n}{\cdot} \varepsilon^2) = \langle \eta - \alpha, \beta\rangle_{\mathbb{R}^n}\,. \end{aligned} \tag{2.14}$$

In the special case $\eta = 0$, a solution of (2.14) is given by $\alpha = (R, 0, 0, \ldots, 0)^T$, $\beta = (0, R, 0, \ldots, 0)^T$, which additionally satisfies (2.12) even in case $n = 2$.
For $\eta \neq 0$ we use η to define the first basis vector $\rho_1 = \frac{\eta}{|\eta|_{R^n}}$ of an orthonormal basis $\{\rho_1, \ldots, \rho_n\}$ on $\mathbb{R}^n$ (such an ONB always exists). Therewith, we can develop α and β with respect to this basis

$$\alpha = \tilde{\alpha}_1 \frac{\eta}{|\eta|_{R^n}} + \sum_{k=2}^{n} \tilde{\alpha}_k \rho_k\,, \quad \beta = \tilde{\beta}_1 \frac{\eta}{|\eta|_{R^n}} + \sum_{k=2}^{n} \tilde{\beta}_k \rho_k\,.$$

Since norm and inner product in $\mathbb{R}^n$ are invariant under orthonormal coordinate transform, we get equivalence of (2.14) to

$$\begin{array}{l} \sum_{k=1}^n \tilde{\alpha}_k^2 = \sum_{k=1}^n \tilde{\beta}_k^2\,, \quad \sum_{k=1}^n \tilde{\alpha}_k \tilde{\beta}_k = 0\,, \\ (|\eta|_{\mathbb{R}^n} - \tilde{\alpha}_1)^2 + \sum_{k=2}^n \tilde{\alpha}_k^2 = \sum_{k=1}^n \tilde{\beta}_k^2\,, \quad (|\eta|_{\mathbb{R}^n} - \tilde{\alpha}_1)\tilde{\beta}_1 + \sum_{k=2}^n \tilde{\alpha}_k \tilde{\beta}_k = 0\,. \end{array} \tag{2.15}$$

In case $n = 2$ we set

$$\tilde{\alpha} = (\frac{|\eta|_{\mathbb{R}^n}}{2}, 0)\,, \quad \tilde{\beta} = (0, \frac{|\eta|_{\mathbb{R}^n}}{2})$$

to satisfy (2.15).
If $n \geq 3$ we can make use of the additional degrees of freedom to satisfy both (2.15) and (2.12), by setting

$$\tilde{\alpha} = (\frac{|\eta|_{\mathbb{R}^n}}{2}, 0, R, 0 \ldots, 0)\,, \quad \tilde{\beta} = (0, \sqrt{R^2 + \frac{|\eta|^2_{\mathbb{R}^n}}{4}}, 0, 0 \ldots, 0)\,.$$

Namely, with this choice (2.15) is easily checked and we get

$$|\Im(\varepsilon^1)|_{\mathbb{R}^n} = |\Im(\varepsilon^2)|_{\mathbb{R}^n} = |\beta|_{\mathbb{R}^n} = |\tilde{\beta}|_{\mathbb{R}^n} = \sqrt{R^2 + \frac{|\eta|^2_{\mathbb{R}^n}}{4}} \geq R$$

$\diamondsuit$

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded domain. Then the set of products of harmonic functions (2.9) is dense in $L^2(\Omega, \mathbb{C})$.*

Proof.
Assume that the set (2.9) is not dense in $L^2(\Omega, \mathbb{C})$, then by the Hahn-Banach theorem there exists a nonzero $f \in L^2(\Omega, \mathbb{C})$ which is orthogonal in $L^2(\Omega, \mathbb{C})$ to the set (2.9), i.e.,

$$\int_\Omega f \overline{u^1 u^2}\, dx = 0 \quad \forall u^1, u^2 \text{ harmonic in } \Omega\,. \tag{2.16}$$

Let $\eta \in \mathbb{R}^n$ be arbitrarily fixed, and choose $\varepsilon^1, \varepsilon^2 \in \mathbb{C}^n$ according to Lemma 2.4 such that (2.11) is satisfied. Then, due to Lemma 2.3 the functions u^1, u^2 defined by $u^j(x) = e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon^j}$ $j = 1, 2$ are harmonic, so that we can insert them into (2.16) to obtain

$$0 = \int_\Omega f(x) \overline{e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon_1} e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon_2}}\, dx = \int_{\mathbb{R}^n} \mathbf{I}_\Omega(x) f(x) \overline{e^{\imath x \overset{\mathbb{C}^n}{\cdot} \eta}}\, dx$$

where $\mathbf{I}_\Omega$ denotes the characteristic function on Ω. Since $\eta \in \mathbb{R}^n$ was arbitrary, this means that the Fourier transform of f (continued by zero outside Ω) vanishes and therewith f itself is equal to zero. Or, in other words, since the set of functions $\{x \mapsto e^{\imath x \overset{\mathbb{C}^n}{\cdot} \eta} \mid \eta \in \mathbb{R}^n\}$ is dense in $L^2(\mathbb{R}^n, \mathbb{C})$, f has to be equal to zero. This is a contradiction.
$\diamondsuit$

2.4. Completeness of products of almost exponential solutions of the Schrödinger equation

As we have seen in the proof of Theorem 2.5, even the subset of exponential harmonic functions, i.e., functions of the form (2.10) (with appropriate ε, see Lemma 2.3) is sufficient to define a dense subset of $L^2(\Omega, \mathbb{C})$ via (2.9). We will now modify the denseness result of the previous section in the sense that the Laplace equation is replaced by the Schrödinger equation and exponential solutions by "almost" exponential solutions. More precisely, we consider solutions v of the Schrödinger equation (2.6) that are of the form

$$v(x) = e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}(1 + w(x)) \tag{2.17}$$

with small w and $\varepsilon \overset{\mathbb{C}^n}{\cdot} \varepsilon = 0$. Since

$$\begin{aligned}\Delta v(x) + b(x)v(x) &= -\varepsilon \overset{\mathbb{C}^n}{\cdot} \varepsilon\, e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}(1 + w(x)) + 2\imath\varepsilon \overset{\mathbb{C}^n}{\cdot} \nabla w(x) e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon} \\ &\quad + \Delta w(x) e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon} + b(x) e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}(1 + w(x)) \\ &= e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon}\Big(\Delta w(x) + 2\imath\varepsilon \overset{\mathbb{C}^n}{\cdot} \nabla w(x) + b(x)(1 + w(x))\Big),\end{aligned}$$

this v solves (2.6) if and only if w solves

$$\Delta w + 2\imath\varepsilon \overset{\mathbb{C}^n}{\cdot} \nabla w + b(1 + w) = 0\,. \tag{2.18}$$

We will prove a result analogous to Lemma 2.4 (see Theorem 2.8 below), whose proof is deeper, though, in the sense that we require the following definition and theorem of the bounded invertibility of differential operators.

Definition 2.6. For some multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we use the notation D^α for a combination of the the partial derivatives according to the entries of α:

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n}$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$.

Let

$$A = \sum_{|\alpha| \le m} c_\alpha D^\alpha$$

be a linear differential operator of order m with constant (possibly complex valued) coefficients $(c_\alpha)_{\alpha \in \mathbb{N}_0^n,\ |\alpha| \le m}$. Denoting $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_1^{\alpha_1}$, we can write this formally as $A = P(\imath D^{(1,\dots,1)})$, where

$$P(\zeta) = \sum_{|\alpha| \le m} (-\imath)^{|\alpha|} c_\alpha \zeta^\alpha .$$

Moreover, we define the function

$$\psi(\zeta) = \sqrt{\sum_{\alpha \in \mathbb{N}_0^n} |D^\alpha P(\zeta)|^2}$$

where the sum is finite due to the fact that P is a polynomial.

Theorem 2.7. *Let Ω be a bounded domain and A be a linear differential operator of order m with constant coefficients. Then there exists a bounded linear operator $E : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ such that*

$$AEf = f \tag{2.19}$$

and

$$\|Ef\|_{L^2(\Omega,\mathbb{C})} \le C \sup_{x \in \mathbb{R}^n} \frac{1}{\psi(x)} \|f\|_{L^2(\Omega,\mathbb{C})} \tag{2.20}$$

for all $f \in L^2(\Omega, \mathbb{C})$. Here $C = C(n, m, diam(\Omega))$ is a constant depending only on the space dimension, the order of the differential operator, and the size of the domain.

Proof. see Hörmander, Theorem 10.3.7.

♢

The operator E in Theorem 2.7 is called the fundamental solution of the differential operator A.

Therewith we can show the following analogon to Lemma 2.4.

Theorem 2.8. *For $b^1, b^2 \in L^\infty(\Omega, \mathbb{R})$, any $\eta \in \mathbb{R}^n$, $n \ge 3$, and any R sufficiently large,*

$$R \ge 2C \max_{j \in \{1,2\}} \|b^j\|_{L^\infty(\Omega,\mathbb{R})} , \tag{2.21}$$

where $C = C(n, 2, diam(\Omega))$ is the constant from Theorem 2.7, there exist solutions v_R^1, v_R^2 of the form

$$v_R^j(x) = e^{\imath x \overset{\mathbb{C}^n}{\cdot} \varepsilon_R^j}(1 + w_R^j(x)) \tag{2.22}$$

of the Schrödinger equation

$$\Delta v + b^j v = 0 \tag{2.23}$$

such that

$$\varepsilon_R^1 + \varepsilon_R^2 = \eta\,, \quad \varepsilon_R^j \overset{\mathbb{C}^n}{\cdot} \varepsilon_R^j = 0 \quad |\varepsilon_R^j|_{\mathbb{C}^n} > R\,, \quad \|w_R^j\|_{L^2(\Omega,\mathbb{C})} \xrightarrow{R \to 0} 0 \quad j = 1, 2 \tag{2.24}$$

Proof. We fix R satisfying (2.21) and omit the subscript R in the proof. By Lemma 2.4, there exist ε^1, ε^2 with the properties (2.11). Let, for $j = 1,2$, A^j be the differential operator $\Delta + 2\imath\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \nabla$, then $P^j(\zeta) = -\zeta \overset{\mathbb{C}^n}{\cdot} \zeta + 2\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \zeta$, and

$$\psi^j(\zeta)^2 = \underbrace{|-\zeta \overset{\mathbb{C}^n}{\cdot} \zeta + 2\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \zeta|^2}_{|\alpha|=0} + \underbrace{\sum_{k=1}^n |-2\zeta_k + 2\varepsilon^j_k|^2}_{|\alpha|=1} + \underbrace{4n}_{|\alpha|=2}$$

so that for real valued vectors $x \in \mathbb{R}^n$

$$\begin{aligned}
\psi^j(x)^2 &= \underbrace{(\sum_{k=1}^n \Big((-x_k^2 + \Re(\varepsilon^j_k)x_k)^2 + (\Im(\varepsilon^j_k)x_k)^2\Big)}_{|\alpha|=0} \\
&\quad + \underbrace{\sum_{k=1}^n \Big((-2x_k + 2\Re(\varepsilon^j_k))^2 + (\Im(\varepsilon^j_k))^2\Big)}_{|\alpha|=1} + \underbrace{4n}_{|\alpha|=2} \\
&\geq |\Im(\varepsilon^j)|^2_{\mathbb{R}^n} \geq R^2
\end{aligned}$$

by (2.12). It follows from Theorem 2.7 that fundamental solutions E^j for the differential operator A^j exist such that

$$(\Delta + 2\imath\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \nabla)E^j = f$$

and

$$\|E^j f\|_{L^2(\Omega,\mathbb{C})} \leq \frac{C}{R} \|f\|_{L^2(\Omega,\mathbb{C})} \,. \tag{2.25}$$

Every solution w^j of the fixed point equation

$$w^j = -E^j(b^j(1 + w^j)) \tag{2.26}$$

in $L^2(\Omega, \mathbb{C})$ is a solution to (2.23), since

$$\Delta w^j + 2\imath\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \nabla w^j = -(\Delta + 2\imath\varepsilon^j \overset{\mathbb{C}^n}{\cdot} \nabla)E^j(b^j(1 + w^j)) = -(b^j(1 + w^j))\,.$$

Each of the operators

$$T^j : w \mapsto -E^j(b^j(1 + w))$$

is a self mapping as well as a contraction on

$$K = \bigcup_{j\in\{1,2\}} \{w \in L^2(\Omega, \mathbb{C}) \mid \|w\|_{L^2(\Omega,\mathbb{C})} \leq \frac{2C \|b^j\|_{L^\infty(\Omega,\mathbb{R})} \sqrt{meas(\Omega)}}{R}\}\,,$$

since for all $w \in K$, by (2.21),

$$\|w\|_{L^2(\Omega,\mathbb{C})} \leq \sqrt{meas(\Omega)}$$

and therewith

$$\begin{aligned}
\|T^j w\|_{L^2(\Omega,\mathbb{C})} &= \|E^j(b^j(1+w))\|_{L^2(\Omega,\mathbb{C})} \\
&\leq \frac{C}{R}\|b^j(1+w)\|_{L^2(\Omega,\mathbb{C})} \quad \text{by (2.25)} \\
&\leq \frac{C}{R}\|b^j\|_{L^\infty(\Omega,\mathbb{R})}(\sqrt{meas(\Omega)} + \|w\|_{L^2(\Omega,\mathbb{C})}) \\
&\leq \frac{C}{R}\|b^j\|_{L^\infty(\Omega,\mathbb{R})}\sqrt{meas(\Omega)}\,.2\,,
\end{aligned}$$

as well as

$$\begin{aligned}
\|T^j w - T^j \tilde{w}\|_{L^2(\Omega,\mathbb{C})} &= \|E^j(b^j(w-\tilde{w}))\|_{L^2(\Omega,\mathbb{C})} \\
&\leq \frac{C}{R}\|b^j(w-\tilde{w})\|_{L^2(\Omega,\mathbb{C})} \quad \text{by (2.25)} \\
&\leq \frac{C}{R}\|b^j\|_{L^\infty(\Omega,\mathbb{R})}\|w-\tilde{w}\|_{L^2(\Omega,\mathbb{C})} \leq \frac{1}{2}\|w-\tilde{w}\|_{L^2(\Omega,\mathbb{C})} \quad \text{by (2.21).}
\end{aligned}$$

By Banach's fixed point theorem, it follows that there exist fixed points w^j of T^j in K and therewith solutions to (2.23) that (by $w^j \in K$) can be estimated by

$$\|w^j\|_{L^2(\Omega,\mathbb{C})} \leq \frac{2C\,\|b^j\|_{L^\infty(\Omega,\mathbb{R})}\sqrt{meas(\Omega)}}{R} \to 0 \text{ as } R \to \infty\,.$$

$\diamondsuit$

Corollary 2.9. *For any $b^1, b^2 \in L^\infty(\Omega,\mathbb{R}$, and any $n \geq 3$, the set*

$$U = \{v^1.v^2 \mid v^j \text{ solves } \Delta v^j + b^j v^j\,,\ j \in \{1,2\}\} \tag{2.27}$$

is dense in $L^1(\Omega,\mathbb{C})$.

Proof.
Assume that the set U according to (2.27) is not dense in $L^1(\Omega,\mathbb{C})$. Then there exists an $f \in L^\infty(\Omega,\mathbb{C})$, $f \neq 0$ such that

$$\int_\Omega f\overline{v}\,dx = 0 \quad \forall v \in U\,. \tag{2.28}$$

For arbitrary η, by Theorem 2.8 there exist solutions v_R^j, $j = 1,2$ of the form (2.22) with the properties (2.24), so that we can set $v = v_R^1.v_R^2$ in (2.28) to obtain

$$\begin{aligned}
&|\int_\Omega f e^{-\imath x\eta}\, dx| \\
&= \quad |\int_\Omega f e^{-\imath x\eta}\, dx - \int_\Omega f\overline{v_1 v_2}\, dx| \\
&= \quad |\int_\Omega f\Big(e^{-\imath x\eta} - e^{-\imath x(\varepsilon^1+\varepsilon^2)}\overline{(1+w_R^1)(1+w_R^2)}\Big)\, dx| \\
&= \quad |\int_\Omega f e^{-\imath x\eta}\overline{(w_R^1 + w_R^2 + w_R^1 w_R^2)}\, dx| \\
&\leq \quad \|f\|_{L^\infty(\Omega)}\Big(\|w_R^1\|_{L^1(\Omega)} + \|w_R^2\|_{L^1(\Omega)} + \|w_R^1 w_R^2\|_{L^1(\Omega)}\Big) \\
&\leq \quad \|f\|_{L^\infty(\Omega)}\Big(\sqrt{meas(\Omega)}(\|w_R^1\|_{L^2(\Omega)} + \|w_R^2\|_{L^2(\Omega)}) + \|w_R^1\|_{L^2(\Omega)}\|w_R^2\|_{L^2(\Omega)}\Big) \\
&\to \quad 0 \text{ as } \mathbb{R} \to \infty\,,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality as well as (2.24) in the last line. Since $\eta \in \mathbb{R}^n$ was arbitrary, and the set of functions $\{x \mapsto e^{\imath x \overset{\mathbb{C}^n}{\cdot} \eta} \mid \eta \in \mathbb{R}^n\}$ is dense in $L^1(\Omega, \mathbb{C})$, f has to be equal to zero. This is a contradiction.

$\diamondsuit$

Note that it is essential to have $n \geq 3$ here. The latter assertion might not be valid for $n = 2$ (see Lemma 2.4).

2.5. Identifiability for the Schrödinger equation

Theorem 2.10. *There exists at most one $b \in L^\infty(\Omega, \mathbb{R}) \cap \mathcal{R}(G)$ that generates a given Neumann - Dirichlet map Λ_b^S for the Schrödinger equation (2.6).*

Proof. Assume that there exist $b^1, b^2 \in L^\infty(\Omega, \mathbb{R}) \cap \mathcal{R}(G)$ with $\Lambda_{b^1}^S = \Lambda_{b^2}^S$ but $b^1 \neq b^2$. This means, that for all $g \in C^2(\partial\Omega, \mathbb{C})$ the Dirichlet data of the solutions $v^j(g)$ of (2.23) with $j = 1, 2$, both with Neumann data g, i.e., $\frac{\partial v^1(g)}{\partial n}|_{\partial\Omega} = \frac{\partial v^2(g)}{\partial n}|_{\partial\Omega} = g$ and the normalization $\int_\Omega \sqrt{G^{-1}(b^j)}v^j(g)\, dx = 0$ (cf. (2.4), (2.5)), coincide.
Let v^1 be an arbitrary solution of (2.23) with $j = 1$, and set $g := \frac{\partial v^1}{\partial n}$, and $w := v^2(g) - v^1$. Then w satisfies

$$\begin{aligned}
\Delta w + b^2 w &= (b^1 - b^2)v^1 && \text{in } \Omega \\
w &= 0 && \text{on } \partial\Omega \\
\tfrac{\partial w}{\partial n} &= 0 && \text{on } \partial\Omega\,.
\end{aligned}$$

Let now v^2 be an arbitrary solution to (2.23) with $j=2$, then

$$\begin{aligned}
\int_\Omega (b^1-b^2)v^1v^2\ dx &= \int_\Omega (\Delta w + b^2 w)v^2\ dx \\
&= -\int_\Omega \nabla w \nabla v^2\ dx + \int_{\partial\Omega} \frac{\partial w}{\partial n} v^2\ d\Gamma + \int_\Omega b^2 w v^2\ dx \text{ (integr. by parts)} \\
&= \int_\Omega \Delta v^2 w\ dx - \int_{\partial\Omega} \frac{\partial v^2}{\partial n} w\ d\Gamma + \int_\Omega b^2 w v^2\ dx \text{ (integr. by parts)} \\
&= \int_\Omega (\Delta v^2 + b^2 v^2) w\ dx \ = \ 0 \text{ by (2.23) with } j=2.
\end{aligned}$$

This so-called *orthogonality relation*, by the density result Corollary 2.9, yields $b^1-b^2=0$.
$\diamondsuit$

2.6. Identifiability for the EIT problem

As mentioned above, to prove that a is uniquely determined by its Dirichlet-Neumann map, we will assume that a is known on the boundary and the normal derivative of a at the boundary vanishes. This is e.g., justified in the situation that the body under consideration is surrounded by a layer of material with known conductivity.

Theorem 2.11. *Given $a_0 \in C^2(\partial\Omega,\mathbb{R})$, There exists at most one $a \in C^2(\Omega,\mathbb{R})$ that satisfies $a|_{\partial\Omega} = a_0$, $\frac{\partial a}{\partial n}_{\partial\Omega} = 0$, and generates a given Neumann - Dirichlet map Λ^a for (2.1).*

Proof.
We want to show that $\Lambda_{a^1} = \Lambda_{a^2}$ implies $a^1=a^2$. For this purpose let $a^1, a^2 \in C^2(\Omega,\mathbb{R})$, $a^j|_{\partial\Omega} = a_0$, $\frac{\partial a^j}{\partial n}_{\partial\Omega} = 0$, set $b^j = G(a^j)$ according to (2.7), $j=1,2$, and assume that $\Lambda_{a^1} = \Lambda_{a^2}$.

For arbitrary $g \in C^2(\partial\Omega,\mathbb{C})$, consider solutions $v^2(g)$ of (2.23) with Neumann data $\frac{\partial v^j(g)}{\partial n}|_{\partial\Omega} = g$ and the normalization $\int_\Omega \sqrt{G^{-1}(b^j)}v^j(g)\ dx = 0$, $j=1,2$. We have to show that our assumption $\Lambda_{a^1} = \Lambda_{a^2}$ implies $v^1(g)|_{\partial\Omega} = v^2(g)|_{\partial\Omega}$. To do so, we observe that $u^j := (a^j)^{\frac{1}{2}} v^j(g)$ solves (2.1) with $a=a^j$ and the Neumann boundary conditions of u^1 and u^2 coincide:

$$a^1\frac{\partial u^1}{\partial n} = \frac{1}{2}(a^1)^{-\frac{1}{2}} \underbrace{\frac{\partial a^1}{\partial n}}_{=0} v^1 + \underbrace{(a^1)^{\frac{1}{2}}}_{=(a_0)^{\frac{1}{2}}} \underbrace{\frac{\partial v^1(g)}{\partial n}}_{=g} = \frac{1}{2}(a^2)^{-\frac{1}{2}}\frac{\partial a^2}{\partial n}v^2 + (a^2)^{\frac{1}{2}}\frac{\partial v^2(g)}{\partial n}| = a^2\frac{\partial u^2}{\partial n} \text{ on } \partial\Omega\,.$$

Therefore, $\Lambda_{a^1} = \Lambda_{a^2}$ implies $u^1=u^2$ on $\partial\Omega$. From this, $v^1(g) = v^2(g)$ on $\partial\Omega$ immediately follows, by our boundary assumptions on a^1, a^2. Since g was arbitrary, this implies $\Lambda^S_{b^1} = \Lambda^S_{b^2}$.

Hence, by Theorem 2.10, $b^1=b^2$ follows, which by Lemma 2.2 implies $a^1=a^2$.
$\diamondsuit$

3. An inverse source problem for a parabolic PDE

Consider the parabolic initial boundary value problem

$$\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta u &=& q(x,t) & \text{in } \Omega \times (0,T) \\ u(x,0) &=& u_0(x) & x \in \Omega \\ u(x,t) &=& h(x,t) & x \in \partial\Omega \,. \end{array} \tag{3.1}$$

This can be viewed as a model for the evolution of the temperature u inside a domain Ω, where the boundary temperature h and the initial temperature u_0 are known and the term $q(x,t)$ represents the heat sources in the domain. The direct problem is here to compute u in $\Omega \times (0,T)$, given q, u_0, h. Under certain conditions on the data, there exists a smooth solution to this problem.

Theorem 3.1. *If $q \in C^1(\overline{\Omega \times (0,T)})$, $h \in C^{3,\frac{3}{2}}(\overline{\partial\Omega \times (0,T)})$, $u_0 \in C^3(\overline{\Omega})$ and the compatibility conditions*

$$h(x,0) = u_0(x)\,, \quad \frac{\partial h}{\partial t}(x,0) - \Delta u_0(x) = q(x,0) \quad x \in \partial\Omega$$

hold, then there exists a solution $u \in C^{2,1}(\overline{\Omega \times (0,T)})$ of (3.1).

Proof: see Ladyshenskaja-Solonnikov-Ural'ceva.

$\diamondsuit$

Moreover, the following maximum principle holds

Theorem 3.2. *Let*

$$\frac{\partial u}{\partial t} - \Delta u \geq 0 \text{ in } Q\,.$$

If the maximum M of u is assumed in an interior point $P = (y, \tilde{t})$ of Q, then $u \equiv M$ in the connected component of the set $\{(x,t) \in Q \mid t = \tilde{t}\}$ that contains P. Moreover, for any point $\tilde{P}$ that can be connected with P by a combination of time constant paths with space constant paths going forwards in time, there holds $u(\tilde{P}) = M$.

Proof: see Protter-Weinberger.

$\diamondsuit$

A simple consequence of this theorem is uniqueness of a solution to (3.1).

3.1. Inverse source problem

We are interested in the situation that the source term takes the form

$$q(x,t) = \alpha(x,t)f(x)\,. \tag{3.2}$$

Here α is assumed to be known and we wish to identify a spatial distribution of heat sources f. For example $\alpha \equiv 1$ stands for a time independent source term, $\alpha = \alpha(t)$ with $\alpha'(t) \geq 0$ represents the situation of spatially distributed heat sources that are gradually switched on. We will show that f can be uniquely determined from just one (as opposed to the previous chapter) additional set of measurements of u namely final temperature measurements

$$u_T(x) := u(x,T) \quad x \in \Omega\,.$$

In the following we will assume that

$$\begin{array}{l} \Omega \text{ is a bounded connected } n \text{ dimensional } C^3 \text{ smooth domain} \\ \alpha, \frac{\partial\alpha}{\partial t} \in C^1(\overline{\Omega \times (0,T)}) \text{ and } \alpha > 0\,,\ \frac{\partial\alpha}{\partial t} \geq 0 \text{ in } \Omega \times (0,T) \end{array} \tag{3.3}$$

as well as

$$f \in C^1(\overline{\Omega})\,.$$

3.2. Orthogonality

Also here, an orthogonality relation plays an important role in the uniqueness proof.

Lemma 3.3. *If (3.3) holds, $f^1, f^2 \in C^1(\overline{\Omega})$, the assumptions of Theorem (3.1) are satisfied with $q = \alpha f^j$, and u^j solves*

$$\begin{array}{rcll} \frac{\partial u}{\partial t} - \Delta u &=& \alpha(x,t)f^j(x) & \textit{in } \Omega \times (0,T) \\ u(x,0) &=& u_0(x) & x \in \Omega \\ u(x,t) &=& h(x,t) & x \in \partial\Omega\,. \end{array} \tag{3.4}$$

$j = 1,2$, with additionally

$$u^1(x,T) = u^2(x,T)\,, \tag{3.5}$$

then

$$\int_0^T \int_\Omega \alpha(x,t)(f^1(x) - f^2(x))v(x,t)\,dx\,dt = 0 \tag{3.6}$$

holds for any solution v of the adjoint problem

$$\begin{array}{rcll} \frac{\partial v}{\partial t} + \Delta v &=& 0 & \textit{in } \Omega \times (0,T) \\ v(x,t) &=& 0 & x \in \partial\Omega\,. \end{array} \tag{3.7}$$

with final data

$$\phi := v(\cdot,T) \in \mathcal{D}(\Omega)\,. \tag{3.8}$$

Here, $\mathcal{D}(\Omega)$ denotes the space of compactly supported infinitely often differentiable functions on Ω.

Proof. The difference $w := u^1 - u^2$ satisfies the parabolic PDE (3.1) with homogeneous initial and boundary data $u_0 = h = 0$ and source term $q = \alpha(f^1 - f^2)$. Multiplying this PDE with v, integrating over $\Omega \times (0,T)$ and using integration by parts with respect to time and space (Green's identity), respectively, in order to move all derivatives from w to v, we obtain

$$\begin{aligned}
\int_0^T \int_\Omega \alpha(f^1 - f^2) v \, dx \, dt &= \int_0^T \int_\Omega (\frac{\partial w}{\partial t} - \Delta w) v \, dx \, dt \\
&= -\int_0^T \int_\Omega w \frac{\partial v}{\partial t} \, dx \, dt + (\int_\Omega w(x,T) v(x,T) \, dx - \int_\Omega w(x,0) v(x,0) \, dx) \\
&\quad - \int_0^T \int_\Omega w \Delta v \, dx \, dt + \int_0^T \int_{\partial\Omega} (-\frac{\partial w}{\partial n} v + w \frac{\partial v}{\partial n}) \, d\Gamma \, dt \\
&= -\int_0^T \int_\Omega w (\frac{\partial v}{\partial t} + \Delta v) \, dx \, dt = 0 \,,
\end{aligned}$$

since w satisfies homogeneous initial and boundary conditions as well as homogeneous end conditions due to (3.5), and since (3.7) holds for v.
♢

3.3. Monotonicity principles

For solutions to (3.1), the following monotonicity principle holds:

Theorem 3.4. *Let the conditions of Theorem 3.1 and (3.3) hold and let u solve (3.1). If $q \geq 0$, $h \geq 0$, $u_0 \geq 0$, then there exists a number $\theta \in [0,T]$ such that*

$$u = 0 \ \text{in} \ \Omega \times (0,\theta] \ \text{and} \ u > 0 \ \text{in} \ \Omega \times (\theta, T)$$

Proof: see Isakov
♢

Backwards parabolic problems like (3.7) with end conditions can be transformed to usual parabolic problems with initial conditions by replacing t with $T - t$. Therewith the previously cited existence and uniqueness results also apply to v in (3.7) (note that $\phi \in \mathcal{D}(\Omega)$ implies the compatibility conditions

$$\phi(x,T) = 0 \,, \quad \Delta\phi(x) = 0 \quad x \in \partial\Omega \tag{3.9}$$

to hold.) Additionally we have

Lemma 3.5. *For any compact subset K of $\Omega \times (0,T)$ and any subset Ω_1 of Ω with $meas(\Omega_1) > 0$ there exists a positive constant $\epsilon = \epsilon(K, \Omega_1) > 0$ such that*

$$v \geq \epsilon \ \text{on} \ K \,,$$

for any solution v to (3.7) with final value ϕ satisfying

$$\phi \geq 0 \text{ on } \Omega\,, \quad \phi \geq 1 \text{ on } \Omega_1\,.$$

Proof: see Isakov

$\diamondsuit$

Corollary 3.6. *For any compact subset S of Ω there exists a positive constant $\epsilon = \epsilon(S, \Omega_1) > 0$ such that*

$$v(\cdot, 0) \geq \epsilon \text{ on } S\,,$$

for v, ϕ be as in Lemma 3.5.

Proof. The assertion follows by application of the previous Lemma to (3.7) on a slightly augmented time interval $(-\eta, T)$ in place of $(0, T)$ with some $\eta > 0$.

$\diamondsuit$

3.4. Uniqueness

To conclude uniqueness from the orthogonality relation (3.6) together with monotonicity arguments, we require a few more technical ingredients:

Lemma 3.7. *Let v solve (3.7) with $\phi \in \mathcal{D}(\Omega)$. Then $w = \frac{\partial v}{\partial t}$ solves (3.7) with ϕ replaced by $-\Delta\Phi$.*

Proof. The result is obtained by formally differentiating (3.7) with respect to t and observing that for $\phi \in \mathcal{D}$ the compatibility conditions (3.9) are automatically satisfied not only by ϕ but also by $-\Delta\phi$.

$\diamondsuit$

Theorem 3.8. *Let A be a closed subset of $\mathbb{R}^n$, G open with $A \subset G$. There exists an infinitely often differentiable function β with the properties*

- $0 \leq \beta(x) \leq 1 \quad \forall x \in \mathbb{R}^n$
- $\beta(x) = 1 \quad \forall x \in A$
- $\beta(x) = 0 \quad \forall x \in \mathbb{R}^n \setminus G$

Corollary 3.9. *Let $\tilde{\Omega}$ be an open subset of $\Omega \subseteq \mathbb{R}^n$, then there exists an open subset $\hat{\Omega} \subset \tilde{\Omega}$ and a sequence of functions $(\phi_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}(\Omega)$ such that*

- $0 \leq \phi_k(x) \leq 1 \quad \forall x \in \Omega$

- $\phi_k(x) = 1 \quad \forall x \in \hat{\Omega}$
- $\phi_k \to \mathbf{I}_{\tilde{\Omega}}$ *as* $k \to \infty$ *in* L^1

Proof. We set $G := \tilde{\Omega}$ and choose a sequence of closed sets A_k containing $\hat{\Omega}$ such that $meas((\tilde{\Omega} \setminus A_k) \cup (A_k \setminus \tilde{\Omega})) \to 0$ as $k \to \infty$.
$\diamondsuit$

Therewith we can prove unique identifiability of the souce distribution f from final temperature measurements.

Theorem 3.10. *Under assumption (3.3) there exists at most one source distribution* $f \in C^1(\overline{\Omega})$ *that induces a given final temperature* $u(\cdot, T)$ *where* u *solves (3.1) with (3.2).*

Proof. Assume that there exist two different source distributions f^1, f^2 leading to two (possibly different) temperatures u^1, u^2 in Ω but the same final temperature. We set $f := f^1 - f^2$, $\Omega_+ := \{x \in \Omega \mid f(x) > 0\}$ $\Omega_- := \{x \in \Omega \mid f(x) < 0\}$. Since f is continuous, both Ω_+ and Ω_- are open sets. Moreover both sets are nonempty for the following reasons: If Ω_- were empty, then $f \geq 0$ would hold on all of Ω and therewith, by the monotonicity principle, the difference $u^1 - u^2$ (which satisfies (3.1) with $q = \alpha f$ and homogeneous initial and boundary conditions) would have to be nonnegative in $\Omega \times (0, T)$. But we have assumed that this difference vanishes for $t = T$, so there has to hold $\theta = T$ in Theorem 3.4 applied to $u^1 - u^2$, which implies $u^1 - u^2 \equiv 0$ on all of $\Omega \times (0, T)$. The same arguments can be repeated with Ω_- replaced by Ω_+ and f by $-f$. Therewith, we have shown

$$\Omega_+, \Omega_- \text{ are open and nonempty.}$$

Now we choose an open set $\Omega_1 \subset \Omega_+$ with $meas(\Omega_1) > 0$ and make use of Corollary 3.9 with $\tilde{\Omega} = \Omega_+$ and $\hat{\Omega} := \Omega_1$. We denote by v_k the solution of (3.7) with $\phi := \phi_k$ and make use of Lemma 3.7, which enables us to insert $\frac{\partial v_k}{\partial t}$ in place of v in the orthogonality relation (3.6) of Lemma 3.3 to obtain

$$\begin{aligned}
0 = &\int_0^T \int_\Omega \alpha(x,t) f(x) \frac{\partial v_k}{\partial t}(x,t)\, dx\, dt = (\text{intergration by parts wrt. } t) \\
= &- \int_0^T \int_\Omega \frac{\partial \alpha}{\partial t}(x,t) f(x) v_k(x,t)\, dx\, dt + \int_\Omega \alpha(x,T) f(x) v_k(x,T)\, dx - \int_\Omega \alpha(x,0) f(x) v_k(x,0)\, dx \\
= &- \int_0^T \int_{\Omega_+} \frac{\partial \alpha}{\partial t}(x,t) f(x)\, dx\, dt + \int_{\Omega_+} \alpha(x,T) f(x) \phi_k(x)\, dx - \int_{\Omega_+} \alpha(x,0) f(x)\, dx \\
&+ \underbrace{\int_0^T \int_{\Omega_+} \frac{\partial \alpha}{\partial t}(x,t) f(x)(1 - v_k(x,t))\, dx\, dt}_{\geq 0} + \underbrace{\int_{\Omega_+} \alpha(x,0) f(x)(1 - v_k(x,0))\, dx}_{\geq 0} \\
&- \int_0^T \int_{\Omega_-} \frac{\partial \alpha}{\partial t}(x,t) f(x) v_k(x,t)\, dx\, dt + \int_{\Omega_-} \alpha(x,T) f(x) \phi_k(x)\, dx - \int_{\Omega_-} \alpha(x,0) f(x) v_k(x,0)\, dx
\end{aligned} \tag{3.10}$$

where the fourth and fifth intergral on the right hand side are nonnegative since $1-\phi_k$ and therewith by Theorem 3.4 also $1-v_k$ is nonnegative, and so are α, $\frac{\partial\alpha}{\partial t}$ by assumption as well as f on Ω_+.

Moreover, we have

$$\lim_{k\to\infty}\int_{\Omega_+}\alpha(x,T)f(x)\phi_k(x)\,dx=\int_{\Omega_+}\alpha(x,T)f(x)\,dx \tag{3.11}$$

since

$$|\int_{\Omega_+}\alpha(x,T)f(x)(\phi_k(x)-1)\,dx|\le\|\alpha\|_{L^\infty}\|f\|_{L^\infty}\|\phi_k-\mathbf{1}_{\Omega_+}\|_{L^1}\,dx\to 0 \text{ as } k\to\infty$$

and similarly we obtain

$$\lim_{k\to\infty}\int_{\Omega_-}\alpha(x,T)f(x)\phi_k(x)\,dx=0\,. \tag{3.12}$$

Taking the limit $k\to\infty$ in (3.10) and using (3.11), (3.12), as well as the fact that

$$-\int_0^T\int_{\Omega_+}\frac{\partial\alpha}{\partial t}(x,t)f(x)\,dx\,dt+\int_{\Omega_+}\alpha(x,T)f(x)\,dx-\int_{\Omega_+}\alpha(x,0)f(x)\,dx=0$$

by the fundametal theorem of calculus (Hauptsatz der Integral-und Differentialrechnung) we arrive at

$$0\ge\lim_{k\to\infty}\int_0^T\int_{\Omega_-}\frac{\partial\alpha}{\partial t}(x,t)(-f(x))v_k(x,t)\,dx\,dt+\int_{\Omega_-}\alpha(x,0)(-f(x))v_k(x,0)\,dx\,. \tag{3.13}$$

Note that both terms on the right hand side are nonnegative since v_k is nonnegative by the monotonicity principle, α and $\frac{\partial\alpha}{\partial t}$ are nonnegative by assumption, and f is nonpositive on Ω_-. To derive a contradiction from (3.13), it therefore suffices to prove that one of the two terms is strictly positive. For this purpose we distinguish between two cases:

a) There exists an open nonempty subset Ω_2 of Ω_- such that $\alpha(\cdot,0)>0$ on Ω_-.

b) There exists a compact subset K of $\Omega_-\times(0,T)$ such that $\frac{\partial\alpha}{\partial t}>0$ on K.

As a matter of fact these are the only two possible cases, since if $\alpha(\cdot,0)$ vanishes on Ω_-, and at the same time $\frac{\partial\alpha}{\partial t}$ vanishes on all compact subsets of $\Omega_-\times(0,T)$, then also $\alpha(x,t)=\alpha(x,0)+\int_0^t\frac{\partial\alpha}{\partial t}(x,\tau)\,d\tau$ has to vanish on all compact subsets of $\Omega_-\times(0,T)$, which gives a contradiction to our asumption (3.3).

Consider first case a): We can choose a compact subset S of Ω_2 with $meas(S)>0$ and apply Corollary 3.6 (using $\phi_k=1$ on Ω_1) to conclude that $v_k(\cdot,0)\ge\epsilon(S,\Omega)>0$ in S. Therewith,

$$\begin{aligned}&\int_0^T\int_{\Omega_-}\frac{\partial\alpha}{\partial t}(x,t)(-f(x))v_k(x,t)\,dx\,dt+\int_{\Omega_-}\alpha(x,0)(-f(x))v_k(x,0)\,dx\\&\quad\ge\int_S\alpha(x,0)(-f(x))v_k(x,0)\,dx\ge\epsilon(S,\Omega)\int_S\alpha(x,0)(-f(x))\,dx=:C_a>0\end{aligned}$$

which gives a contradiction to (3.13).

In case b), we apply Lemma 3.5 to conclude $v_k \geq \epsilon(K,\Omega) > 0$ in K and therewith

$$\begin{aligned}
&\int_0^T \int_{\Omega_-} \frac{\partial\alpha}{\partial t}(x,t)(-f(x))v_k(x,t)\,dx\,dt + \int_{\Omega_-} \alpha(x,0)(-f(x))v_k(x,0)\,dx \\
&\quad \geq \int_K \frac{\partial\alpha}{\partial t}(x,t)(-f(x))v_k(x,t)\,dx\,dt \geq \epsilon(K,\Omega)\int_K \frac{\partial\alpha}{\partial t}(x,t)(-f(x))\,dx\,dt =: C_b > 0
\end{aligned}$$

in contradiction to (3.13).
$\diamondsuit$

4. An inverse coefficient problem for a hyperbolic PDE

In this chapter we consider a spatially one dimensional exmple of a parameter identification problem for the wave equation

$$u_{tt} - u_{xx} + cu = f \text{ in } Q := (0,\infty) \times (0,T)\,, \tag{4.1}$$

with initial and/or boundary conditions

$$u_x(0,t) = h(t)\,, \quad t \in (0,T)\,, \tag{4.2}$$

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x) \quad x \in (0,\infty)\,. \tag{4.3}$$

The searched for parameter c is this time again contained in the differential operator defining the PDE (as opposed to Chapter 3) but only along with zero order terms of the measured state u (as opposed to Chapter 2).

Before stating the inverse problem, we will formulate a generalized solution concept that is appropriate for nonsmooth initial and/or boundary data: We call u a generalized solution to (4.1), (4.2), (4.3), if u is piecewise continuous on Q, continuous on $\{0\} \times (0,T)$ and satisfies

$$\int_Q u(v_{tt} - v_{xx} + cv)\,dx\,dt = \int_Q fv\,dx\,dt + \int_0^\infty u_1 v(\cdot,0)\,dx - \int_0^T hv(0,\cdot)\,dt - \int_0^\infty u_0 v_t(\cdot,0)\,dx \tag{4.4}$$

for all $v \in C^2(\overline{Q})$ with $v(\cdot,T) = v_t(\cdot,T) = v_x(0,\cdot) = 0$. Note that this solution concept is still more general than the usual weak solution concept in Sobolev spaces,

$$\begin{array}{l} u \in H^1(Q)\,, \quad u(\cdot,0) = u_0\,, \quad \text{and} \\ \int_Q(-u_t v_t + u_x v_x + cv)\,dx\,dt = \int_Q fv\,dx\,dt + \int_0^\infty u_1 v(\cdot,0)\,dx - \int_0^T hv(0,\cdot)\,dt \\ \forall v \in H^1(Q)\,, \quad v(\cdot,T) = 0\,, \end{array} \tag{4.5}$$

with $c \in L^\infty(Q)$, $u_0 \in H^1(0,\infty)$, $u_1 \in L^2(0,\infty)$, $f \in L^2(Q)$, $h \in H^{\frac{1}{2}}(\{0\} \times (0,T))$, (see, e.g., Lions&Magenes), since we will deal with nonsmooth boundary conditions $h \notin H^{\frac{1}{2}}(\{0\} \times (0,T)$ and even $h \notin H^{-\frac{1}{2}}(\{0\} \times (0,T)$ for the inverse problem. More precisely, we will consider a delta pulse concentrated at zero as Neumann boundary condition, while restricting ourselves to homogeneous initial conditions and right hand side for simplicity:

$$h = \delta_0\,, \quad u_0 = u_1 = 0\,, \quad f = 0\,. \tag{4.6}$$

Existence of a solution in this generalized sense can be shown as follows:

Theorem 4.1. *For $c \in C(0,\infty)$, there exists a solution $u \in \hat{u} + C(\overline{Q})$ of (4.4) with homogeneous initial data and u is unique in $\hat{u} + C(\overline{Q})$, where*

$$\hat{u}(x,t) = \begin{cases} 1 & \text{for } x < t \\ 0 & \text{else.} \end{cases}$$

Proof. We observe that $\hat{u}$ solves (4.4) with $c = 0$ and the delta pulse boundary conditions (4.6) and (4.6). We now search for a solution of the form $u = \hat{u} + U$; from the properties of $\hat{u}$ it follows that U has to satisfy

$$\begin{aligned} U_{tt} - U_{xx} + cU &= -c\hat{u} \text{ in } Q := (0,\infty) \times (0,T)\,, \\ U_x(0,t) &= 0\,, \quad t \in (0,T)\,, \\ U(x,0) = U_t(x,0) &= 0 \quad x \in (0,\infty)\,, \end{aligned}$$

in a generalized sense. Note that this problem is solvable even in the weak sense (4.5), since the right hand side is in $L^2(Q)$. For the sake of completeness, we here carry out an existence proof based on a fixed point argument. We extend U and c to the negative x axis by symmetric reflection $U(-x,t) = U(x,t)$, $c(-x,t) = c(x,t)$ (note that therewith U satisfies the homogeneous Neumann boundary conitions). Using d'Alembert's formula for the solution of $U_{tt} - U_{xx} = c(\hat{u} + U)$, we arrive at the Volterra type integral equation

$$U(x,t) = -\frac{1}{2}\int_{\triangle(x,t)} c(\hat{u} + U)\, d(y,s)\,, \tag{4.7}$$

where $\triangle(x,t)$ is the characteristic triangle

$$\triangle(x,t) = \{(y,s) \mid |x-y| < |t-s| \ \wedge \ s < t\}\,. \tag{4.8}$$

We will now show that (4.7) has a unique solution $U \in C(\overline{Q})$. For this purpose, we define the linear operator $B : C(\overline{Q}) \to C(\overline{Q})$ by

$$(BV)(x,t) = -\frac{1}{2}\int_{\triangle(x,t)} cV\, d(y,s)\,.$$

Using induction (see exercises!) it can be shown that

$$\|B^k V\|_{L^\infty} \le \frac{C^{2k}T^{2k}}{(2k)!}\|V\|_{L^\infty}\,,$$

where

$$C := \sqrt{\|c\|_{L^\infty}}\,.$$

Hence, the operator $I - B : C(\overline{Q}) \to C(\overline{Q})$ is invertible with inverse $(I-B)^{-1} = \sum_{k=0}^\infty B^k$, whose norm is bounded by

$$\|(I-B)^{-1}\|_{C(\overline{Q})\to C(\overline{Q})} \le \sum_{k=0}^\infty \frac{C^{2k}T^{2k}}{(2k)!} = e^{CT}\,.$$

Hence the unique solution U of (4.7) can be obtained from $(I-B)U = B\hat{u}$ via the inverse operator $(I-B)^{-1}$,

$$U = (I-B)^{-1} B\hat{u} \tag{4.9}$$

and obeys the bound

$$\|U\|_{L^\infty} \leq \frac{1}{2} C^2 T^2 e^{CT} . \tag{4.10}$$

Moreover, for solutions U^{c^1}, U^{c_2} with different coefficients $c_1, c_2 \in C(Q)$, we get the Lipschitz estimate

$$\|U^{c^1} - U^{c_2}\|_{L^\infty} \leq (1 + \frac{1}{2}\|c_1\|_{L^\infty} T^2 e^{\sqrt{\|c_1\|_{L^\infty}} T}) \frac{1}{2} T^2 e^{\sqrt{\|c_2\|_{L^\infty}} T} \|c_1 - c_2\|_{L^\infty} , \tag{4.11}$$

which can be seen by subtracting the identities (4.9) for U^{c^1}, U^{c_2}:

$$\begin{aligned} U^{c^1} - U^{c_2} &= (I - B^{c^1})^{-1} B^{c^1} \hat{u} - (I - B^{c^2})^{-1} B^{c^2} \hat{u} \\ &= \Big((I - B^{c^1})^{-1} - (I - B^{c^2})^{-1} \Big) B^{c^1} \hat{u} + (I - B^{c^2})^{-1} \Big(B^{c^1} - B^{c^2} \Big) \hat{u} \\ &= (I - B^{c^2})^{-1} \Big(B^{c^1} - B^{c^2} \Big) (I - B^{c^1})^{-1} B^{c^1} \hat{u} + (I - B^{c^2})^{-1} \Big(B^{c^1} - B^{c^2} \Big) \hat{u} . \end{aligned}$$

Differentiating (4.7) wrt. x and t, respectively, we even get $U \in C^1$.

$\diamondsuit$

As a consequence of the finite speed of propagation for hyperbolic PDEs, the following Lemma holds

Lemma 4.2. *Let u be a weak solution, i.e., let (4.5) hold and let, for some $(x,t) \in \mathbb{R}^+ \times [0,T]$*

$f = 0$ on $\triangle(x,t) \cap Q$, $u_0 = u_1 = 0$ on $\triangle(x,t) \cap \mathbb{R} \times \{0\}$, $h = 0$ on $\triangle(x,t) \cap \{0\} \times [0,T]$

with $\triangle(x,t)$ as in (4.8). Then also

$$u = 0 \text{ on } \triangle(x,t) \cap Q .$$

Remark 4.3. Note that by $\hat{u} = 0$ in $\{(x,t) \,|\, x > t\}$ Lemma 4.2 implies that for U as in the proof of Theorem 4.1

$$U(x,t) = 0 \quad \text{for } x > t . \tag{4.12}$$

holds and therefore, by continuity of U, also

$$U(x,x) = 0 \quad \forall x \geq 0 \tag{4.13}$$

4.1. The inverse problem

We here consider the inverse problem of identifying $c \in C(0,\infty)$ in (4.5) from additional boundary measurements

$$u(0,t) = g(t) \quad t \in (0,T) . \tag{4.14}$$

From the proof of Theorem 4.1 it follows that g has to satisfy the initial conditions

$$g(0) = \lim_{t\to 0^+} \hat{u}(0,t) + U(0,t) = 1 , \quad g'(0) = \lim_{t\to 0^+} \hat{u}_t(0,t) + U_t(0,t) = 0 . \tag{4.15}$$

4.2. Formulation as a Volterra integral equation

Theorem 4.4. *Let (4.15) hold. Then the inverse problem is equivalent to the nonlinear Volterra integral equation*

$$c(\tau) + 2\int_0^\tau c(s)U_t^c(s, 2\tau - s)\,ds = -2g''(\tau) \quad \tau \in (0, T/2)\,. \tag{4.16}$$

Proof. We assume that c solves the inverse problem, fix $\tau \in (0, T/2)$, and define v by

$$v(x,t) = \begin{cases} 1 & \text{for } x + t < 2\tau < T \\ 0 & \text{else.} \end{cases}$$

This function v can be approximated by smooth functions v_ϵ constructed as

$$v_\epsilon(x,t) = \phi_\epsilon(x+t)$$

with

$$\phi_\epsilon \in C^\infty\,, \quad \phi_\epsilon \equiv 1 \text{ on } (0, 2\tau - \epsilon)\,, \quad \phi_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} \mathbf{I}_{(0,2\tau)} \text{ in } L^1\,, \quad \phi_\epsilon' \stackrel{\epsilon\to 0}{\longrightarrow} -\delta_{2\tau} \text{ in } L^1\,,$$

which implies

$$\begin{gathered} v_\epsilon \in C^\infty\,, \quad v_\epsilon = v \text{ on } \{(x,t)\,|\,x + t \le 2\tau - \epsilon\}\,, \quad v_\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} v \text{ in } L^1\,, \\ v_{\epsilon\,x}(0,\cdot) \stackrel{\epsilon\to 0}{\longrightarrow} -\delta_{2\tau} \text{ in } L^1\,, \quad v_\epsilon(\cdot, T) = v_{\epsilon\,t}(\cdot, T) = 0\,. \end{gathered}$$

Since U is a weak solution in the sense of (4.5) of

$$\begin{aligned} U_{tt} - U_{xx} &= -c(\hat{u} + U) \quad \text{in } Q \\ U_x(0,\cdot) &= 0 \\ U(\cdot,0) = U_t(\cdot,0) &= 0\,, \end{aligned}$$

we obtain

$$\begin{aligned} &-\int_Q c(\hat{u} + U)v_\epsilon\,dx\,dt \\ &\quad= \int_Q (-U_t v_{\epsilon\,t} + U_x v_{\epsilon\,x})\,dx\,dt \\ &\quad= \int_Q U(v_{\epsilon\,tt} - v_{\epsilon\,xx})\,dx\,dt - \int_0^\infty (U(x,T)v_{\epsilon\,t}(x,T) - U(x,0)v_{\epsilon\,t}(x,0))\,dx \\ &\qquad- \int_0^T U(0,t)v_{\epsilon\,x}(0,t)\,dt \\ &\quad= -\int_0^T U(0,t)v_{\epsilon\,x}(0,t)\,dt\,, \end{aligned}$$

where we have used integration by parts (note that v_ϵ is sufficiently smooth) and the fact that v_ϵ being a function of $x+t$ solves the homogeneous wave equation. Taking the limit $\epsilon \to 0$ and using the above mentioned properties of v_ϵ, as well as the boundary values $g = u(0,\cdot) = \hat{u}(0,\cdot) + U(0,\cdot) = 1 + U(0,\cdot)$, we arrive at

$$\int_Q c(\hat{u} + U)v\,dx\,dt = -U(0,2\tau) = -g(2\tau) + 1\,. \tag{4.17}$$

Since $\hat{u}+U$ vanishes on $\{(x,t)\,|\,x \geq t\}$ (see the proof of Theorem 4.1), and by the definition of v, the integral on the left hand side only goes over the triangle with vertices at $(0,0)$, (τ,τ), $(0,2\tau)$, thus (4.17) becomes

$$\int_0^\tau c(x) \int_x^{2\tau - x} (1 + U(x,t))\,dt\,dx = -g(2\tau) + 1\,.$$

Differentiating both sides with respect to τ gives

$$\begin{aligned} -2g'(2\tau) &= c(\tau) \int_\tau^{2\tau-\tau} (1 + U(x,t))\,dt\,dx + \int_0^\tau c(x) 2(1 + U(x, 2\tau - x))\,dx \\ &= 2\int_0^\tau c(x) 2(1 + U(x, 2\tau - x))\,dx\,. \end{aligned}$$

Another differentiation wrt τ and the fact that $U(\tau,\tau) = 0$ yields

$$-4g''(2\tau) = 2c(\tau) + 2\int_0^\tau c(x) 2U_t(x, 2\tau - x)\,dx\,, \tag{4.18}$$

i.e., (4.16).

To show the other direction, we assume that c solves (4.16). Then the boundary values $\tilde{g} = u(0,\cdot)$ of the solution u according to Theorem 4.1 corresponding to c have to coincide with g: Namely, by (4.15), the steps from (4.17) to (4.18) can be reversed. On the other hand, we can repeat the arguments that led to (4.17) with g replaced by $\tilde{g}$. Thus

$$\tilde{g}(2\tau) - 1 = \int_Q c(\hat{u} + U)v\,dx\,dt = g(2\tau) - 1$$

for all $\tau \leq \frac{T}{2}$, which implies that $\tilde{g} = g$ and therewith c solves the inverse problem.
$\diamondsuit$

Remark 4.5. Considering the equivalent formulation (4.16) of the inverse problem, we see that it involves only values of $c(x)$ for $x \in (0, \frac{T}{2})$. Indeed, also $U_t^c(x, 2\tau - x)$ for $x \in (0,\tau)$, $\tau \in (0, \frac{T}{2})$, as appearing in the integral only depends on values of c in the interval $x \in (0, \frac{T}{2})$: The set $B = \{(x, 2\tau - x)\,|\,x \in (0,\tau)\,,\ \tau \in (0, \frac{T}{2})$ is equal to the triangle with corners $(0,0)$, $(\frac{T}{2}, \frac{T}{2})$, $(0,T)$. We first of all consider the lower part of this triangle, namely the one with corners $(0,0)$, $(\frac{T}{2},\frac{T}{2})$, $(0,\frac{T}{2})$. Since we know that $U^c=0$ on $\{x\geq t\}$, we can replace the initial- boundary value problem

$$\begin{aligned} U_{tt}-U_{xx}+cU &= -c\hat{u} \text{ in } (0,\infty)\times(0,\tfrac{T}{2})\,, \\ U_x(0,t) &= 0\,, \quad t\in(0,\tfrac{T}{2})\,, \\ U(x,0)=U_t(x,0) &= 0 \quad x\in(0,\infty)\,, \end{aligned}$$

on the semi-unbounded domain $(0,\infty)\times(0,\frac{T}{2})$ by the IBVP

$$\begin{aligned} U_{tt}-U_{xx}+cU &= -c\hat{u} \text{ in } (0,\tfrac{T}{2})\times(0,\tfrac{T}{2})\,, \\ U_x(0,t)=U(\tfrac{T}{2},t) &= 0\,, \quad t\in(0,\tfrac{T}{2})\,, \\ U(x,0)=U_t(x,0) &= 0 \quad x\in(0,\tfrac{T}{2})\,, \end{aligned}$$

on the bounded domain $(0,\frac{T}{2})\times(0,\frac{T}{2})$, that is also uniquely weakly solvable with solution by uniqueness coinciding with U^c on $(0,\frac{T}{2})\times(0,\frac{T}{2})$. This solution obviously only involves values $c(x)$ for $x\in(0,\frac{T}{2})$. To treat the upper part of the triangle B, namely the one with corners $(\frac{T}{2},\frac{T}{2})$, $(0,\frac{T}{2})$, $(0,T)$ we just shift the initial time from zero to $\frac{T}{2}$ and use the values of the just determined U^c from the lower part on $x\in[0,\frac{T}{2}]$ as initial values. Indeed, by Lemma 4.2, these values on $[0,\frac{T}{2}]\times\{\frac{T}{2}\}$ suffice for determining U^c un this upper part of B as one easily convinces oneself by making a plot. Again, only values of $c(x)$ for $x\in(0,\frac{T}{2})$ are involved.

4.3. Uniqueness

The equivalent formulation from Theorem 4.4 can now be used to prove identifiability.

Theorem 4.6. *Let $g\in C^2(0,T)$ and (4.15) hold.*
Then a solution c to the inverse problem is unique on $(0,\frac{T}{2})$.
Moreover, there exists a $T_0\in(0,T]$ such that a solution c of the inverse problem exists on $(0,\frac{T_0}{2})$.

Proof. First of all, we restrict out attention to a sufficiently small subinterval $(0,T_0)$ of $(0,T)$ with T_0 satisfying (4.20), (4.21) for some $R>2\|g''\|_{L^\infty}$ (e.g., $R:=4\|g''\|_{L^\infty}$).

Obviously all assertions made so far remain valid with T replaced by T_0. To prove existence and uniqueness of c on $(0,\frac{T_0}{2})$, we will show that the operator $A: B_R(0)\subset L^\infty(\mathbb{R}^+)\to L^\infty(\mathbb{R}^+)$ defined by

$$(A(c))(\tau)=\begin{cases} -2g''(2\tau)-2\int_0^\tau c(s)U_t^c(s,2\tau-s)\,ds & \text{for } \tau\leq\frac{T_0}{2} \\ 0 & \text{else} \end{cases} \tag{4.19}$$

is a self-mapping and a contraction for all fixed $R>2\|g''\|_{L^\infty}$ as long as $T_0=T_0(R)$ is sufficiently small, such that

$$2\|g''\|_{L^\infty}+T_0R^2(1+\frac{1}{2}RT_0^2e^{\sqrt{R}T_0})\leq R \tag{4.20}$$

and

$$T_0R(1+\frac{1}{2}RT_0^2e^{\sqrt{R}T_0})(2+R\frac{1}{2}T_0e^{\sqrt{R}T_0})<1\,. \tag{4.21}$$

Namely, from (4.7) it follows by differentiation with respect to t that

$$\begin{aligned} U_t^c(x,t) &= -\frac{1}{2}\frac{d}{dt}\int_0^t\int_{x-t+s}^{x+t-s} c(y)(\hat{u}(y,s)+U^c(y,s))\,dy\,ds \\ &= -\frac{1}{2}\int_0^t\Big(c(x+t-s)(\hat{u}(x+t-s,s)+U^c(x+t-s,s)) \\ &\qquad +c(x-t+s)(\hat{u}(x-t+s,s)+U^c(x-t+s,s))\Big)\,ds \end{aligned}$$

Inserting this into (4.19), we can estimate

$$\begin{aligned} \|Ac\|_{L^\infty(0,T_0/2)} &\le 2\|g''\|_{L^\infty}+2\frac{T_0}{2}\|c\|_{L^\infty(0,T_0/2)}\|U_t^c\|_{L^\infty((0,T_0/2)\times(0,T_0))} \\ &\le 2\|g''\|_{L^\infty}+T_0\|c\|_{L^\infty(0,T_0/2)}\|c\|_{L^\infty(0,3T_0/2)}\|\hat{u}+U\|_{L^\infty((0,3T_0/2)\times(0,T_0))} \\ &\le 2\|g''\|_{L^\infty}+T_0\|c\|^2_{L^\infty(0,3T_0/2)}(1+\frac{1}{2}\|c\|_{L^\infty(\mathbb{R}^+)}T_0^2e^{\sqrt{\|c\|_{L^\infty(\mathbb{R}^+)}}T_0})\,, \end{aligned}$$

where we have used (4.10) in the last inequality. Thus by (4.20), $\|c\|_{L^\infty}\le R$ implies $\|Ac\|_{L^\infty}\le R$. To show contractivity, we observe that for $c_1,c_2\in B_R(0)\subset L^\infty(\mathbb{R}^+)$, $\tau\le\frac{T_0}{2}$,

$$(A(c_1)-A(c_2))(\tau)=-2\int_0^\tau\Big(c_1(s)-c_2(s)U_t^{c_1}(s,2\tau-s)+c_2(s)(U_t^{c_1}(s,2\tau-s)-U_t^{c_2}(s,2\tau-s))\Big)\,ds$$

Similarly to above we get

$$\begin{aligned} &U_t^{c_1}(x,t)-U_t^{c_2}(x,t) \\ &= -\frac{1}{2}\int_0^t\Big((c_1(x+t-s)-c_2(x+t-s))(\hat{u}(x+t-s,s)+U^{c_1}(x+t-s,s)) \\ &\qquad +c_2(x+t-s)(U^{c_1}(x+t-s,s)-U^{c_2}(x+t-s,s)) \\ &\qquad +(c_1(x-t+s)-c_2(x-t+s))(\hat{u}(x-t+s,s)+U^{c_1}(x-t+s,s)) \\ &\qquad +c_2(x-t+s)(U^{c_1}(x-t+s,s)-U^{c_2}(x-t+s,s)\Big)\,ds\,. \end{aligned}$$

Using (4.11), we get

$$\begin{aligned} &\|A(c_1)-A(c_2)\|_{L^\infty} \\ &\le 2\frac{T_0}{2}\Big(\|U_t^{c_1}\|_{L^\infty}\|c_1-c_2\|_{L^\infty}+\|c_2\|_{L^\infty}\|U_t^{c_1}-U_t^{c_2}\|_{L^\infty}\Big) \\ &\le 2\frac{T_0}{2}\Big(\|c_1\|_{L^\infty}(1+\frac{1}{2}\|c_1\|_{L^\infty(\mathbb{R}^+)}T_0^2e^{\sqrt{\|c_1\|_{L^\infty(\mathbb{R}^+)}}T_0}) \\ &\quad +\|c_2\|_{L^\infty}(1+\frac{1}{2}\|c_1\|_{L^\infty(\mathbb{R}^+)}T_0^2e^{\sqrt{\|c_1\|_{L^\infty(\mathbb{R}^+)}}T_0}) \\ &\quad +\|c_2\|^2_{L^\infty}(1+\frac{1}{2}\|c_1\|_{L^\infty}T_0^2e^{\sqrt{\|c_1\|_{L^\infty}}T_0})\frac{1}{2}T_0^2e^{\sqrt{\|c_2\|_{L^\infty}}T_0}\Big)\;\|c_1-c_2\|_{L^\infty}\,. \end{aligned}$$

This by (4.21) yields contractivity.

Hence, the fixed point equation $c = A(c)$, which is just (4.16) on $(0, \frac{T_0}{2})$ and $c = 0$ on $(\frac{T_0}{2}, \infty)$, has a solution and this solution is unique in $B_R(0)$. Global uniqueness (so on all of L^∞ and not only $B_R(0)$ can be seen ba assuming $c_1 \neq c_2$ being two solutions, setting $R > \max\{2\|g''\|_\infty, \|c_1\|_\infty, \|c_1\|_\infty\}$ and repeating the proof above, which implies that c_1 and c_2 have to coincide.

By remark 4.5, the fixed point equation $c = A(c)$ is equivalent to (4.16).

The uniqueness on the small time interval $(0, \frac{T_0}{2})$ enables us to conclude uniqueness on all of $(0, \frac{T}{2})$ as follows. Wlog we can assume that there exists an $N \in \mathbb{N}_0$ such that $NT_0 = T$. We have just shown that

$$\left.\begin{array}{c} U^{c_1} = U^{c_2} \text{ and } U_t^{c_1} = U_t^{c_2} \text{ on } \{0\} \times \mathbb{R}^+ \\ \text{and } g_1(2\cdot) = g_2(2\cdot) \text{ on } [0, \frac{T}{2}] \end{array}\right\} \Rightarrow c_1 = c_2 \text{ on } [0, \frac{T_0}{2}] \tag{4.22}$$

By induction, this implies that $c_1 = c_2$ on $[0, \frac{T}{2}]$. Namely, from $c_1 = c_2$ on $[0, \frac{T_0}{2}]$, and the fact that $w := U^{c_1} - U^{c_2}$ solves

$$w_{tt} - w_{xx} + c_2 w = (c_1 - c_2)(\hat{u} + U^{c_1}) = 0 \text{ in } \{(x,t) \mid x \leq \frac{T_0}{2}\} \cup \{(x,t) \mid x \geq t\} =: S\,,$$

it follows with Lemma (4.2) that $U^{c_1} = U^{c_2}$ on M, where M is the union of all cones $\triangle(x,t)$ that are completely contained in S. It is straightforward (make a plot) to see that $M \supset \{(y,s) \mid s \leq \frac{T_0}{2}\}$. Hence, $U^{c_1} = U^{c_2}$ as well as $U_t^{c_1} = U_t^{c_2}$ holds on the line $\{(x, \frac{T_0}{2}) \mid 0 \leq x\}$. Thus we can replace $t = 0$ with $t = \frac{T_0}{2}$ and use (4.22) once more to conclude $c_1 = c_2$ also on $[\frac{T_0}{2}, 2\frac{T_0}{2}]$. Repeating this N times (note that after that, we cannot guarantee $g_1(2\cdot) = g_2(2\cdot)$ any more), we arrive at $c_1 = c_2$ on $[0, \frac{T}{2}]$. For carrying out this "layer stripping" type proof, it is important that we had proven implication (4.22) using shortness of the interval $[0, \frac{T_0}{2}]$ only but not its position on the time line.

$\diamondsuit$

5. An inverse scattering problem

In this chapter we mainly follow the exposition of Chapter 5 in [Kirsch].

Acoustic waves travelling in a medium can be described by the wave equation

$$\frac{\partial^2 \tilde{p}}{\partial t^2} + \gamma \frac{\partial \tilde{p}}{\partial t} = c^2 \rho_0 \nabla \left(\frac{1}{\rho_0} \nabla \tilde{p} \right) \tag{5.1}$$

for the pressure fluctuation $\tilde{p}$, where γ is a (constant) damping parameter, c the speed of sound in the medium and ρ_0 the density. Now we assume that terms involving $\nabla \rho_0$ are negligible and that $\tilde{p}$ is time harmonic, i.e., of the form

$$\tilde{p}(x,t) = \Re(u(x) e^{-\imath \omega t})$$

with frequency $\omega > 0$ and a complex valued function u depending only on the spatial variable Substituting this into the wave equation (5.1) yields the three- dimensional Helmholtz equation for u:

$$\Delta u + \frac{\omega^2}{c(x)^2}(1 + \imath \frac{\gamma}{\omega}) u = 0 \,.$$

We define the wave number k and the index of refraction n by

$$k := \frac{\omega}{c_0} > 0 \quad n(x) := \frac{c_0^2}{c(x)^2}(1 + \imath \frac{\gamma}{\omega}) \,.$$

The Helmholtz equation then takes the form

$$\Delta u + k^2 n u = 0 \tag{5.2}$$

where n is a complex valued function with $\Re(n(x)) \geq 0$ and $\Im(n(x)) \geq 0$. In free space, $c = c_0$ is constant and $\gamma = 0$, hence, $n = 1$.

This equation holds in every source free domain in $\mathbb{R}^3$. We assume in this chapter that there exists $a > 0$ such that $c(x) = c_0$ and $\gamma(x) = 0$ for all x with $|x| > a$, i.e., $n(x) = 1$ for x with $|x| > a$. This means that the inhomogeneous medium $\{x \in \mathbb{R}^3 \,|\, n(x) \neq 1\}$ is bounded and contained in a ball $B_a(0)$ of radius a.

We further assume that the sources lie outside the ball $\overline{B_a(0)}$. These sources generate incident fields u^i, that satisfy the unperturbed Helmholtz equation

$$\Delta u^i + k^2 u^i = 0 \text{ in } \mathbb{R}^3 \,. \tag{5.3}$$

Especially, we here consider plane waves, i.e., incident fields

$$u^i = e^{\imath k\hat{\theta}\cdot x} \tag{5.4}$$

for a unit vector $\hat{\theta} \in \mathbb{R}^3$. The corresponding pressure field $\tilde{p}$ is the one of a plane wave that travels in the direction $\hat{\theta}$ with velocity c_0.

The incident field will be disturbed by the medium described by the index of refraction n and will produce a scattered wave u^s. The total field $u = u^i + u^s$ satisfies the Helmholtz equation (5.2) outside the sources (i.e., in $B_a(0)$). Furthermore, we expect the scattered field to behave as a spherical wave far away from the medium. This can be decribed by the following Sommerfeld radiation condition

$$\frac{\partial u^s(x)}{\partial r} - \imath k u^s(x) = O(r^{-2}) \qquad \text{as } r = |x| \to \infty\,, \tag{5.5}$$

uniformly in $\frac{x}{|x|} \in S^2$, where S^2 denotes the unit sphere.

5.1. The forward problem

We can now formulate the direct scattering problem: Given the wave number $k > 0$, the index of refraction $n \in C^2(\mathbb{R}^3)$ with $n(x) = 1$ for $|x| \geq a$, $\Re(n(x)) \geq 0$, $\Im(n(x)) \geq 0$, and the incident field u^i according to (5.4) with $\hat{\theta} \in S^2$, determine a solution u to the Helmholtz equation (5.2) in $\mathbb{R}^3$ such that $u^s = u - u^i$ satisfies the Sommerfeld radiation condition (5.5).

The proof of uniqueness of u relies on the following very important theorem

Lemma 5.1. *(Rellich) Let u satisfy the unperturbed Helmholtz equation $\Delta u + k^2 u = 0$ for $|x| > a$. Assume furthermore, that*

$$\lim_{R\to\infty} \int_{|x|=R} |u(x)|^2 ds = 0\,.$$

Then $u = 0$ for $|x| > a$.

Proof. see Lemma 2.11 in [ColtonKress]

$\diamondsuit$

Theorem 5.2. *The problem (5.2), (5.5) has at most one solution.*

Proof. see [Kirsch]

$\diamondsuit$

Now, let

$$\Phi(x, y) = \frac{e^{\imath k|x-y|}}{4\pi|x - y|} \qquad \text{for } x, y \in \mathbb{R}^3, x \neq y\,. \tag{5.6}$$

be the fundamental solution or free space Green's function of the Helmholtz equation. Properties of the fundamental solution are summarized in the following theorem.

Theorem 5.3. *For each $y \in \mathbb{R}^3$, $\Phi(\cdot, y)$ solves the unperturbed Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \setminus \{y\}$.*
It satisfies the radiation condition

$$\frac{x}{|x|} \nabla_x \Phi(x,y) - \imath k \Phi(x,y) = O(|x|^{-2})$$

uniformly in $\frac{x}{|x|} \in S^2$ and $y \in Y$ for every bounded subset $Y \subset \mathbb{R}^3$.
In addition,

$$\Phi(x,y) = \frac{e^{\imath k|x|}}{4\pi|x|} e^{-\imath k \hat{x} \cdot y} + O(|x|^{-2}) \tag{5.7}$$

uniformly in $\hat{x} = \frac{x}{|x|} \in S^2$ and $y \in Y$.

Proof. exercise
$\diamondsuit$
With this fundamental solution, we can construct volume potentials:

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. For every $\phi \in C(\overline{\Omega})$, the volume potential*

$$v(x) = \int_\Omega \phi(y) \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,, \tag{5.8}$$

exists as an improper integral. Furthermore, $v \in C^{1,\alpha}(\mathbb{R}^3)$ and

$$\nabla v(x) = \int_\Omega \phi(y) \nabla_x \Phi(x,y)\, dy\,, \quad x \in \mathbb{R}^3\,. \tag{5.9}$$

If, in addition, ϕ is Hölder continuous, i.e., $\phi \in C^\alpha(\Omega)$ for some $\alpha \in (0,1]$, then $v \in C^{2,\alpha}(\Omega)$ solves

$$\Delta v + k^2 v = -\phi \tag{5.10}$$

in Ω, and there exists $c > 0$ depending only on Ω, such that

$$\|v\|_{C^{2,\alpha}(\Omega)} \le c \|\phi\|_{C^\alpha(\Omega)}\,.$$

If $\phi \in C^\alpha(\Omega)$ is of compact support, then $v \in C^{2,\alpha}(\mathbb{R}^3)$ solves $\Delta v + k^2 v = -\phi$ in $\mathbb{R}^3$, where ϕ is extended by zero into allof $\mathbb{R}^3$.

Now we can transform the scattering problem into a Fredholm integral equation of the first kind

Theorem 5.5. .

a) Let $u \in C^2(\mathbb{R}^3)$ be a solution of the scattering problem (5.2), (5.5). Then $u|_{B_a(0)}$ solves the Lippmann-Schwinger integral equation

$$u(x) = u^i(x) - k^2 \int_{|y|<a} (1 - n(y)) \Phi(x,y) u(y)\, dy \quad x \in B_a(0)\,. \tag{5.11}$$

b) If, on the other hand, $u \in C(\overline{B_a(0)})$ is a solution of the integral equation (5.11), then u can be extended by the right hand side of (5.11) to a solution $u \in C^2(\mathbb{R}^3)$ of the scattering problem (5.2), (5.5).

Proof.

a): Let u be a solution of (5.2), (5.5) and $v(x)$ the right hand side of (5.11) for $x \in \mathbb{R}^3$. Since $u \in C^2(\mathbb{R}^3)$, we conclude by Theorem 5.4, that $v \in C^{2,\alpha}$ and

$$\Delta v + k^2 v = k^2(1-n)u = k^2 u + \Delta u\,.$$

Therefore, $w = v - u$ satisfies the unperturbed Helmholtz equation $\Delta w + k^2 w = 0$ in $\mathbb{R}^3$. Moreover, by Theorem 5.3, and (5.5),

$$w = u^i - u - k^2 \int_{|y|<a} (1 - n(y))\Phi(\cdot, y)u(y)\,dy$$

satisfies the radiation condition (5.5). The uniqueness Theorem 5.2 therefore yields $w = 0$, i.e., $v = u$.

b): Let $u \in C(\overline{B_a(0)})$ be a solution of the integral equation (5.11) and extend u by the right hand side of (5.11) to all of $\mathbb{R}^3$. A first application of Theorem 5.4 yields $u \in C^{1,\alpha}(\mathbb{R}^3)$. Then, since $n \in C^2$, a second application of Theorem 5.4 yields $u \in C^{2,\alpha}(\mathbb{R}^3)$. Furthermore, by (5.10) and the fact that u^i solves the unperturbed Helmholtz equation, we get

$$\Delta u + k^2 u = 0 + k^2(1-n)u$$

thus (5.2) holds in $\mathbb{R}^3$. The radiation condition for the scattered field $u - u^i$ again follows from Theorem 5.3.

$\diamondsuit$

As a corollary, we immediately have the following

Theorem 5.6. *Under the given assumptions on k, n, and $\hat{\theta}$, there exists a unique solution u of the scattering problem (5.2), (5.5) or, equivalently, the integral equation (5.11).*

Proof.

We apply the Fredholm alternative (for a compact operator T, either $\mathcal{N}(I+T) \neq \{0\}$ or $(I+T)u = f$ is solvable for all f) to the integral equation $u = u^i - Tu$, where the operator $T : C(\overline{B_a(0)}) \to C(\overline{B_a(0)})$ is defined by

$$(Tu)(x) = k^2 \int_{|y|<a} (1-n(y))\Phi(x,y)u(y)\,dy \quad x \in B_a(0)\,. \tag{5.12}$$

This integral operator is compact, since the kernel is weakly singular. Therefore it is sufficient to prove uniqueness of a solution to (5.11). This follows by Theorems 5.2, 5.5.

$\diamondsuit$

As another application of the Lippmann-Schwinger equation, we derive the following asymptotic behaviour of u.

Theorem 5.7. *Let u be the solution to the scattering problem (5.2), (5.5). Then*

$$u(x) = u^i(x) + \frac{e^{\imath k|x|}}{|x|} u_\infty(\hat{x};\hat{\theta}) + O(|x|^{-2}) \ \text{as} \ |x| \to \infty \tag{5.13}$$

uniformly in $\hat{x} = \frac{x}{|x|}$, where

$$u_\infty(\hat{x};\hat{\theta}) = -\frac{k^2}{4\pi}\int_{|y|<a} (1-n(y))e^{-\imath k\hat{x}\cdot y}u(y)\,dy \quad \hat{x}\in S^2\,, \tag{5.14}$$

and $\hat{\theta}$ denotes the direction of the incident field. The function $u_\infty : S^2 \to \mathbb{C}$ is called the far field pattern or scattering amplitude of u. It is analytic on S^2 and determines u^s outside of $B_a(0)$ uniquely, i.e., $u_\infty = 0$ if and only if $u^s(x) = 0$ for $|x| > a$.

Proof.
Formula (5.13) with (5.14) follow directly from the asymptotic behaviour of the fundamental solution. The analyticity of u_∞ follows from the formula (5.14). Finally, if $u_\infty = 0$, then an application of Rellich's Lemma yields that $u^s = u - u^i = 0$ for all $|x| > a$.
$\diamondsuit$

The existence of a far field pattern, i.e., a function u_∞ with (5.13) is not restricted to scattering problems. Indeed, Theorem 5.8 assures the existence of the far field pattern for every radiating solution of the Helmholtz equation.

We now draw some conclusions from the Lippmann-Schwinger integral equation $u + Tu = u^i$. We estimate the norm of the integral operator T according to (5.12)

$$\|Tu\|_\infty \le k^2\,\|1-n\|_\infty\,\|u\|_\infty \max_{|x|\le a}\int_{|y|<a} |\Phi(x,y)|\,dy$$

hence

$$\|T\| \le k^2\,\|1-n\|_\infty \max_{|x|\le a} \underbrace{\int_{|y|<a}\frac{1}{4\pi|x-y|}\,dy}_{=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\frac{1}{4\pi r}r^2\sin(\phi)\,dr\,d\phi\,d\psi} = \frac{(ka)^2}{2}\,\|1-n\|_\infty\,.$$

We conclude that $\|T\| < 1$ provided $(ka)^2\,\|1-n\|_\infty < 2$, and hence in this case the Banach contraction mapping theorem yields existence and uniqueness of a solution to the Lippmann-Schwinger equation. (We know this already from Fredholm theory — even in the case $(ka)^2\,\|1-n\|_\infty \ge 2$.) Additionally, we can represent the solution in a Neumann series in the form

$$u = \sum_{j=0}^{\infty}(-1)^j T^j u^i\,.$$

The first two terms in this series are

$$u^B(x) = u^i(x) - k^2\int_{\mathbb{R}^3}(1-n(y))\Phi(x,y)u^i(y)\,dy \quad x\in\mathbb{R}^3\,. \tag{5.15}$$

u^B is called the Born approximation. It provides a good approximation to u in $B_a(0)$ for small values of $(ka)^2 \|1-n\|_\infty$, since

$$\|u-u^B\|_\infty \leq \sum_{j=2}^{\infty} \|T\|^j \|u^i\|_\infty = \frac{\|T\|^2}{1-\|T\|} \leq \frac{(ka)^4}{2} \|1-n\|_\infty^2$$

for $(ka)^2 \|1-n\|_\infty \leq 1$. Considering the far field pattern for the Born approximation, we see from the asymptotic form (5.7) that

$$u_\infty^B(\hat{x};\hat{\theta}) = \frac{k^2}{4\pi} \int_{\mathbb{R}^3} (n(y)-1) e^{\imath k \hat{\theta}\cdot y} e^{-\imath k \hat{x}\cdot y}\, dy = \sqrt{\frac{\pi}{2}} (\mathcal{F}m)(k(\hat{x}-\hat{\theta})) \tag{5.16}$$

with $\mathcal{F}$ the Fourier transform. From this, the reciprocity principle follows:

$$u_\infty^B(-\hat{\theta};-\hat{x}) = u_\infty^B(\hat{x};\hat{\theta}) \quad \hat{x},\hat{\theta} \in S^2\,.$$

It can be shown (which we will not do here, though) that this relation also holds for u itself, by using the following Green's representation theorem

Theorem 5.8. *(Green's representation theorem) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$ its exterior. Let the boundary $\partial\Omega$ be sufficiently smooth so that Gauss' theorem holds, and let n be the outer unit normal vector.*

a) Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then

$$\begin{aligned} u(x) &= \int_{\partial\Omega} \left(\Phi(x,y) \frac{\partial}{\partial n} u(y) - u(y) \frac{\partial}{\partial n(y)} \Phi(x,y) \right) ds(y) \\ &\quad - \int_{\Omega} \Phi(x,y)(\Delta u(y) + k^2 u(y))\, dy \qquad x \in \Omega \end{aligned} \tag{5.17}$$

b) Let $u^s \in C^2(\Omega^c) \cap C^1(\overline{\Omega^c})$ be a solution of the Helmholtz equation $\Delta u^s + k^2 u^s = 0$ in Ω^c, and let u^s satisfy the radiation condition (5.5). Then

$$\int_{\partial\Omega} \left(\Phi(x,y) \frac{\partial}{\partial n} u^s(y) - u^s(y) \frac{\partial}{\partial n(y)} \Phi(x,y) \right) ds(y) = \begin{cases} 0 & x \in \Omega \\ -u^s(x) & x \notin \overline{\Omega}\,. \end{cases} \tag{5.18}$$

The far field pattern of u^s has the representation

$$u_\infty(\hat{x},\hat{\theta}) = -\frac{1}{4\pi} \int_{\partial\Omega} \left(e^{-\imath k \hat{x}\cdot y} \frac{\partial}{\partial n} u^s(y) - u^s(y) \frac{\partial}{\partial n(y)} e^{-\imath k \hat{x}\cdot y} \right) ds(y)$$

for $x \in S^2$.

Proof. see Theorems 2.1 and 2.4 in [ColtonKress].

5.2. The Inverse problem

In this section we will consider uniqueness of the following inverse problem: Determine the refraction index $n \in C^2$ from measurements of the far field pattern $u_\infty(\cdot, \hat{\theta} : S^2 \to \mathbb{C}$ either

a) at fixed wave number k and for all directions $\hat{\theta} \in S^2$ of incident fields or

b) for an interval of wave numbers $[\underline{k}, \overline{k}]$ with $\underline{k} < \overline{k}$ and a fixed incident field direction $\hat{\theta}$.

Both cases can be treated in a very straightforward manner under the simplifications made by the Born approximation. For the general case we will here only consider the first case and refer, e.g., to [ColtonKress] for the second one.

Let $n_1, n_2 \in C^2(\mathbb{R}^3)$ be two refraction indices with $n_1 = n_2 = 1$ on $\{|x| > a\}$ and u_1^B, u_2^B the corresponding Born approximations to the total fields according to (5.15) If the two Born approximation far field patterns coincide

$$u_{1\infty}^B(\hat{x}, \hat{\theta}) = u_{2\infty}^B(\hat{x}, \hat{\theta}) \quad \forall \hat{x} \in S^2 \tag{5.19}$$

for some $\hat{\theta} \in S^2$, then, according to (5.16), this means that the Fourier transforms of $m_1 := n_1 - 1$ and $m_2 := n_2 - 1$ coincide on a sphere with center $k\hat{\theta}$ and radius $k > 0$. This, however, is not enough to conclude that m_1 and m_2 coincide.

If one assumes that (5.19) holds for *all* $\hat{\theta} \in S^2$, i.e., case a), then the Fourier transforms of m_1 and m_2 coincide on the set $\{k(\hat{x} - \hat{\theta}) \,|\, \hat{x}, \hat{\theta} \in S^2\}$ which describes a ball in $\mathbb{R}^3$ with center zero and radius $2k$. Since m_1, m_2 are compactly supported, their Fourier transforms are analytic functions and the unique continuation principle for analytic functions yields that $\mathcal{F}m_1 = \mathcal{F}m_2$ on all of $\mathbb{R}^3$, hence $m_1 = m_2$ and the refraction indices n_1, n_2 have to coincide. This proves uniqueness in the Born approximation setting and case a).

In case b), the Fourier transforms $\mathcal{F}m_1, \mathcal{F}m_2$ coincide on the set $\{k(\hat{x} - \hat{\theta}) \,|\, \hat{x} \in S^2\,, \quad k \in [\underline{k}, \overline{k}]\}$, which also obviously has nonempty interior, so that we can the analyticity argument from above to conclude uniqueness.

We will now show the uniqueness proof for measurements of the actual far field pattern in case a). It is — similarly to the EIT problem — based on density and orthogonality arguments. As a matter of fact, we can re-use some of the results proved in Chapter 2.

The first step of proof is a density result stating that for fixed index of refraction the span of all total fields that correspond to scattering problems with plane incident fields is dense in the space of solutions to the perturbed Helmholtz equation.

Lemma 5.9. *Let $n \in C^2(\mathbb{R}^3, \mathbb{C})$ with $n = 1$ on $\{|x| > a\}$, and let $u(\cdot, \hat{\theta})$ denote the total field corresponding to the incident field $u^i(x; \hat{\theta}) = e^{\imath\hat{\theta}\cdot x}$. Let $k \geq 0$, $b > a$ and define the space H by*

$$H = \{v \in C^2(B_b(0)) \,|\, \Delta v + k^2 n v = 0 \text{ in } B_b(0)\}\,. \tag{5.20}$$

Then

$$span\{u(\cdot,\hat{\theta})|_{B_a(0)}\,|\,\hat{\theta}\in S^2\}$$

is dense in $H|_{B_a(0)}$ *with respect to the* $L^2(B_a(0))$ *norm.*

Proof. Assume non-density, then there exists a nonzero $v \in \overline{H}$ with

$$\langle v, u(x,\hat{\theta})\rangle_{L^2} = \int_{B_a(0)} v(x)\overline{u(x,\hat{\theta})}\,dx = 0 \quad \forall \hat{\theta}\in S^2 .$$

The Lippmann-Schwinger equation yields $u(\cdot;\hat{\theta}) = (I+T)^{-1}u^i(\cdot,\hat{\theta})$, thus

$$0 = \langle v, (I+T)^{-1}u^i(\cdot,\hat{\theta})\rangle_{L^2} = \langle \underbrace{(I+T^*)^{-1}v}_{=:w}, u^i(\cdot,\hat{\theta})\rangle_{L^2} \quad \forall\hat{\theta}\in S^2$$

The so defined w solves the adjoint equation

$$w(x) + k^2(1-\overline{n(x)})\int_{B_a(0)}\overline{\Phi(x,y)}w(y)\,dy = v(x) \qquad x\in B_a(0)\,.$$

Now set

$$\tilde{w}(x) := \overline{\int_{B_a(0)}\overline{\Phi(x,y)}w(y)\,dy}\,, \qquad x\in\mathbb{R}^3\,.$$

Then $\tilde{w}$ is a volume potential with density $\overline{w}$. It is readily checked that $\tilde{w}$ satisfies the Helmholtz equation $\Delta\tilde{w}+k^2\tilde{w}=0$ for $|x|>a$. Moreover, its far field pattern vanishes, since

$$\overline{\tilde{w}_\infty(\hat{\theta})} = \frac{1}{4\pi}\int_{B_a(0)} w(y)e^{\imath k\hat{\theta}\cdot y}\,dy = \frac{1}{4\pi}\langle w, u^i(x,-\hat{\theta})\rangle_{L^2} = 0 \quad \forall\hat{\theta}\in S^2\,.$$

Therefore, Rellich's Lemma implies $\tilde{w}=0$ on $\{|x|>a\}$. Now let $(v_j)_{j\in\mathbb{N}}\subset H$ be a sequence converging to v in L^2. Then,

$$\begin{aligned}\int_{B_a(0)}\overline{v}v_j\,dx &= \int_{B_a(0)}\overline{w}v_j\,dx + \int_{B_a(0)}k^2(1-n)\tilde{w}v_j\,dx\\ &= \int_{B_a(0)}\overline{w}v_j\,dx + \int_{B_a(0)}\tilde{w}(\Delta v_j + k^2 v_j)\,dx\,,\end{aligned}$$

where we have used the fact that $v_j\in H$ solves $\Delta v_j + k^2 n v_j = 0$. Now we substitute the definition of $\tilde{w}$ and change the order of integration. This yields

$$\begin{aligned}&\int_{B_a(0)}\overline{v}v_j\,dx\\ &\quad = \int_{B_a(0)}\overline{w(y)}\left(v_j(y) + \int_{B_a(0)}\Phi(x,y)(\Delta v_j + k^2 v_j)\,dx\right)\,dy\,.\end{aligned}$$

Green's representation theorem 5.8 yields

$$\int_{B_a(0)} \overline{v} v_j \, dx = \int_{B_a(0)} \overline{w(y)} \int_{|x|=a} \left(\Phi(\cdot, y) \frac{\partial v_j}{\partial n} - v_j \frac{\partial \Phi(\cdot, y)}{\partial n} \right) ds \, dy \,. \tag{5.21}$$

Since v_j satisfies the Helmholtz equation $\Delta v_j + k^2 v_j = 0$ for $a < |x| < b$, we can apply Theorem 5.8 to obtain, with $c \in (a, b)$,

$$\int_{|x|=a} \left(\Phi(\cdot, y) \frac{\partial v_j}{\partial n} - v_j \frac{\partial \Phi(\cdot, y)}{\partial n} \right) ds = \int_{|x|=c} \left(\Phi(\cdot, y) \frac{\partial v_j}{\partial n} - v_j \frac{\partial \Phi(\cdot, y)}{\partial n} \right) ds$$

for all $y \in B_a(0)$. Inserting this into (5.21) and changing the order of integration yields

$$\int_{B_a(0)} \overline{v} v_j \, dx = \int_{|x|=c} \left(\tilde{w} \frac{\partial v_j}{\partial n} - v_j \frac{\partial \tilde{w}}{\partial n} \right) ds = 0 \,,$$

since $\tilde{w}$ vanishes outside $B_a(0)$. Letting j tend to ∞ yields $\|v\|_{L^2} = 0$, a contradiction to our assumption.
$\diamondsuit$

This denseness result enables us to show the following orthogonality relation

Lemma 5.10. *Let $n_1, n_2 \in C^2(\mathbb{R}^3)$ be two indices of refraction with $n_1(x) = n_2(x) = 1$ for $|x| > a$ and assume that*

$$u_{1,\infty}(\hat{x}, \hat{\theta}) = u_{2,\infty}(\hat{x}, \hat{\theta}) \quad \forall \hat{x}, \hat{\theta} \in S^2 \,.$$

Then

$$\int_{B_a(0)} v_1(x) v_2(x) (n_1(x) - n_2(x)) \, dx = 0$$

for all solutions $v_j \in C^2(B_b(0))$ of the Helmholtz equation $\Delta v_j + k^2 n_j v_j = 0$ in $B_b(0)$, $j = 1, 2$, where $b > a$.

Proof. Let v_1 be any fixed solution of $\Delta v_1 + k^2 n_1 v_1 = 0$ in $B_b(0)$. By the density result Lemma 5.9, it suffices to prove the assertion for $v_2 = u_2(\cdot, \hat{\theta})$ and arbitrary $\hat{\theta} \in S^2$. We set $u = u_1(\cdot, \hat{\theta}) - u_2(\cdot, \hat{\theta})$. Then u satisfies the Helmholtz equation

$$\Delta u + k^2 n_1 u = k^2 (n_2 - n_1) u_2 \,.$$

We multiply this equation by v_1 and the Helmholtz equation for v_1 by u, subtract the results and then integrate over $B_a(0)$. This yields, by Green's second identity,

$$\begin{aligned} k^2 \int_{B_a(0)} (n_2 - n_1) u_2 v_1 \, dx &= \int_{B_a(0)} (v_1 \Delta u - u \Delta v_1) \, dx \\ &= \int_{|x|=a} \left(v_1 \frac{\partial u}{\partial n} - u \frac{\partial v_1}{\partial n} \right) ds = 0 \end{aligned}$$

In the last identity we have used the fact that by $u_{1,\infty}(\cdot,\hat{\theta}) = u_{2,\infty}(\cdot,\hat{\theta})$ and Rellich's Lemma $u = 0$ on $|x| \geq a$.
$\diamondsuit$

Now we can make use of the result on completeness of products of solutions to the Schrödinger (=Helmholtz) equation Corollary 2.9 from Chapter 2 to arrive at

Theorem 5.11. *Let $n_1, n_2 \in C^2(\mathbb{R}^3)$ be two indices of refraction with $n_1(x) = n_2(x) = 1$ for $|x| > a$ and assume that*

$$u_{1,\infty}(\hat{x},\hat{\theta}) = u_{2,\infty}(\hat{x},\hat{\theta}) \quad \forall \hat{x},\hat{\theta} \in S^2 \, .$$

Then $n_1 = n_2$.

6. Numerical solution techniques: Operator equation methods

Most often, parameter identification problems can be formulated as operator equations

$$F(x) = y\,, \tag{6.1}$$

where $F : \mathcal{D}(F) \to Y$ with domain $\mathcal{D}(F) \subset X$, X, Y real Hilbert spaces. Measurements are usually contaminated with noise, therefore, we assume that noisy data y^δ with

$$\|y^\delta - y\| \le \delta\,. \tag{6.2}$$

are given.

As examples, consider the problems from the previous chapters:

- Chapter 2: $F : a \mapsto \Lambda_a$, where Λ_a is the Dirichlet-Neumann operator for

$$\nabla(a\nabla u) = 0 \quad \text{in } \Omega$$

- Chapter 3: $F : f \mapsto u(\cdot, T)$, where u solves

$$\begin{aligned} \tfrac{\partial u}{\partial t} - \Delta u &= \alpha(x,t)f(x) && \text{in } \Omega \times (0,T) \\ u(x,0) &= u_0(x) && x \in \Omega \\ u(x,t) &= h(x,t) && x \in \partial\Omega\,. \end{aligned}$$

- Chapter 4: $F : c \mapsto u(0, \cdot)$, where u solves

$$\begin{aligned} u_{tt} - u_{xx} + cu &= 0 \text{ in } (0,\infty) \times (0,T)\,, \\ u_x(0,\cdot) &= \delta_0\,, \quad \text{in } (0,T)\,, \\ u(\cdot,0) = u_t(\cdot,0) &= 0 \quad \text{in } (0,\infty)\,, \end{aligned}$$

- Chapter 5: $F : n \mapsto \{u_\infty(\cdot, \hat\theta) \mid \hat\theta \in S^2\}$, where $u_\infty(\cdot, \hat\theta)$ is the far field according to

$$u(x) = u^i(x) + \frac{e^{\imath k|x|}}{|x|} u_\infty(\hat x; \hat\theta) + O(|x|^{-2}) \text{ as } |x| \to \infty$$

  uniformly in $\hat x = \frac{x}{|x|} \in S^2$, where u solves

$$\Delta u + k^2 n u = 0$$

with the Sommerfeld radiation condition

$$\frac{\partial u^s(x)}{\partial r} - \imath k u^s(x) = O(r^{-2}) \quad \text{as } r = |x| \to \infty \,,$$

uniformly in $\frac{x}{|x|} \in S^2$.

Since in this context, (6.1) is usually ill-posed, regularization methods for nonlinear ill-posed problems have to be applied for its stable approximate solution.

6.1. Preliminaries: Regularization methods for linear problems

We consider an operator equation

$$Tx = y \tag{6.3}$$

where $T \in L(X,Y)$ and X and Y are Hilbert spaces. We assume that $y \in \mathcal{D}(T^\dagger)$ and that the data $y^\delta \in Y$ satisfy

$$\|y - y^\delta\| \leq \delta. \tag{6.4}$$

In this section we are going to consider the convergence of regularization methods of the form

$$R_\alpha y^\delta := q_\alpha(T^*T)T^*y^\delta \tag{6.5}$$

with some functions $q_\alpha \in C([0, \|T^*T\|])$ depending on some regularization parameter $\alpha > 0$. The notation $f(A)$ for some piecewise continuous function f and some selfadjoint non-negative definite operator A can be justified by spectral theory, see, e.g. [EnglHankeNeubauer]. In the case of compact operator A with Eigensystem $(\sigma_j^2; u_j)$, i.e.,

$$Ax = \sum_{j=0}^{\infty} \sigma_j^2 \langle x, u_j \rangle u_j \,,$$

the expression $f(A)$ can be rewritten as

$$f(A)x = \sum_{j=0}^{\infty} f(\sigma_j^2) \langle x, u_j \rangle u_j \,.$$

As this compactness often occurs in the context of ill-posed problems, we restrict ourselves to this case here, for simplicity of exposition.

We denote the reconstructions for exact data $y \in \mathcal{R}(T) + \mathcal{R}(T)^\perp$ and noisy data y^δ by $x_\alpha := R_\alpha y$ and $x_\alpha^\delta := R_\alpha y^\delta$, respectively and use the symbol $x^\dagger := T^\dagger y$ for the exact solution. Here $T^\dagger$ denotes the generalized inverse of T:

$$\forall y \in \mathcal{D}(T^\dagger) = \mathcal{R}(T) + \mathcal{R}(T)^\perp \;:\quad T^\dagger y = (T|_{\mathcal{N}(T) \to \mathcal{R}(T)})^{-1} Q y \,,$$

where Q is the projection onto $\mathcal{R}(T)$. Since $T^*g = T^*Qg = T^*Tx^\dagger$, the reconstruction error for exact data is given by

$$x^\dagger - x_\alpha = (I - q_\alpha(T^*T)T^*T)x^\dagger = r_\alpha(T^*T)x^\dagger \tag{6.6}$$

with

$$r_\alpha(\lambda) := 1 - \lambda q_\alpha(\lambda), \qquad \lambda \in [0, \|T^*T\|]. \tag{6.7}$$

The following tables list the definitions of some regularization methods, as well as the corresponding functions q_α and r_α. Note that for Landweber iteration, the equation has to be scaled such that

$$\|T\| \leq 2 \tag{6.8}$$

- Tikhonov regularization $\min\left\{\|Tx - y^\delta\|^2 + \alpha\|x - x_0\|^2\right\}$, which is equivalent to

$$x_\alpha^\delta = (T^*T + \alpha I)^{-1}(T^*y^\delta + \alpha x_0). \tag{6.9}$$

- iterated Tikhonov regularization

$$\begin{aligned} \varphi_{\alpha,0}^\delta &:= 0 && \text{(6.10a)} \\ \varphi_{\alpha,n+1}^\delta &:= (T^*T + \alpha I)^{-1}(T^*y^\delta + \alpha\varphi_{\alpha,n}^\delta), \qquad n \geq 0 && \text{(6.10b)} \end{aligned}$$

- Landweber iteration

$$\begin{aligned} \varphi_0 &= 0 && \text{(6.11a)} \\ \varphi_{n+1} &= \varphi_n - \mu T^*(T\varphi_n - y^\delta), \qquad n \geq 0, && \text{(6.11b)} \end{aligned}$$

Tikhonov regularization (6.9)	$q_\alpha(\lambda) = \frac{1}{\lambda+\alpha}$	$r_\alpha(\lambda) = \frac{\alpha}{\lambda+\alpha}$
iterated Tikhonov regularization (6.10)	$q_\alpha(\lambda) = \frac{(\lambda+\alpha)^n - \alpha^n}{\lambda(\lambda+\alpha)^n}$	$r_\alpha(\lambda) = \left(\frac{\alpha}{\lambda+\alpha}\right)^n$
Landweber iteration (6.11)	$q_n(\lambda) = \sum_{j=0}^{n-1}(1-\lambda)^j$	$r_n(\lambda) = (1-\lambda)^n$

In all these cases the functions r_α satisfy

$$\lim_{\alpha\to 0} r_\alpha(\lambda) = \begin{cases} 0, & \lambda > 0 \\ 1, & \lambda = 0 \end{cases} \tag{6.12}$$

$$|r_\alpha(\lambda)| \leq C_\mathrm{r} \qquad \text{for } \lambda \in [0, \|T^*T\|]. \tag{6.13}$$

with some constant $C_\mathrm{r} > 0$. The limit function defined by the right hand side of (6.12) is denoted by $r_0(\lambda)$. For Landweber iteration we set $\alpha = 1/n$ and assume that the normalization condition (6.8) holds true. Note that (6.12) is equivalent to $\lim_{\alpha\to 0} q_\alpha(\lambda) = 1/\lambda$ for all $\lambda > 0$. Hence, q_α explodes near 0. For all methods listed in the table above this growth is bounded by

$$|q_\alpha(\lambda)| \le \frac{C_\mathrm{q}}{\alpha} \qquad \text{for } \lambda \in [0, \|T^*T\|] \tag{6.14}$$

with some constant $C_\mathrm{q} > 0$.

Theorem 6.1. *If (6.12) and (6.13) hold true, then the operators R_α defined by (6.5) converge pointwise to $T^\dagger$ on $\mathcal{D}(T^\dagger)$ as $\alpha \to 0$. With the additional assumption (6.14) the norm of the regularization operators can be estimated by*

$$\|R_\alpha\| \le \sqrt{\frac{(C_\mathrm{r}+1)C_\mathrm{q}}{\alpha}}. \tag{6.15}$$

If $\overline{\alpha}(\delta, y^\delta)$ is a parameter choice rule satisfying

$$\overline{\alpha}(\delta, y^\delta) \to 0, \qquad \textit{and} \qquad \delta/\sqrt{\overline{\alpha}(\delta, y^\delta)} \to 0 \qquad \textit{as } \delta \to 0, \tag{6.16}$$

then $(R_\alpha, \overline{\alpha})$ is a regularization method in the sense that

$$\lim_{\delta\to 0} \sup \left\{ \|R_{\overline{\alpha}(\delta,y^\delta)} y^\delta - T^\dagger y\| : y^\delta \in Y, \|y^\delta - y\| \le \delta \right\} = 0 \tag{6.17}$$

for all $y \in \mathcal{D}(T^\dagger)$..

Proof. We first aim to show the pointwise convergence $R_\alpha \to T^\dagger$. Let $y \in \mathcal{D}(T^\dagger)$, $x^\dagger = T^\dagger y$, and $A := T^*T$. Recall from (6.6) that $T^\dagger g - R_\alpha g = r_\alpha(A)x^\dagger$. Using the boundedness condition (6.13), it follows that $\lim_{\alpha\to 0} r_\alpha(A)x^\dagger = r_0(A)x^\dagger$. Since r_0 is real-valued and $r_0^2 = r_0$, the operator $r_0(A)$ is an orthogonal projection. Moreover, $R(r_0(A)) \subset N(A)$ since $\lambda r_0(\lambda) = 0$ for all λ and hence $Ar_0(A) = 0$. By

$$N(T) = R(T^*)^\perp \qquad \text{and} \qquad \overline{R(T)} = N(T^*)^\perp. \tag{6.18}$$

we have $N(T) = N(A)$. Hence, $\|r_0(A)x^\dagger\|^2 = \langle r_0(A)x^\dagger, x^\dagger\rangle = 0$ as $x^\dagger \in N(T)^\perp = N(A)^\perp$. This shows that $\|R_\alpha y - T^\dagger y\| \to 0$ as $\alpha \to 0$.

Using the fact that we can "shift the operator through" in the sense that

$$f(T^*T)T^* = T^* f(TT^*)\,,$$

and the Cauchy-Schwarz inequality we obtain that

$$\|R_\alpha \psi\|^2 = \langle TT^* q_\alpha(TT^*)\psi, q_\alpha(TT^*)\psi\rangle \le \|\lambda q_\alpha(\lambda)\|_\infty \|q_\alpha(\lambda)\|_\infty \|\psi\|^2$$

for $\psi \in Y$. Now (6.15) follows from the assumptions (6.13) and (6.14).

To prove that $(R, \overline{\alpha})$ is a regularization method, we estimate

$$\|x^\dagger - x_\alpha^\delta\| \leq \|x^\dagger - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\|. \tag{6.19}$$

The approximation error $\|x^\dagger - x_\alpha\|$ tends to 0 due to the pointwise convergence of R_α and the first assumption in (6.16). The propagated data noise error $\|x_\alpha - x_\alpha^\delta\| = \|R_{\overline{\alpha}(\delta)}(y - y^\delta)\|$ vanishes asymptotically as $\delta \to 0$ by (6.4), (6.15), and the second assumption in (6.16). $\diamondsuit$

Source conditions

It can be shown (Schock, 1985) that the convergence of any regularization method can be arbitrarily slow in general. On the other hand, estimates on the rate of convergence as the noise level δ tends to 0 can be obtained under a-priori smoothness assumptions on the solution. In a general Hilbert space setting such conditions have the form

$$x^\dagger = f(T^*T)w, \qquad w \in X, \|w\| \leq \rho \tag{6.20}$$

with a continuous function f satisfying $f(0) = 0$. (6.20) is called a *source condition.* The most common choice $f(\lambda) = \lambda^\mu$ with $\mu > 0$ leads to source conditions of Hölder type,

$$x^\dagger = (T^*T)^\mu w, \qquad w \in X, \|w\| \leq \rho. \tag{6.21}$$

Since T is typically a smoothing operator, (6.20) and (6.21) can be seen as abstract smoothness conditions. In (6.21) the case $\mu = 1/2$ is of special importance, since

$$R((T^*T)^{1/2}) = R(T^*) \tag{6.22}$$

as shown in the exercises. To take advantage of the source condition (6.21) we assume that there exist constants $0 \leq \mu_0 \leq \infty$ and $C_\mu > 0$ such that

$$\sup_{\lambda \in [0, \|T^*T\|]} |\lambda^\mu r_\alpha(\lambda)| \leq C_\mu \alpha^\mu \qquad \text{for } 0 \leq \mu \leq \mu_0. \tag{6.23}$$

The constant μ_0 is called the *qualification* of the family of regularization operators (R_α) defined by (6.5). A straightforward computation shows that the qualification of (iterated) Tikhonov regularization is $\mu_0 = 1$ (or $\mu_0 = n$, respectively), and that the qualification of Landweber iteration and the truncated singular value decomposition is $\mu_0 = \infty$. By the following theorem, μ_0 is a measure of the maximal degree of smoothness, for which the method converges of optimal order.

Theorem 6.2. *Assume that (6.21) and (6.23) hold. Then the approximation error and its image under T satisfy*

$$\|x^\dagger - x_\alpha\| \leq C_\mu \alpha^\mu \rho, \qquad \textit{for } 0 \leq \mu \leq \mu_0, \tag{6.24}$$

$$\|Tx^\dagger - Tx_\alpha\| \leq C_{\mu+1/2} \alpha^{\mu+1/2} \rho, \qquad \textit{for } 0 \leq \mu \leq \mu_0 - \frac{1}{2}. \tag{6.25}$$

If the regularization parameter α *is chosen according to*

$$\alpha^{\mu+\frac{1}{2}} \sim \delta \tag{6.26}$$

then the optimal convergence rate

$$\|x_\alpha^\delta - x^\dagger\| \leq \tilde{C}_\mu \delta^{\frac{2\mu}{2\mu+1}} \text{ for } 0 \leq \mu \leq \mu_0 \tag{6.27}$$

is obtained.

Note that (6.26) is an a priori parameter choice rule, that requires information on μ (i.e.,on the smoothness of the exact solution) in order to be optimal. The same optimal rates (6.27) (with $0 \leq \mu \leq \mu_0$ replaced by $0 \leq \mu \leq \mu_0 - \frac{1}{2}$) can be obtained by using instead of (6.26) Morozov's discrepancy principle

$$\alpha = \max \ s.t. \ \|Tx_\alpha^\delta - y^\delta\| \leq \delta \,,$$

which is an a posteriori choice in the sense that it does not require knowledge of μ. Another possibility for choosing the regularization parameter a posteriori is given by the so-called balancing principle (or Lepskii rule), see [Perverzev, 2000].

6.2. Tikhonov regularization for nonlinear problems

Let X, Y be Hilbert spaces and $F : D(F) \subset X \to Y$ a continuous operator. We want to solve the operator equation

$$F(x) = y \tag{6.28}$$

given noisy data $y^\delta \in Y$ satisfying $\|y^\delta - y\| \leq \delta$. Let $x^\dagger$ denote the exact solution. We assume that the solution to (6.28) with exact data $g = F(x^\dagger)$ is unique, i.e. that

$$F(x) = g \qquad \Rightarrow \qquad x = x^\dagger \tag{6.29}$$

although many of the results below can be obtained in a modified form without this assumption.

The straightforward generalization of linear Tikhonov regularization leads to the minimization problem

$$\|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 = \min! \tag{6.30}$$

over $x \in D(F)$ where x_0 denotes some initial guess of $x^\dagger$. The mapping $D(F) \to \mathbb{R}$, $x \mapsto \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2$ is called *(nonlinear) Tikhonov functional.* Note that as opposed to the linear case, the element $0 \in X$ does not have a special role any more.

As opposed to the linear case it is not clear under the given assumptions if the minimization problem (6.30) has a solution. We will have to impose additional assumptions on F to ensure existence. Moreover, even if (6.30) has a unique solution for $\alpha = 0$ there may be more than one global minimizer of (6.30) for $\alpha > 0$.

As in the linear case, it is sometimes useful to consider other penalty terms in (6.30) than $\alpha\|x - x_0\|^2$ (e.g., total variation, maximum entropy, seminorms, Hilbert scales).

Convergence analysis

For proving existence of a minimizer and convergence, we assume that F is weakly closed, i.e.,

$$((\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(F) \ \wedge\ \psi_n \hookrightarrow \psi \ \wedge\ F(\psi_n) \hookrightarrow f) \quad \Longrightarrow \quad \psi \in \mathcal{D}(F) \ \wedge\ F(\psi) = f\,.$$

Theorem 6.3. *Assume that F is weakly closed. Then the Tikhonov functional (6.30) has a global minimum for all $\alpha > 0$.*

Proof. Let $I := \inf_{x\in D(F)} \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2$ denote the infimum of the Tikhonov functional and choose a sequence (x_n) in $D(F)$ such that

$$\|F(x_n) - y^\delta\|^2 + \alpha\|x_n - x_0\|^2 \le I + \frac{1}{n}. \tag{6.31}$$

Since $\alpha > 0$, x_n is bounded. Hence, there exists a weakly convergent subsequence $x_{n(k)}$ with a weak limit $x \in X$. Moreover, it follows from (6.31) that $F(x_{n(k)})$ is bounded. Therefore, there exists a further subsequence such that $F(x_{n(k(l))})$ is weakly convergent. Now the weak closedness of F implies that $x \in D(F)$ and that $F(x_{n(k(l))}) \rightharpoonup F(x)$ as $l \to \infty$. It follows from the fact that the norm is weakly lower semicontinuous, i.e.,

$$\varphi_n \rightharpoonup \varphi \ \Rightarrow \limsup_{n\to\infty} \|\varphi_n\| \ge \|\varphi\|\,,$$

that

$$\|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 \le \limsup_{n\to\infty} \left\{\|F(x_n) - y^\delta\|^2 + \alpha\|x_n - x_0\|^2\right\} \le I.$$

Hence, x is a global minimum of the Tikhonov functional.
♢

We do not know if a solution to (6.30) is unique. Nevertheless, it can be shown that an arbitrary sequence of minimizers converges to the exact solution $x^\dagger$ as the noise level δ tends to 0. This means that nonlinear Tikhonov regularization is a regularization method.

Theorem 6.4. *Assume that F is weakly closed and that (6.29) holds true. Let $\alpha = \overline{\alpha}(\delta)$ be chosen such that*

$$\overline{\alpha}(\delta) \to 0 \quad \textit{and} \quad \delta^2/\overline{\alpha}(\delta) \to 0 \qquad \textit{as } \delta \to 0. \tag{6.32}$$

If y^{δ_k} is some sequence in Y such that $\|y^{\delta_k} - y\| \le \delta_k$ and $\delta_k \to 0$ as $k \to \infty$, and if $x_{\alpha_k}^{\delta_k}$ denotes a solution to (6.30) with $y^\delta = y^{\delta_k}$ and $\alpha = \alpha_k = \overline{\alpha}(\delta_k)$, then $\|x_{\alpha_k}^{\delta_k} - x^\dagger\| \to 0$ as $k \to \infty$.

Proof. Since $x_{\alpha_k}^{\delta_k}$ minimizes the Tikhonov functional, we have

$$\begin{aligned}\|F(x_{\alpha_k}^{\delta_k}) - y^{\delta_k}\|^2 + \alpha_k\|x_{\alpha_k}^{\delta_k} - x_0\|^2 &\leq \|F(x^\dagger) - y^{\delta_k}\|^2 + \alpha_k\|x^\dagger - x_0\|^2 \\ &\leq \delta_k^2 + \alpha_k\|x^\dagger - x_0\|^2.\end{aligned}$$

The assumptions $\delta_k \to 0$ and $\alpha_k \to 0$ imply that

$$\lim_{k\to\infty} F(x_{\alpha_k}^{\delta_k}) = y, \tag{6.33}$$

and the assumption $\delta_k^2/\alpha_k \to 0$ yields

$$\limsup_{k\to\infty} \|x_{\alpha_k}^{\delta_k} - x_0\|^2 \leq \limsup_{k\to\infty} \left\{\delta_k^2/\alpha_k + \|x^\dagger - x_0\|^2\right\} = \|x^\dagger - x_0\|^2. \tag{6.34}$$

It follows from (6.34) that there exists a weakly convergent subsequence of $x_{\alpha_k}^{\delta_k}$ with some weak limit $x \in X$. By virtue of the boundedness of $\|F(x_{\alpha_k}^{\delta_k})\|$ and the weak closedness of F we have $x \in D(F)$ and $F(x) = y$, so $x = x^\dagger$ by (6.29).

It remains to show that $\|x_{\alpha_k}^{\delta_k} - x^\dagger\| \to 0$. Assume on the contrary that there exists $\epsilon > 0$ such that

$$\|x_{\alpha_k}^{\delta_k} - x^\dagger\| \geq \epsilon \tag{6.35}$$

for some subsequence of $(x_{\alpha_k}^{\delta_k})$ which may be assumed to be identical to $(x_{\alpha_k}^{\delta_k})$ without loss of generality. By the argument above, we may further assume that $x_{\alpha_k}^{\delta_k} \rightharpoonup x^\dagger$. Since

$$\|x_{\alpha_k}^{\delta_k} - x^\dagger\|^2 = \|x_{\alpha_k}^{\delta_k} - x_0\|^2 + \|x_0 - x^\dagger\|^2 + 2\langle x_{\alpha_k}^{\delta_k} - x_0, x_0 - x^\dagger\rangle,$$

it follows from (6.34) that

$$\limsup_{k\to\infty} \|x_{\alpha_k}^{\delta_k} - x^\dagger\|^2 \leq 2\|x^\dagger - x_0\|^2 + 2\langle x^\dagger - x_0, x_0 - x^\dagger\rangle = 0.$$

This contradicts (6.35).
$\diamondsuit$

As in the linear case we need a source condition to establish estimates on the rate of convergence as $\delta \to 0$. Source conditions for nonlinear problems usually involve $x^\dagger - x_0$ instead of $x^\dagger$ because of the loss of the special role of $0 \in X$.

Theorem 6.5. *Assume that F is weakly closed and Fréchet differentiable, that $D(F)$ is convex, and that there exists a Lipschitz constant $L > 0$ such that*

$$\|F'[x] - F'[\tilde{x}]\| \leq L\|x - \tilde{x}\| \tag{6.36}$$

for all $x, \tilde{x} \in D(F)$. Moreover, assume that the source condition

$$\begin{aligned} x^\dagger - x_0 &= F'[x^\dagger]^* w, && \text{(6.37a)} \\ L\|w\| &< 1 && \text{(6.37b)} \end{aligned}$$

is satisfied for some $w \in Y$ and that a parameter choice rule $\alpha = c\delta$ with some $c > 0$ is used. Then there exists a constant $C > 0$ independent of δ such that every global minimum x_α^δ of the Tikhonov functional satisfies the estimates

$$\|x_\alpha^\delta - x^\dagger\| \leq C\sqrt{\delta}, \tag{6.38a}$$

$$\|F(x_\alpha^\delta) - y\| \leq C\delta. \tag{6.38b}$$

Proof. As in the proof of the Theorem 6.4 we use the inequality

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x_0\|^2 \leq \delta^2 + \alpha\|x^\dagger - x_0\|^2$$

for global minimum x_α^δ of the Tikhonov functional. Since this estimate would not give the optimal rate for $\|F(x_\alpha^\delta) - y^\delta\|^2$, we add $\alpha\|x_\alpha^\delta - x^\dagger\|^2 - \alpha\|x_\alpha^\delta - x_0\|^2$ on both sides to obtain

$$\begin{aligned} \|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 &\leq \delta^2 + 2\alpha\langle x^\dagger - x_0, x^\dagger - x_\alpha^\delta\rangle \\ &= \delta^2 + 2\alpha\langle w, F'[x^\dagger](x^\dagger - x_\alpha^\delta)\rangle. \end{aligned}$$

Here (6.37a) has been used in the second line. Using the Cauchy-Schwarz inequality and inserting the inequality

$$\begin{aligned} \|F'[x^\dagger](x^\dagger - x_\alpha^\delta)\| &\leq \frac{L}{2}\|x_\alpha^\delta - x^\dagger\|^2 + \|F(x_\alpha^\delta) - F(x^\dagger)\| \\ &\leq \frac{L}{2}\|x_\alpha^\delta - x^\dagger\|^2 + \|F(x_\alpha^\delta) - y^\delta\| + \delta, \end{aligned}$$

which follows from (6.36) yields

$$\|F(x_\alpha^\delta) - y^\delta\|^2 + \alpha\|x_\alpha^\delta - x^\dagger\|^2 \leq \delta^2 + 2\alpha\delta\|w\| + 2\alpha\|w\|\,\|F(x_\alpha^\delta) - y^\delta\| + \alpha L\|w\|\|x_\alpha^\delta - x^\dagger\|^2,$$

and hence

$$\left(\|F(x_\alpha^\delta) - y^\delta\| - \alpha\|w\|\right)^2 + \alpha(1 - L\|w\|)\|x_\alpha^\delta - x^\dagger\|^2 \leq (\delta + \alpha\|w\|)^2.$$

Therefore,

$$\|F(x_\alpha^\delta) - y^\delta\| \leq \delta + 2\alpha\|w\|$$

and, due to (6.37b),

$$\|x_\alpha^\delta - x^\dagger\| \leq \frac{\delta + \alpha\|w\|}{\sqrt{\alpha(1 - L\|w\|)}}.$$

With the parameter choice rule $\alpha = c\delta$, this yields (6.38).
$\diamondsuit$

By virtue of (6.22), condition (6.37a) is equivalent to

$$x^\dagger - x_0 = (F'[x^\dagger]^* F'[x^\dagger])^{1/2}\tilde{w}, \qquad L\|\tilde{w}\| < 1$$

for some $\tilde{w} \in X$. Hence, for linear problems Theorem 6.5 reduces to a special case of Theorem 6.2.

Theorem 6.5 was obtained by Engl, Kunisch and Neubauer, 1989. Other Hölder-type source conditions with a-priori parameter choice rules are treated in [Neubauer, 1989], and a posteriori parameter choice rules were investigated in [Scherzer, Engl, Kunisch, 1993]. For further references and results we refer to Chapter 10 in [EnglHankeNeubauer].

6.3. Nonlinear Landweber Iteration

Assume that F has a continuous Fréchet-derivative $F'(\cdot)$.
The *nonlinear Landweber iteration* is defined via

$$x_{k+1}^\delta = x_k^\delta + F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\,, \quad k \in \mathbb{N}_0\,, \tag{6.39}$$

where y^δ are noisy data satisfying the estimate (6.2), $x_0^\delta = x_0$ is an initial guess which may incorporate a-priori knowledge of an exact solution.

Denote by x_k the Landweber iterates for exact data, $y = y^\delta$.

As a stopping rule, we use the *discrepancy principle*, i.e., we determine $k_* = k_*(\delta, y^\delta)$ such that

$$\|y^\delta - F(x_{k_*}^\delta)\| \leq \tau\delta < \|y^\delta - F(x_k^\delta)\|\,, \qquad 0 \leq k < k_*\,, \tag{6.40}$$

where $\tau > 1$ is appropriately chosen.

The convergence and convergence rates results quoted here are taken from [Hanke, Neubauer, Scherzer, Numerische Mathematik, 1995]

6.3.1. Basic Conditions

Local convergence: consider solution as well as iterates in a (closed) ball $\mathcal{B}_{2\rho}(x_0)$.
Scaling:

$$\|F'(x)\| \leq 1\,, \qquad x \in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)\,, \tag{6.41}$$

Nonlinearity condition:

$$\begin{aligned}&\|F(x) - F(\tilde{x}) - F'(x)(x - \tilde{x})\| \leq \eta\|F(x) - F(\tilde{x})\|\,, \qquad \eta < \tfrac{1}{2}\,,\\ &x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)\,.\end{aligned} \tag{6.42}$$

From (6.42) it follows immediately with the triangle inequality that:

$$\frac{1}{1+\eta}\|F'(x)(\tilde{x} - x)\| \leq \|F(\tilde{x}) - F(x)\| \leq \frac{1}{1-\eta}\|F'(x)(\tilde{x} - x)\| \tag{6.43}$$

In particular, this implies that

$$F(x) = F(\tilde{x}) \;\Leftrightarrow\; x - \tilde{x} \in \mathcal{N}(F'(x))$$

provided that $x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0)$.
Solvability: in $\mathcal{B}_\rho(x_0)$: $y \in F(\mathcal{B}_\rho(x_0))$. Implies existence of unique solution of minimal distance to x_0 (x_0-*minimum-norm-solution*) denoted by $x^\dagger$, which satisfies

$$x^\dagger - x_0 \in \mathcal{N}(F'(x^\dagger))^\perp\,. \tag{6.44}$$

6.3.2. Convergence of the Landweber Iteration

Lemma 6.6. *(Stability)*
For fixed k, the Landweber iterate x_k^δ depends continuously on the data y^δ.

Proof. x_k^δ is the result of a combination of continuous operations.
♢

Proposition 6.7. *(Monotonicity)*
Assume that the conditions (6.41) and (6.42) hold and that $F(x) = y$ has a solution $x_* \in \mathcal{B}_\rho(x_0)$. If $x_k^\delta \in \mathcal{B}_\rho(x_*)$, and

$$\|y^\delta - F(x_k^\delta)\| > 2\,\frac{1+\eta}{1-2\eta}\,\delta\,, \tag{6.45}$$

then

$$\|x_{k+1}^\delta - x_*\| < \|x_k^\delta - x_*\|$$

and $x_k^\delta, x_{k+1}^\delta \in \mathcal{B}_\rho(x_*) \subset \mathcal{B}_{2\rho}(x_0)$.

Proof. Assume that $x_k^\delta \in \mathcal{B}_\rho(x_*)$ which is a subset of $\mathcal{B}_{2\rho}(x_0)$ by the triangle inequality $\Rightarrow$ (6.41) and (6.42) are applicable.

$$\begin{aligned}
&\|x_{k+1}^\delta - x_*\|^2 - \|x_k^\delta - x_*\|^2 \\
&\quad= 2\langle x_{k+1}^\delta - x_k^\delta, x_k^\delta - x_*\rangle + \|x_{k+1}^\delta - x_k^\delta\|^2 \\
&\quad= 2\langle y^\delta - F(x_k^\delta), F'(x_k^\delta)(x_k^\delta - x_*)\rangle + \|F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\|^2 \\
&\quad\le 2\langle y^\delta - F(x_k^\delta), y^\delta - F(x_k^\delta) - F'(x_k^\delta)(x_* - x_k^\delta)\rangle - \|y^\delta - F(x_k^\delta)\|^2 \\
&\quad\le \|y^\delta - F(x_k^\delta)\|\,(2\delta + 2\eta\,\|y - F(x_k^\delta)\| - \|y^\delta - F(x_k^\delta)\|\,) \\
&\quad\le \|y^\delta - F(x_k^\delta)\|\,(2(1+\eta)\delta - (1-2\eta)\,\|y^\delta - F(x_k^\delta)\|\,)
\end{aligned} \tag{6.46}$$

Assertions now follows from (6.45). holds.
♢

The estimate (6.45) suggests to choose τ in the stopping rule (6.40) such that

$$\tau > 2\,\frac{1+\eta}{1-2\eta} > 2 \tag{6.47}$$

Corollary 6.8. *(Estimate of k_*) Let the assumptions of Proposition 6.7 hold and let k_* be chosen according to the stopping rule (6.40), (6.47). Then*

$$k_*(\tau\delta)^2 < \sum_{k=0}^{k_*-1} \|y^\delta - F(x_k^\delta)\|^2 \le \frac{\tau}{(1-2\eta)\tau - 2(1+\eta)}\,\|x_0 - x_*\|^2\,. \tag{6.48}$$

In particular, if $y^\delta = y$ (i.e., if $\delta = 0$), then

$$\sum_{k=0}^{\infty} \|y - F(x_k)\|^2 < \infty \,. \tag{6.49}$$

Therefore, if x_k converges, then the limit is a solution of $F(x) = y$.

Proof. From $x_0^\delta = x_0 \in \mathcal{B}_\rho(x_*)$, we get by induction that Proposition 6.7 is applicable and $x_{k+1}^\delta \in \mathcal{B}_{2\rho}(x_0)$ for all $0 \le k < k_*$. By (6.46)

$$\|x_{k+1}^\delta - x_*\|^2 - \|x_k^\delta - x_*\|^2 \le \|y^\delta - F(x_k^\delta)\|^2(\tfrac{2}{\tau}(1+\eta) + 2\eta - 1)$$

Summing up both sides for k from 0 to $k_* - 1$, we obtain

$$(1 - 2\eta - \tfrac{2}{\tau}(1+\eta)) \sum_{k=0}^{k_*-1} \|y^\delta - F(x_k^\delta)\|^2 \le \|x_0 - x_*\|^2 - \|x_{k_*}^\delta - x_*\|^2$$

which together with (6.40) yields (6.48).
Case $\delta = 0$: (6.48) holds for arbitrary k_*, let τ tend to ∞:

$$\sum_{k=0}^{\infty} \|y - F(x_k)\|^2 \le \frac{1}{(1-2\eta)} \|x_0 - x_*\|^2 \,.$$

$\diamondsuit$

Theorem 6.9. *(Convergence of Landeweber with exact data)*
Assume that the conditions (6.41) and (6.42) hold and that $F(x) = y$ is solvable in $\mathcal{B}_\rho(x_0)$. Then the nonlinear Landweber iteration applied to exact data y converges to a solution of $F(x) = y$. If $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in \mathcal{B}_\rho(x^\dagger)$, then x_k converges to $x^\dagger$ as $k \to \infty$.

Proof. Unique x_0-minimum-norm-solution, $x^\dagger$, exists in $\mathcal{B}_\rho(x_0)$. For

$$e_k := x_k - x^\dagger \,,$$

Proposition 6.7 implies that $\|e_k\|$ monotonically decreases to some $\varepsilon \ge 0$.
We show that $\{e_k\}$ is a Cauchy sequence: Given $j \ge k$, choose l with $k \le l \le j$ such that

$$\|y - F(x_l)\| \le \|y - F(x_i)\| \qquad \text{for all}\, k \le i \le j \,. \tag{6.50}$$

Triangle inequality $\Rightarrow$

$$\|e_j - e_k\| \le \|e_j - e_l\| + \|e_l - e_k\| \,, \tag{6.51}$$

with

$$\begin{aligned} \|e_j - e_l\|^2 &= 2\langle e_l - e_j, e_l\rangle + \underbrace{\|e_j\|^2}_{\to\varepsilon^2 \ as \ k\to\infty} - \underbrace{\|e_l\|^2}_{\to\varepsilon^2 \ as \ k\to\infty} \,, \\ \|e_l - e_k\|^2 &= 2\langle e_l - e_k, e_l\rangle + \underbrace{\|e_k\|^2}_{\to\varepsilon^2 \ as \ k\to\infty} - \underbrace{\|e_l\|^2}_{\to\varepsilon^2 \ as \ k\to\infty} \,. \end{aligned} \tag{6.52}$$

It remains to show that $\langle e_l - e_j, e_l\rangle \to 0$, $\langle e_l - e_k, e_l\rangle \to 0$ as $k \to \infty$:

$$
\begin{aligned}
|\langle e_l - e_j, e_l\rangle| &= \Big|\sum_{i=l}^{j-1}\langle F'(x_i)^*(y - F(x_i)), e_l\rangle\Big| \le \sum_{i=l}^{j-1}|\langle y - F(x_i), F'(x_i)(x_l - x^\dagger)\rangle| \\
&\le \sum_{i=l}^{j-1}\|y - F(x_i)\|\,\|F'(x_i)(x_l - x_i + x_i - x^\dagger)\| \\
&\le \sum_{i=l}^{j-1}\|y - F(x_i)\|\Big(\|y - F(x_i) - F'(x_i)(x^\dagger - x_i)\| \\
&\qquad + \|y - F(x_l)\| + \|F(x_i) - F(x_l) - F'(x_i)(x_i - x_l)\|\Big) \\
&\le (1+\eta)\sum_{i=l}^{j-1}\|y - F(x_i)\|\,\|y - F(x_l)\| + 2\eta\sum_{i=l}^{j-1}\|y - F(x_i)\|^2 \\
&\le (1+3\eta)\sum_{i=l}^{j-1}\|y - F(x_i)\|^2, \qquad (6.53)
\end{aligned}
$$

due to (6.50). Analogously

$$
|\langle e_l - e_k, e_l\rangle| \le (1+3\eta)\sum_{i=k}^{l-1}\|y - F(x_i)\|^2\,. \qquad (6.54)
$$

Since $\|y - F(x_i)\|^2$ is summable the right hand sides in (6.53) and (6.54) have to go to zero as $k \to \infty$.
Altogether, we have shown that e_k and therefore x_k is a Cauchy sequence and therefore has a limit x_* which by Corollary 6.8, has to be a solution of $F(x) = y$.

If $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in \mathcal{B}_\rho(x^\dagger)$, then by the iteration rule (6.39)

$$
\forall k \in \mathbb{N}\ :\ x_{k+1} - x_k \in \mathcal{R}(F'(x_k)^*) \subset \mathcal{N}(F'(x_k))^\perp \subset \mathcal{N}(F'(x^\dagger))^\perp
$$

$$
\Rightarrow\quad \forall k \in \mathbb{N}\ : x_k - x_0 \in \mathcal{N}(F'(x^\dagger))^\perp \overset{k\to\infty}{\Rightarrow} x_* - x_0 \in \mathcal{N}(F'(x^\dagger))^\perp \overset{(6.44)}{\Rightarrow} x_* = x^\dagger
$$

$\diamondsuit$

Theorem 6.10. *(Convergence of Landeweber with noisy data)*
Let the assumptions of Theorem 6.9 hold and let $k_* = k_*(\delta, y^\delta)$ be chosen according to the stopping rule (6.40), (6.47). Then the Landweber iterates $x^\delta_{k_*}$ converge to a solution of $F(x) = y$. If $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in \mathcal{B}_\rho(x^\dagger)$, then $x^\delta_{k_*}$ converges to $x^\dagger$ as $\delta \to 0$.

Proof. According to Theorem 6.9, the limit x_* of the Landweber iteration with precise data y exists. Let (δ_n) be a sequence of noise levels with $\delta_n \to 0$ as $n \to \infty$, $y_n := y^{\delta_n}$ such that (6.2) holds with $\delta := \delta_n$, denote $k_n = k_*(\delta_n, y_n)$ according to discrepancy principle for $y^\delta := y_n$, $\delta := \delta_n$.

Case 1: k_n has a finite accumulation point k:
Wlog $k_n = k$ for all $n \in \mathbb{N}$. Discrepancy principle $\Rightarrow$

$$\|y_n - F(x_k^{\delta_n})\| \le \tau\delta_n \,. \tag{6.55}$$

k fixed $\Rightarrow$ x_k^δ depends continuously on y^δ, hence,

$$x_k^{\delta_n} \to x_k\,, \quad F(x_k^{\delta_n}) \to F(x_k) \quad \text{as} \quad n \to \infty\ ,.$$

and by (6.55)

$$F(x_k) = y\,.$$

$\Rightarrow$ Iteration with exact data terminates at k $\Rightarrow$ $x_* = x_k$.

Case 2: $k_n \to \infty$ as $n \to \infty$:
Wlog $k_n \nearrow \infty$. Proposition 6.7 yields

$$\begin{aligned} &\forall n \ge m : && \qquad (6.56)\\ &\|x_{k_n}^{\delta_n} - x_*\| \le \|x_{k_{n-1}}^{\delta_n} - x_*\| \le \ldots \le \|x_{k_m}^{\delta_n} - x_*\| \\ &\qquad\qquad \le \|x_{k_m}^{\delta_n} - x_{k_m}\| + \|x_{k_m} - x_*\|\,. && \qquad (6.57)\end{aligned}$$

$\varepsilon > 0$ arbitrarily fixed:
By Theorem 6.9, choose $m = m(\varepsilon)$ so that $\|x_{k_m} - x_*\| \le \varepsilon/2$
Keep m fixed, let $n \to \infty$ using stability of Landweber for fixed k_m
$\Rightarrow$ $\|x_{k_m}^{\delta_n} - x_{k_m}\| < \varepsilon/2$, for $n > n(\varepsilon)$ sufficiently large.
Thus, by (6.57) $x_{k_n}^{\delta_n} \to x_*$ as $n \to \infty$.
$\diamondsuit$

6.4. Iteratively Regularized Gauss-Newton Method

In this section we deal with the iteratively regularized Gauss-Newton method

$$x_{k+1}^\delta = x_k^\delta + (F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1}(F'(x_k^\delta)^*(y^\delta - F(x_k^\delta)) + \alpha_k(x_0 - x_k^\delta))\,, \tag{6.58}$$

where, as always, $x_0^\delta = x_0$ is an initial guess for the true solution, α_k is a sequence of positive numbers tending towards zero, and y^δ are noisy data satisfying the estimate (6.2). This method is quite similar to Levenberg-Marquardt iterations

$$x_{k+1}^\delta = x_k^\delta + (F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)^{-1} F'(x_k^\delta)^*(y^\delta - F(x_k^\delta))\,, \tag{6.59}$$

cf. [Hanke, Inverse Problems, 1997], but for the latter, optimal convergence rates are to some extent still an open problem.

Note that the approximate solution x_{k+1}^δ minimizes the functional

$$\phi(x) := \|y^\delta - F(x_k^\delta) - F'(x_k^\delta)(x - x_k^\delta)\|^2 + \alpha_k \|x - x_0\|^2 .$$

This means that x_{k+1}^δ solves the Tikhonov functional where the nonlinear function F is linearized around x_k^δ.

Similarly to Landweber iteration, for a fixed number of iterations the process (6.58) is a stable algorithm if $F'(\cdot)$ is continuous.

Literature: [Bakushinskii,Comput. Math. Math. Phys., 1992], [BK,Neubauer,Scherzer, IMA J.Numer.Anal, 1997], [BK, Inverse Problems, 1997] [Hohage, Inverse Problems, 1997]

6.4.1. Convergence analysis with a priori stopping rule

Assumption 6.11. Let ρ be a positive number.

(i) The equation $F(x) = y$ is solvable in $\mathcal{B}_\rho(x_0)$ and $\mathcal{B}_{2\rho}(x_0) \subset \mathcal{D}(F)$. The x_0-minimum-norm-solution is denoted by $x^\dagger$.

(ii) $x^\dagger$ satisfies the smoothness condition

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger))^\mu v , \quad v \in \mathcal{N}(F'(x^\dagger))^\perp . \tag{6.60}$$

for some $0 \le \mu \le 1$, cf. (6.21) for the linear case.

(iii) If $\mu < \frac{1}{2}$, the Fréchet-derivative F' satisfies the following conditions

$$\begin{aligned} F'(\tilde{x}) &= R(\tilde{x}, x)F'(x) + Q(\tilde{x}, x) && (6.61)\\ \|I - R(\tilde{x}, x)\| &\le c_R && (6.62)\\ \|Q(\tilde{x}, x)\| &\le c_Q \|F'(x^\dagger)(\tilde{x} - x)\| && (6.63) \end{aligned}$$

for $x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0)$, where c_R and c_Q are nonnegative constants with $c_R + c_Q > 0$.

If $\mu \ge \frac{1}{2}$, the Fréchet-derivative F' is Lipschitz continuous in $\mathcal{B}_{2\rho}(x_0)$, i.e.,

$$\|F'(\tilde{x}) - F'(x)\| \le L \|\tilde{x} - x\| , \qquad x, \tilde{x} \in \mathcal{B}_{2\rho}(x_0) \tag{6.64}$$

for some $L > 0$.

(iv) The sequence $\{\alpha_k\}$ in (6.58) satisfies

$$\alpha_k > 0 , \qquad 1 \le \frac{\alpha_k}{\alpha_{k+1}} \le r , \qquad \lim_{k\to\infty} \alpha_k = 0 , \tag{6.65}$$

for some $r > 1$.

It can be show thatconditions (6.61) – (6.63) imply that F is constant on $x^\dagger + \mathcal{N}(F'(x^\dagger)) \cap \mathcal{B}_\rho(x^\dagger)$ assuming that ρ, c_R and c_Q are sufficiently small (compare (6.43)). Note that these conditions are slightly stronger than the convergence condition (6.42) for Landweber iterates. Moreover, the condition $x^\dagger - x_0 \in \mathcal{N}(F'(x^\dagger))^\perp$, which is an immediate consequence of Assumption 6.11 (ii), is not restrictive. It is automatically satisfied for the then unique x_0-minimum-norm-solution.

Before we can prove convergence or convergence rates for the iteration process (6.58) we need some preparatory lemmata. The first one gives an estimate on the error in the special case of the forward operator being linear $F(x) = Kx$:

Lemma 6.12. *Let $K \in \mathcal{L}(X,Y)$, $s \in [0,1]$, and let $\{\alpha_k\}$ be a sequence satisfying $\alpha_k > 0$ and $\alpha_k \to 0$ as $k \to \infty$. Then it holds that*

$$w_k(s) := \alpha_k^{1-s} \|(K^*K + \alpha_k I)^{-1}(K^*K)^s v\| \le s^s(1-s)^{1-s} \|v\| \le \|v\| \tag{6.66}$$

and that

$$\lim_{k\to\infty} w_k(s) = \begin{cases} 0\,, & 0 \le s < 1\,, \\ \|v\|\,, & s = 1\,, \end{cases} \tag{6.67}$$

for any $v \in \mathcal{N}(A)^\perp$.

Proof. The assertions follow with spectral theory.
$\diamondsuit$

In the second lemma we derive an estimate for $\|e_{k+1}^\delta\|$ and $\|Ke_{k+1}^\delta\|$ assuming that $x_k^\delta \in \mathcal{B}_\rho(x^\dagger)$.

Lemma 6.13. *Let Assumption 6.11 hold and assume that $x_k^\delta \in \mathcal{B}_\rho(x^\dagger)$. Moreover, we put $K := F'(x^\dagger)$ and $e_k^\delta := x_k^\delta - x^\dagger$.*

(i) If $0 \le \mu < \frac{1}{2}$, we obtain the estimates

$$\begin{aligned}\|e_{k+1}^\delta\| &\le \alpha_k^\mu w_k(\mu) + c_R \alpha_k^\mu w_k(\mu+\tfrac{1}{2}) \\ &\quad + c_Q \,\|Ke_k^\delta\|\,\alpha_k^{\mu-\frac{1}{2}}(\tfrac{1}{2}w_k(\mu) + w_k(\mu+\tfrac{1}{2})) \\ &\quad + \alpha_k^{-\frac{1}{2}}(c_R\|Ke_k^\delta\| + \tfrac{3}{4}c_Q\,\|e_k^\delta\|\,\|Ke_k^\delta\| + \tfrac{1}{2}\delta)\,,\end{aligned} \tag{6.68}$$

$$\begin{aligned}\|Ke_{k+1}^\delta\| &\le (1+2c_R(1+c_R))\alpha_k^{\mu+\frac{1}{2}} w_k(\mu+\tfrac{1}{2}) \\ &\quad + \|Ke_k^\delta\|\Big(c_Q(1+c_R)\alpha_k^\mu(w_k(\mu)+\tfrac{1}{2}w_k(\mu+\tfrac{1}{2})) \\ &\qquad + c_Q c_R \alpha_k^\mu w_k(\mu+\tfrac{1}{2}) + (1+c_R)(2c_R + \tfrac{3}{2}c_Q\,\|e_k^\delta\|)\Big) \\ &\quad + \|Ke_k^\delta\|^2 c_Q\Big(c_Q\alpha_k^{\mu-\frac{1}{2}}(\tfrac{1}{2}w_k(\mu)+w_k(\mu+\tfrac{1}{2})) \\ &\qquad + \alpha_k^{-\frac{1}{2}}(c_R + \tfrac{3}{4}c_Q\,\|e_k^\delta\|)\Big) \\ &\quad + (1+c_R+\tfrac{1}{2}c_Q\alpha_k^{-\frac{1}{2}}\,\|Ke_k^\delta\|)\,\delta\,.\end{aligned} \tag{6.69}$$

(ii) If $\frac{1}{2} \le \mu \le 1$, we obtain the estimates

$$\begin{aligned}\|e_{k+1}^\delta\| &\le \alpha_k^\mu w_k(\mu) + L\,\|e_k^\delta\|\,(\tfrac{1}{2}\alpha_k^{\mu-\frac{1}{2}}w_k(\mu) + \|(K^*K)^{\mu-\frac{1}{2}}v\|) \qquad (6.70)\\ &\quad + \tfrac{1}{2}\alpha_k^{-\frac{1}{2}}(\tfrac{1}{2}L\,\|e_k^\delta\|^2 + \delta)\,, \\ \|Ke_{k+1}^\delta\| &\le \alpha_k\,\|(K^*K)^{\mu-\frac{1}{2}}v\| + L^2\,\|e_k^\delta\|^2(\tfrac{1}{2}\alpha_k^{\mu-\frac{1}{2}}w_k(\mu) + \|(K^*K)^{\mu-\frac{1}{2}}v\|) \\ &\quad + L\alpha_k^{\frac{1}{2}}\,\|e_k^\delta\|\,(\alpha_k^{\mu-\frac{1}{2}}w_k(\mu) + \tfrac{1}{2}\,\|(K^*K)^{\mu-\frac{1}{2}}v\|) \qquad (6.71)\\ &\quad + (\tfrac{1}{2}L\alpha_k^{-\frac{1}{2}}\,\|e_k^\delta\| + 1)(\tfrac{1}{2}L\,\|e_k^\delta\|^2 + \delta)\,.\end{aligned}$$

Proof. We put $K_k := F'(x_k^\delta)$. Due to (6.60), we can rewrite (6.58) as follows

$$\begin{aligned}e_{k+1}^\delta &= -\alpha_k(K^*K+\alpha_k I)^{-1}(K^*K)^\mu v \\ &\quad - \alpha_k(K_k^*K_k+\alpha_k I)^{-1}\Big(K_k^*(K-K_k) \\ &\qquad\qquad + (K^*-K_k^*)K\Big)(K^*K+\alpha_k I)^{-1}(K^*K)^\mu v \\ &\quad + (K_k^*K_k+\alpha_k I)^{-1}K_k^*(y^\delta - F(x_k^\delta) + K_k e_k^\delta)\,.\end{aligned} \tag{6.72}$$

(i) Here we only carry out the proof for the case that $0 \le \mu < \frac{1}{2}$ with $Q = 0$ in Assumption 6.11 (iii). Since $x_k^\delta \in \mathcal{B}_\rho(x^\dagger)$, Assumption 6.11 (i) implies that (6.61) –(6.63) are applicable. By the nonlinearity assumptions, we obtain that

$$\|F(x_k^\delta) - F(x^\dagger) - K_k e_k^\delta\| = \|\int_0^1 (R(x^\dagger+\theta e_k^\delta, x^\dagger) - R(x_k^\delta, x^\dagger))\,d\theta K e_k^\delta\| \le 2c_R\,\|Ke_k^\delta\|\,. \tag{6.73}$$

The well-known estimates (see (6.66))

$$\|(K_k^* K_k + \alpha_k I)^{-1}\| \leq \alpha_k^{-1} \qquad \wedge \qquad \|(K_k^* K_k + \alpha_k I)^{-1} K_k^*\| \leq \tfrac{1}{2}\alpha_k^{-\frac{1}{2}}$$

and

$$K_k^*(K - K_k) + (K^* - K_k^*)K \quad = \quad K_k^*(R^*(x^\dagger, x_k^\delta) - R(x_k^\delta, x^\dagger))K$$

imply that

$$\begin{aligned} &\|\alpha_k (K_k^* K_k + \alpha_k I)^{-1}(K_k^*(K - K_k) + (K^* - K_k^*)K)(K^*K + \alpha_k I)^{-1}(K^*K)^\mu v\| \\ &\quad \leq \tfrac{1}{2}\alpha_k^{-\frac{1}{2}} \|R^*(x^\dagger, x_k^\delta) - R(x_k^\delta, x^\dagger)\|)\, \|K(K^*K + \alpha_k I)^{-1}(K^*K)^\mu v\| \,. \end{aligned}$$

This together with (6.2), (6.66), (6.72), and $F(x^\dagger) = y$ yields the estimate (6.68). Note that

$$\|K(K^*K + \alpha_k I)^{-1}(K^*K)^\mu v\| = \alpha_k^{\mu - \frac{1}{2}} w_k(\mu + \tfrac{1}{2})\,.$$

Since, due to (6.61), $K = R(x^\dagger, x_k^\delta)K_k$, we obtain together with (6.72) that

$$\begin{aligned} K e_{k+1}^\delta \quad = \quad & -\alpha_k K(K^*K + \alpha_k I)^{-1}(K^*K)^\mu v \\ & -\alpha_k R(x^\dagger, x_k^\delta)K_k(K_k^* K_k + \alpha_k I)^{-1} \\ & \qquad (K_k^*(R^*(x^\dagger, x_k^\delta) - R(x_k^\delta, x^\dagger))K) \\ & \qquad (K^*K + \alpha_k I)^{-1}(K^*K)^\mu v \\ & -R(x^\dagger, x_k^\delta)K_k(K_k^* K_k + \alpha_k I)^{-1}K_k^* \\ & \qquad (F(x_k^\delta) - F(x^\dagger) - K_k e_k^\delta + y - y^\delta)\,. \end{aligned}$$

Now the estimate (6.69) for $\|K e_{k+1}^\delta\|$ follows together with (6.2), (6.62), (6.63), (6.66) and (6.73).

We will consider now the following a-priori stopping rule, where the iteration is stopped after $k_* = k_*(\delta)$ steps with

$$\begin{cases} \eta \alpha_{k_*}^{\mu+\frac{1}{2}} \leq \delta < \eta \alpha_k^{\mu+\frac{1}{2}}\,, \qquad 0 \leq k < k_*\,, & 0 < \mu \leq 1\,, \\ k_*(\delta) \to \infty \quad \text{and} \quad \eta \geq \delta \alpha_{k_*}^{-\frac{1}{2}} \to 0 \quad \text{as} \quad \delta \to 0\,, & \mu = 0\,, \end{cases} \tag{6.74}$$

for some $\eta > 0$. Note that, due to (6.65), this guarantees that $k_*(\delta) < \infty$, if $\delta > 0$ and that $k_*(\delta) \to \infty$ as $\delta \to 0$. In the noise free case ($\delta = 0$) we can set $k_*(0) := \infty$ and $\eta := 0$.

We will prove in the next theorem that this a-priori stopping rule yields convergence and convergence rates for the iteratively regularized Gauss-Newton method (6.58) provided that $\|v\|$, ρ, c_R, c_Q, L and η are sufficiently small.

Theorem 6.14. *Let Assumption 6.11 hold and let $k_* = k_*(\delta)$ be chosen according to (6.74). Moreover, let ρ, η, $\|v\|$ and , in case $\mu \le \frac{1}{2}$, c_R be sufficiently small. Then we obtain that*

$$\|x^\delta_{k_*} - x^\dagger\| = \begin{cases} o(1)\,, & \mu = 0\,, \\ O\Big(\delta^{\frac{2\mu}{2\mu+1}}\Big)\,, & 0 < \mu \le 1\,. \end{cases}$$

For the noise free case ($\delta = 0$, $\eta = 0$) we obtain that

$$\|x_k - x^\dagger\| = \begin{cases} o(\alpha_k^\mu)\,, & 0 \le \mu < 1\,, \\ O(\alpha_k)\,, & \mu = 1\,, \end{cases}$$

and that

$$\|F(x_k) - y\| = \begin{cases} o\Big(\alpha_k^{\mu+\frac{1}{2}}\Big)\,, & 0 \le \mu < \frac{1}{2}\,, \\ O(\alpha_k)\,, & \frac{1}{2} \le \mu \le 1\,. \end{cases}$$

Proof. We here again only give the idea of proof in the case $\mu \le \frac{1}{2}$ and $Q = 0$, where we have, by Lemma 6.13,

$$\begin{aligned} \|e^\delta_{k+1}\| &\le \alpha_k^\mu w_k(\mu) + c_R \alpha_k^\mu w_k(\mu + \tfrac{1}{2}) \\ &\quad + \alpha_k^{-\frac{1}{2}}(c_R\|Ke^\delta_k\| + \tfrac{1}{2}\delta)\,, \end{aligned}$$

$$\begin{aligned} \|Ke^\delta_{k+1}\| &\le (1 + 2c_R(1 + c_R))\alpha_k^{\mu+\frac{1}{2}} w_k(\mu + \tfrac{1}{2}) \\ &\quad + (1 + c_R)\Big(2c_R\|Ke^\delta_k\| + \delta\Big) \end{aligned}$$

Setting

$$\gamma_k := \max\left\{ \frac{\|e^\delta_k\|}{\alpha_k^\mu}, \frac{\|Ke^\delta_k\|}{\alpha_k^{\mu+\frac{1}{2}}} \right\}$$

we get, using (6.65)

$$\begin{aligned} \gamma_{k+1} &\le \max\Bigg\{ r^\mu\Big(w_k(\mu) + c_R w_k(\mu + \tfrac{1}{2}) + c_R\gamma_k + \tfrac{1}{2}\frac{\delta}{\alpha_k^{\mu+\frac{1}{2}}}\,, \\ &\qquad r^{\mu+\frac{1}{2}}\Big((1 + 2c_R(1 + c_R))w_k(\mu + \tfrac{1}{2}) + (1 + c_R)2c_R\gamma_k + \frac{\delta}{\alpha_k^{\mu+\frac{1}{2}}} \Bigg\} \\ &\le a_k + b\gamma_k \end{aligned}$$

where

$$\begin{aligned} a_k &= \max\left\{ r^\mu\Big(w_k(\mu) + c_R w_k(\mu + \tfrac{1}{2}) + \tfrac{1}{2}\frac{\delta}{\alpha_k^{\mu+\frac{1}{2}}}\,, r^{\mu+\frac{1}{2}}\Big((1 + 2c_R(1 + c_R))w_k(\mu + \tfrac{1}{2}) + \frac{\delta}{\alpha_k^{\mu+\frac{1}{2}}} \right\} \\ &\quad \to 0 \text{ as } k \to \infty \text{ in case } \delta = 0 \end{aligned}$$

and
$$b = r^{\mu+\frac{1}{2}}(1 + c_R)2c_R < 1$$
for c_R sufficiently small. Moreover, $\frac{\delta}{\alpha_k^{\mu+\frac{1}{2}}} \leq \eta$, for $k \leq k_*$, hence if ρ, η, $\|v\|$ are sufficiently small, then a_k is so small that

$$\sum_{j=0}^{k} b^{k-j} a_j + b^k \gamma_0 \leq \rho$$

for $k \leq k_*$. Thus, one can inductively conclude from $\gamma_{k+1} \leq a_k + b\gamma_k$

$$\gamma_k \leq \sum_{j=0}^{k} b^{k-j} a_j + b^k \gamma_0 \leq \overline{\gamma} \text{ and } x_k^\delta \in \mathcal{B}_{2\rho}(x_0)\,,$$

which implies
$$\|e_{k^*}^\delta\| \leq \overline{\gamma}\alpha_{k_*}^\mu \leq \overline{\gamma}\eta^{-\mu}\delta^{\frac{2\mu}{2\mu+1}}\,.$$

$\diamondsuit$

The same convergence rates can also be established with the discrepancy principle as a stopping rule, but only for $\mu \leq \frac{1}{2}$.

Since α_k can be chosen as $\alpha_k := 2^{-n}$, the iteratively regularized Gauss-Newton method converges much faster than the Landweber iteration method. However, each iteration step is more expensive due to the fact that one has to invert the operator $(F'(x_k^\delta)^* F'(x_k^\delta) + \alpha_k I)$.

Without going into details, we just mention generalizations of the results in the previous sections into several directions.

- First of all, one can consider regularization methods other than Tikhonov regularization for the linear subproblems in each Newton step, cf. [Bakushinski, 1994], [BK,Inverse Problems, 1997].
- Moreover, the theory can be augmented to different regularity assumptions than the Hölder type source conditions (6.60). Especially, logarithmic source conditions are more appropriate for severely ill-posed problems, cf.[Hohage, Inverse Problems, 1997], [Hohage, Dissertation, 1999].
- A convergence analysis can be carried out also with a nonlinearity condition alternative to the assumptions made so far. [BK,Inverse Problems, 1997].

7. Numerical solution techniques: The factorization method for EIT

see

`http://www.numerik.mathematik.uni-mainz.de/~hanke/index_engl.html`

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