Mathematical Description and Modelling of Piezoelectric Systems

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Introduction

- Piezoelectric devices represent an important new group of actuators and sensors for active vibration control. This technology allows to construct spatially distributed devices.
- This fact requires special control techniques to improve the dynamical behavior of this kind of smart structures.
- A deeper insight in the mathematical structure of the models of smart structures will be given by the PCHD-approach.
- The controller design is based on a method for infinite dimensional Hamiltonian systems, which requires the collocation of sensors and actuators and therefore, certain distributed and/or integral quantities must be measured by means of the piezoelectric sensors.
- Some facts of modern control theory and of differential geometry are required

PCHD-Systems, ODE

Choice of a state space:
$\mathcal{X}$: q-dimensionale manifold with coordinates (x^α), $\alpha = 1, \ldots, q$.

Tangential- $\mathcal{T}(\mathcal{X})$, cotangential bundle $\mathcal{T}^*(\mathcal{X})$:

holonomic bases $\{\partial_\alpha\}$, $\{\mathrm{d}x^\alpha\}$ with coordinates $(x^\alpha, \dot{x}^\alpha)$, $(x^\alpha, \dot{x}_\alpha)$.

Structure matrix: $J = -J^T$, dissipative effects: $R = R^T$, $R \geq 0$,

$$J; R : \mathcal{T}^*(\mathcal{X}) \to \mathcal{T}(\mathcal{X}) \qquad \dot{x}^\alpha = \left(J^{\alpha\beta}(x) - R^{\alpha\beta}(x)\right)\dot{x}_\beta \ .$$

Ports: The input space $\mathcal{U} = \mathrm{span}\{e_\varsigma\}$, a vector space with coordinates (u^ς), $\varsigma = 1, \ldots, m$ and a map B,

$$B : \mathcal{U} \to \mathcal{T}(\mathcal{X}) \ , \qquad \dot{x}^\alpha = B^\alpha_\varsigma(x)\, u^\varsigma \ .$$

ODE (2)

The system

$$\dot{x}^\alpha = v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + B_\varsigma^\alpha u^\varsigma \ ,$$
$$y_\varsigma = B_\varsigma^\alpha \partial_\alpha H$$

meets with $v_H = v_H^\alpha \partial_\alpha$,

$$v_H(H) = -\partial_\alpha (H) R^{\alpha\beta}\partial_\beta H + y_\varsigma u^\varsigma \leq y_\varsigma u^\varsigma$$

with the output space $\mathcal{Y} = \mathcal{U}^* = \mathrm{span}\{e^\varsigma\}$ with coordinates (y_ς) and Hamiltonian $H \in C^\infty(\mathcal{X})$.

Exterior derivative:

$$\mathbb{R} \rightarrow C^\infty(\mathcal{X}) \stackrel{\mathrm{d}}{\rightarrow} \wedge^1(\mathcal{T}^*(\mathcal{X})) \cdots \stackrel{\mathrm{d}}{\rightarrow} \wedge^q(\mathcal{T}^*(\mathcal{X})) \stackrel{\mathrm{d}}{\rightarrow} \{0\} \ .$$

ODE (3)

Interior product:

$$\rfloor : \mathcal{T}(\mathcal{X}) \times \wedge^r (\mathcal{T}^*(\mathcal{X})) \to \wedge^{r-1}(\mathcal{T}^*(\mathcal{X})) \ , \quad r \geq 0 \ .$$

Fact: v_H is no vector field. Choose a section σ of the bundle $\mathcal{U} \times \mathcal{X} \xrightarrow{\rho} \mathcal{X}$, then $\left(v_H^\alpha \circ \sigma\right) \partial_\alpha \in \Gamma(\mathcal{T}(\mathcal{X}))$ is met.

$$\begin{array}{ccc} \rho^*(\mathcal{T}(\mathcal{X})) & \stackrel{\tau^*(\rho)}{\rightarrow} & \mathcal{T}(\mathcal{X}) \\ \downarrow \rho^*(\tau) & & \downarrow \tau \\ \mathcal{U} \times \mathcal{X} & \stackrel{\rho}{\rightarrow} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} (x^\alpha, u^\varsigma, \dot{x}^\alpha) & \stackrel{\tau^*(\rho)}{\rightarrow} & (x^\alpha, \dot{x}^\alpha) \\ \uparrow v_H(x,u) & & v^* \uparrow v_H \circ g(x) \\ (x^\alpha, u^\varsigma) & \stackrel{g(x)}{\leftarrow} & (x^\alpha) \end{array}$$

One gets $v_H \in \Gamma(\rho^*(\tau))$ with

$$\rho^*(\mathcal{T}(\mathcal{X})) \stackrel{\rho^*(\tau)}{\rightarrow} \mathcal{U} \times \mathcal{X} \ .$$

Coordinate Transformations

The structure of a PCHD-system is preserved by transformations of the type

$$\begin{aligned} \bar{x}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}(x) \\ \bar{u}^{\bar{\varsigma}} &= M_{\varsigma}^{\bar{\varsigma}}(x)\left(u^{\varsigma} + f^{\varsigma}(x)\right) \\ \bar{y}_{\bar{\varsigma}} &= \bar{M}_{\bar{\varsigma}}^{\varsigma}(x)\left(y_{\varsigma} + \partial_{\alpha} H_f B_{\varsigma}^{\alpha}\right) \ , \qquad M_{\sigma}^{\bar{\varsigma}} \bar{M}_{\varsigma}^{\sigma} = \delta_{\varsigma}^{\bar{\varsigma}} \ , \end{aligned}$$

provided that one finds a function $H_f \in C^{\infty}(\mathcal{X})$ such that

$$\left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_{\beta} H_f + B_{\varsigma}^{\alpha} f^{\varsigma} = 0 \ .$$

Remark: For $f^{\varsigma} = 0$ one chooses $H_f = 0$.

C.trans. (2)

Further relations

$$
\begin{aligned}
\bar{H} &= \left(H + H_f\right) \circ \varphi^{-1} \\
\bar{J}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} J^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ \varphi^{-1} \\
\bar{R}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} R^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ \varphi^{-1} \\
\bar{B}^{\bar{\alpha}}_{\bar{\varsigma}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} B^{\alpha}_{\varsigma} \bar{M}^{\varsigma}_{\bar{\varsigma}}\right) \circ \varphi^{-1} \; .
\end{aligned}
$$

Remark 1: The transformation for y, u is affine, therefore we get

$$y_\varsigma u^\varsigma \neq \bar{y}_{\bar{\varsigma}} \bar{u}^{\bar{\varsigma}} \; .$$

Remark 2: A function H_f, which meets the PDE with $f^\varsigma = 0$, does not change the equations.

An Alternative

If there exist functions $H_\varsigma(x) \in C^\infty(\mathcal{X})$, such that

$$B_\sigma^\alpha = -J^{\alpha\beta}\partial_\beta H_\varsigma \ , \quad R^{\alpha\beta}\partial_\beta H_\varsigma = 0 \ , \quad \partial_\beta(H_\varsigma) J^{\alpha\beta}\partial_\beta H_\varsigma = 0$$

is met, then one can rewrite the system as

$$\begin{aligned} \dot{x}^\alpha &= v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta\left(H - H_\varsigma u^\varsigma\right) \ , \\ y_\varsigma &= v_H(H_\varsigma) \ . \end{aligned}$$

Remark: Often one chooses the output $Y_\varsigma = H_\varsigma$ instead of y_ς. Furthermore,

$$y_\varsigma = \frac{\mathrm{d}}{\mathrm{d}t} Y_\varsigma$$

is met along a solution of the system.

D-Control

Damping injection $\mathbb{R}^{m\times m} \ni D \geq 0$:

$$\begin{aligned} \bar{x}^{\alpha} &= x^{\alpha} \\ \bar{u}^{\varsigma} &= D^{\varsigma\tau} y_{\tau} + u^{\varsigma} = D^{\varsigma\tau} B^{\alpha}_{\varsigma} \partial_{\alpha} H + u^{\varsigma} \\ \bar{y}_{\varsigma} &= y_{\varsigma} \end{aligned}$$

One derives the PCHD-system

$$\begin{aligned} \dot{x}^{\alpha} &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_{\beta} H + B^{\alpha}_{\varsigma} \left(-D^{\varsigma\tau} y_{\tau} + \bar{u}^{\varsigma}\right) \\ &= \left(J^{\alpha\beta} - R^{\alpha\beta} - B^{\alpha}_{\varsigma} D^{\varsigma\tau} B^{\beta}_{\tau}\right) \partial_{\beta} H + B^{\alpha}_{\varsigma} \bar{u}^{\varsigma} \ , \end{aligned}$$

which meets

$$v_H(H) = -\partial_{\alpha}(H) \left(R^{\alpha\beta} + B^{\alpha}_{\varsigma} D^{\varsigma\tau} B^{\beta}_{\tau}\right) \partial_{\beta} H + y_{\varsigma} \bar{u}^{\varsigma} \leq y_{\varsigma} \bar{u}^{\varsigma} \ .$$

P/I-Regler

Choose the controller

$$\dot{\tilde{x}}^\varsigma = \tilde{u}^\varsigma$$
$$\tilde{y}_\varsigma = \partial_\varsigma \tilde{H} = P_{\varsigma\tau}\tilde{x}^\tau$$

which is a PCHD system with $\tilde{J} = 0$, $\tilde{R} = 0$, $\tilde{H}\left(\tilde{x}\right) = \frac{1}{2}\tilde{x}^\varsigma P_{\varsigma\tau}\tilde{x}^\tau$, $\mathbb{R}^{m\times m} \ni P \geq 0$,

Connect the system

$$\dot{x}^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta H + B^\alpha_\varsigma u^\varsigma$$
$$y_\varsigma = B^\alpha_\varsigma \partial_\alpha H$$

with the controller using the interconnection

$$u^\varsigma = -\delta^{\varsigma\tau}\tilde{y}_\tau + v^\varsigma \;, \qquad \tilde{u}^\varsigma = \delta^{\varsigma\tau} y_\tau \;.$$

P/I (2)

$$
\begin{aligned}
\dot{x}^{\alpha} &= J^{\alpha\beta}\partial_{\beta}\left(H+\tilde{H}\right) - B^{\alpha}_{\tau}\delta^{\tau\varsigma}\partial_{\varsigma}\left(H+\tilde{H}\right) - R^{\alpha\beta}\partial_{\beta}\left(H+\tilde{H}\right)\\
\dot{\tilde{x}}^{\varsigma} &= \delta^{\varsigma\tau}B^{\alpha}_{\tau}\partial_{\alpha}\left(H+\tilde{H}\right)
\end{aligned}
$$

The transformation

$$
\begin{aligned}
\bar{x}^{\alpha} &= x^{\alpha}\\
\bar{\tilde{x}}^{\varsigma} &= \tilde{x}^{\varsigma} - \delta^{\varsigma\tau}Y_{\tau}\left(x\right)\\
\bar{\tilde{H}} &= \tilde{H}\left(\bar{\tilde{x}}+Y\right)
\end{aligned}
$$

leads to a lengthy PCHD-system.

Assumption: If relation $\partial_{\alpha}\left(Y_{\varsigma}\right)J^{\alpha\beta}\partial_{\beta}H = y_{\varsigma} = B^{\alpha}_{\varsigma}\partial_{\alpha}H$ and

$$
R\partial_{\beta}\left(H\right)\partial_{\alpha}Y_{\varsigma} = B^{\alpha}_{\upsilon}\partial_{\alpha}Y_{\varsigma} = \partial_{\alpha}\left(Y_{\varsigma}\right)J^{\alpha\beta}\partial_{\beta}Y_{\tau} = 0
$$

P/I (3)

are met, then one gets

$$\begin{aligned} \dot{x}^\alpha &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta H - B^\alpha_\varsigma P^{\varsigma\tau} \left(\bar{\tilde{x}}^\tau + \delta^{\varsigma\tau} Y_\tau\right) + B^\alpha_\varsigma v^\varsigma \\ \dot{\bar{\tilde{x}}}^{\bar{\varsigma}} &= 0 \end{aligned}$$

or

$$\dot{x}^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right) \partial_\beta \left(H + \left(\bar{\tilde{x}}^\tau + \delta^{\varsigma\tau} Y_\tau\right) \frac{P^{\varsigma\tau}}{2} \left(\bar{\tilde{x}}^\tau + \delta^{\varsigma\tau} Y_\tau\right)\right) + B^\alpha_\varsigma v^\varsigma \; .$$

H_2-Control

Find a control law, such that the objective functional

$$\frac{1}{2}\int_0^\infty \left(y_\varsigma D^{\varsigma\tau} y_\tau + u^\varsigma \bar{D}_{\varsigma\tau} u^\tau\right) \mathrm{d}t$$

with $D^{\varsigma\chi}\bar{D}_{\chi\tau} = \delta^\varsigma_\tau$ is minimized.

The HJB-inequality for this problem is

$$\inf_u \left(2 v_H\left(V\right) + y_\varsigma D^{\varsigma\tau} y_\tau + u^\varsigma \bar{D}_{\varsigma\tau} u^\tau\right) \leq 0 \; .$$

Provided that H is a positive definite function then the choice

$$V = H$$

leads to

H_2-Control (2)x

$$\inf_u \left(2\partial_\alpha (H) R^{\alpha\beta} \partial_\beta H + 2 y_\varsigma u^\varsigma + y_\varsigma D^{\varsigma\tau} y_\tau + u^\varsigma \bar{D}_{\varsigma\tau} u^\tau \right) \leq 0$$

$$-2\partial_\alpha (H) R^{\alpha\beta} \partial_\beta H \leq 0$$

with the optimal control law

$$y_\varsigma = -\bar{D}_{\varsigma\tau} u^\tau , \quad u^\varsigma = -D^{\varsigma\tau} y_\tau .$$

H_∞-Control

The system is given by

$$\begin{aligned}
\dot{x}^\alpha &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta\left(H - H_\varsigma u^\varsigma - \hat{H}_\varsigma \hat{u}^\varsigma\right) \\
y_\varsigma &= \partial_\alpha\left(H_\varsigma\right) J^{\alpha\beta}\partial_\beta H \\
\hat{y}_\varsigma &= \partial_\alpha\left(\hat{H}_\varsigma\right) J^{\alpha\beta}\partial_\beta H \ ,
\end{aligned}$$

with $K \in \mathbb{R}^{qm}$, where u is the control and $\hat{u}$ is the disturbance input.

Find a control law, such that the objective functional

$$\frac{1}{2}\int_0^\infty \left(\lambda^2 y_\varsigma \delta^{\varsigma\tau} y_\tau + u^\varsigma \delta_{\varsigma\tau} u^\tau - \gamma^2 \hat{u}^\varsigma \delta_{\varsigma\tau} \hat{u}^\tau\right) \mathrm{d}t$$

is minimized with respect to u and maximized with respect to $\hat{u}$.

H_∞-Control (2)

The HJBI-inequality for this problem is

$$\inf_u \sup_{\hat{u}} \left(2v_H\left(V\right) + \lambda^2 y_\varsigma \delta^{\varsigma\tau} y_\tau + u^\varsigma \delta_{\varsigma\tau} u^\tau - \gamma^2 \hat{u}^\varsigma \delta_{\varsigma\tau} \hat{u}^\tau\right) \leq 0\ .$$

Provided that H is a positive definite function, then the choice $V = H$ leads to

$$\begin{aligned} \inf_u \sup_{\hat{u}} \Big(2\partial_\alpha\left(H\right)) R^{\alpha\beta} \partial_\beta H + 2y_\varsigma u^\varsigma + 2\hat{y}_\varsigma \hat{u}^\varsigma & \\ + \lambda^2 y_\varsigma \delta^{\varsigma\tau} y_\tau + u^\varsigma \delta_{\varsigma\tau} u^\tau - \gamma^2 \hat{u}^\varsigma \delta_{\varsigma\tau} \hat{u}^\tau\Big) & \leq 0 \\ -2\partial_\alpha\left(H\right)) R^{\alpha\beta} \partial_\beta H + \left(\lambda^2 - 1\right) y_\varsigma \delta^{\varsigma\tau} y_\tau + \frac{1}{\gamma^2} \hat{y}_\varsigma \delta^{\varsigma\tau} \hat{y}_\tau & \leq 0 \end{aligned}$$

with the optimal choice

H_∞-Control (3)

$$\gamma^2 \hat{u}^\varsigma \delta_{\varsigma\tau} = \hat{y}_\tau \ , \quad y_\varsigma = -\delta_{\varsigma\tau} u^\tau \ , \quad u^\varsigma = -\delta^{\varsigma\tau} y_\tau \ .$$

In the case $H_\varsigma = \hat{H}_\varsigma$ the condition simplifies to

$$-2\partial_\alpha \left(H \right)) R^{\alpha\beta} \partial_\beta H + \left(\lambda^2 - 1 + \frac{1}{\gamma^2} \right) y_\varsigma \delta^{\varsigma\tau} y_\tau \ \ \leq \ \ 0$$

E.g. one chooses

$$\lambda^2 \leq \frac{\gamma^2 - 1}{\gamma^2} \ .$$

PCHD-Systeme, PDE

Choice of a state space:

$\mathcal{B}$ an orientable smooth p-dimensional manifold with coordinates $\left(X^i\right)$, $i = 1, \ldots, p$.

A compact p-dimensional manifold $\mathcal{D} \subset \mathcal{B}$ with boundary $\partial\mathcal{D}$ and volume form dX.

Inclusion map: $i : \partial\mathcal{D} \to \mathcal{D}$ with coordinates $\left(\bar{X}^{\bar{i}}\right)$, $\bar{i} = 1, \ldots, p-1$ for $\partial\mathcal{D}$ and volume form $\mathrm{d\bar{X}}$.

Bundle: $\mathcal{X} \xrightarrow{\pi} \mathcal{D}$ with coordinates $\left(X^i, x^\alpha\right)$, $\alpha = 1, \ldots, q$.

A section $\sigma \in \Gamma\left(\mathcal{X}\right)$,

$$\pi \circ \sigma = \mathrm{id}_{\mathcal{D}}\,, \quad \pi\left(X^i, x^\alpha\right) = \left(X^i\right)$$

represents the state of a distributed system.

PDE (2)

The first jet $j(\sigma)$ of σ, the functions σ^α and $\partial_i \sigma^\alpha$, is a section of the $J(\mathcal{X}) \stackrel{\pi^1}{\to} \mathcal{D}$ with the first jet-manifold $J(\mathcal{X})$, or $J(\pi)$, of $\mathcal{X}$. We use the adapted coordinates $\left(X^i, x^\alpha, x^\alpha_{1_i}\right)$.

Analogously one defines the n-th order jet $j^n(\sigma)$ of σ with $\partial_I \sigma^\alpha$, $\#I \leq n$.

The index $I = i_1, \ldots, i_p$ is an ordered multi-index with $\#I = \sum_{j=1}^{p} i_p$,

which meets $I + J = i_1 + j_i, \ldots, i_p + j_p$ and $1_i = i_1, \ldots, i_j, \ldots, i_p$, $i_j = \delta^i_j$ as well as

$$\partial_I = (\partial_1)^{i_1} \cdots (\partial_p)^{i_p} \quad .$$

The coordinates of $J^n(\mathcal{X})$ are denoted by $\left(X^i, x^\alpha_I\right)$, $0 \leq \#I \leq n$.

PDE (3)

With the projections

$$\pi^n_m \left(X^i, x^\alpha, x^\alpha_I\right) = \left(X^i, x^\alpha, x^\alpha_J\right) \ , \ 0 \leq \#I \leq n \ , \ 0 \leq \#J \leq m \ , \ m < n$$

one derives the bundles

$$J^n \left(\mathcal{X}\right) \xrightarrow{\pi^n_m} J^m \left(\mathcal{X}\right) \ ,$$

with the identity $J^0 \left(\mathcal{X}\right) = \mathcal{X}$.

With the projection

$$\pi^n_m \left(X^i, x^\alpha, x^\alpha_I\right) = \left(X^i, x^\alpha\right)$$

one gets

$$J^n \left(\mathcal{X}\right) \xrightarrow{\pi^n} \mathcal{D} \ .$$

PDE (4)

The total derivative d_i with respect to X^i,
$d_i : C^\infty\left(J^n\left(\mathcal{X}\right)\right) \to C^\infty\left(J^{n+1}\left(\mathcal{X}\right)\right)$,

$$d_i = \partial_i + x^\alpha_{I+1_i}\partial^I_\alpha\,, \qquad \partial^I_\alpha = \partial_{x^\alpha_I}\,.$$

meets

$$\partial_i\left(f \circ j^n\left(\sigma\right)\right) = \left(d_i f\right) \circ j^{n+1}\left(\sigma\right)$$

with $f \in C^\infty\left(J^n\left(\mathcal{X}\right)\right)$, $\sigma \in \Gamma\left(\mathcal{X}\right)$.

Remark: A section $\gamma \in \Gamma\left(J\left(\pi\right)\right)$ does not necessarily met $j\left(\sigma\right) = \gamma$ with $\sigma \in \Gamma\left(\pi\right)$, since the integrability conditions

$$\gamma^\alpha_I = \partial_I \varsigma^\alpha$$

must be met.

PDE (5)

A section $\sigma \in \Gamma(\pi)$ defines a state. Obviously, σ fixes $j(\sigma)$.

Let us choose a section $\gamma \in \Gamma(\pi^1)$ with $j(\sigma) = \gamma$, and a section ϑ of $i^*(\pi) \; : \; i^*(\mathcal{X}) \stackrel{\pi|_{\partial\mathcal{D}}}{\longrightarrow} \partial\mathcal{D}$.

Obviously, the state follows from

$$\sigma = \pi_0^1 \circ \gamma \; ,$$

where the functions σ^α are solutions of the PDE

$$\partial_i \sigma^\alpha = \gamma^\alpha_{1_i}$$

with the boundary conditions given by ϑ.

PDE (6)

This is only possible for distributed parameter systems (PDEs) !

Ideal rod with Hamiltonian desnsity $H = \frac{\left(x^2\right)^2}{2\rho} + \frac{E\left(x^1_{1_1}\right)^2}{2}$:

$$\dot{x}^1 = \frac{1}{\rho} x^2 \,, \qquad \dot{x}^2 = E x^1_{2_1} \,.$$

Die Lie-Bäcklund Transformation

$$\bar{x}^1 = x^1_{1_1} \,, \qquad \bar{x}^2 = x^2$$

leads to

$$\dot{\bar{x}}^1 = \frac{1}{\rho} \bar{x}^2_{1_1} \,, \qquad \dot{\bar{x}}^2 = E \bar{x}^1_{1_1} \,, \qquad \bar{H} = \frac{1}{2\rho} \left(\bar{x}^2\right)^2 + \frac{E}{2} \left(\bar{x}^1\right)^2 \,.$$

PDE (7)

Telegraphers equation with Hamiltonian density

$$H = \frac{1}{2C}\left(x^1\right)^2 + \frac{1}{2L}\left(x^2\right)^2$$

and

$$\dot{x}^1 = -\frac{1}{L} x^2_{1_1}\ , \qquad \dot{x}^2 = -\frac{1}{C} x^1_{1_1}\ .$$

Remark: We confine ourselves to the case, where a state is defined by $\sigma \in \Gamma\left(\pi\right)$.

PDE (8)

Given the bundle $\mathcal{X} \xrightarrow{\pi} \mathcal{D}$ one gets several important bundles by standard constructions:

The tangent bundles $\mathcal{T}(\mathcal{X})$, $\mathcal{T}(\mathcal{D})$,
the cotangent bundles $\mathcal{T}^*(\mathcal{X})$, $\mathcal{T}^*(\mathcal{D})$.

The vertical tangential bundle of $\mathcal{X}$,

$$\mathcal{V}(\mathcal{X}) = \ker(\pi_*) \subset \mathcal{T}(\mathcal{X}) \ ,$$

and the horizontal cotangent bundle

$$\mathcal{H}^*(\mathcal{X}) = \pi^*(\mathcal{T}^*(\mathcal{D})) \subset \mathcal{T}^*(\mathcal{X}) \ ,$$

which annuls $\mathcal{V}(\mathcal{E})$, or

$$\mathcal{V}(\mathcal{E}) \rfloor \mathcal{H}^*(\mathcal{E}) = \{0\} \ .$$

PDE (9)

Given these vector bundles one gets the further vector bundles

$$\wedge_r^0 (\mathcal{T}^* (\mathcal{X})) = \wedge_r (\mathcal{H}^* (\mathcal{X})) \ , \quad r \leq p$$

and

$$\wedge_p^1 (\mathcal{T}^* (\mathcal{X})) = (\mathcal{T}^* (\mathcal{X})) \wedge (\wedge_p (\mathcal{H}^* (\mathcal{X}))) \ ,$$

with

$$\mathcal{V} (\mathcal{X}) \rfloor \wedge_r^1 (\mathcal{T}^* (\mathcal{X})) = \wedge_r^0 (\mathcal{T}^* (\mathcal{X})) \ .$$

The interior product

$$\rfloor : \mathcal{V} (\mathcal{X}) \times \wedge_p^1 (\mathcal{T}^* (\mathcal{X})) \to \wedge_p^0 (\mathcal{T}^* (\mathcal{X})) \ , \quad r = \dot{x}^\alpha \dot{r}_\alpha$$

replaces the canonical product of the lumped parameter case. (X^i, x^α, r), $(X^i, x^\alpha, \dot{r}_\alpha)$ are coordinates for $\wedge_p^0 (\mathcal{T}^* (\mathcal{X}))$, $\wedge_p^1 (\mathcal{T}^* (\mathcal{X}))$.

PDE (10)

The total derivative d_i and the horizontal exterior derivatve d_h are connected by

$$\mathrm{d}_h = \mathrm{d}X^i \wedge d_i \ ,$$

or by

$$j^{n+1}(\sigma)^*(\mathrm{d}_h\omega) = \mathrm{d}\left(j^n(\sigma)^*(\omega)\right)$$

is met for all $\sigma \in \Gamma(\mathcal{X})$, $\omega \in \Gamma(\wedge\mathcal{T}^*(J^n(\mathcal{X})))$.

Remark: The sequence

$$\mathbb{R} \ \rightarrow \ C^\infty(\mathcal{X}) \ \stackrel{\mathrm{d}_h}{\rightarrow} \ \wedge_1^0(\mathcal{T}^*(\mathcal{X})) \ \cdots \ \stackrel{\mathrm{d}_h}{\rightarrow} \ \wedge_p^0(\mathcal{T}^*(\mathcal{X})) \ \stackrel{\mathrm{d}_h}{\rightarrow} \ \{0\}$$

is exact.

PDE (11)

Consider d_h as a map

$$\mathrm{d}_h : \pi^{n,*}\left(\wedge_r\left(\mathcal{H}^*\left(\mathcal{X}\right)\right)\right) \longrightarrow \pi^{n+1,*}\left(\wedge_{r+1}\left(\mathcal{H}^*\left(\mathcal{X}\right)\right)\right) \ ,$$

then one derives a version of Stokes's Theorem, adapted to bundles:

$$\int_{\partial\mathcal{D}} j^n\left(\sigma\right)^*\left(\omega\right) = \int_{\mathcal{D}} j^{n+1}\left(\sigma\right)^*\left(\mathrm{d_h}\omega\right)$$

is met for all $\sigma \in \Gamma\left(\mathcal{X}\right)$, $\omega \in \Gamma\left(\pi^{n,*}\left(\wedge_{p-1}\left(\mathcal{T}^*\left(\mathcal{D}\right)\right)\right)\right)$.

The Hamiltonian of the lumped parameter case is replaced by the Hamiltonian density

$$H\mathrm{dX} \in \pi_0^{1,*}\left(\wedge_p^0\left(\mathcal{T}^*\left(\mathcal{D}\right)\right)\right) \ .$$

PDE (12)

The exterior derivative of

$$\begin{aligned}
\mathrm{d}\left(H\mathrm{d}X\right) &= \partial_\alpha H \mathrm{d}x^\alpha \wedge \mathrm{d}X + \partial_\alpha^i H \mathrm{d}x^\alpha_{1_i} \wedge \mathrm{d}X \\
&= \partial_\alpha H \mathrm{d}x^\alpha \wedge \mathrm{d}X + \partial_\alpha^i H d_i \left(\mathrm{d}x^\alpha \wedge \mathrm{d}X\right) \\
&= \underbrace{\left(\left(\partial_\alpha - d_i \partial_\alpha^i\right) H\right)}_{\delta_\alpha} \mathrm{d}x^\alpha \wedge \mathrm{d}X + \mathrm{d}_h \left(\partial_\alpha^i H \partial_i \rfloor \mathrm{d}x^\alpha \wedge \mathrm{d}X\right)
\end{aligned}$$

induces two new maps: The variational derivative δ,

$$\delta : \pi_0^{1,*}\left(\wedge_p^0\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)\right) \to \pi_0^{2,*}\left(\wedge_p^1\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)\right) \ .$$

Using the coordinates $\left(X^i, x^\alpha, \dot{r}_\alpha\right)$ for $\wedge_p^1\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)$ we get

$$\dot{r}_\alpha = \delta_\alpha\left(H\right) \quad \text{with} \quad \delta_\alpha = \partial_\alpha + \sum_I \left(-1\right)^{\#I} d_I \partial_\alpha^I \ .$$

PDE (13)

Remark: The sequence

$$\wedge^0_{p-1}\left(\mathcal{T}^*\left(\mathcal{X}\right)\right) \overset{\mathrm{d}_h}{\longrightarrow} \wedge^0_{p}\left(\mathcal{T}^*\left(\mathcal{X}\right)\right) \overset{\delta}{\longrightarrow} \wedge^1_{p}\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)$$

is exact.

Now, on the boundary $\partial\mathcal{D}$ the bundle

$$\wedge^1_{p-1}\left(\mathcal{T}^*\left(i^*\left(\mathcal{X}\right)\right)\right)$$

with local coordinates $\left(\hat{X}^i, \hat{x}^\alpha, \dot{\hat{r}}_\alpha\right)$ takes over the function of $\wedge^1_{p}\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)$.

PDE (14)

From

$$i^* \left(\partial^i_\alpha H \partial_i \rfloor \mathrm{d}x^\alpha \wedge \mathrm{d}X\right) = i^* \left(\partial^i_\alpha H\right) \gamma_i \mathrm{d}x^\alpha \wedge \mathrm{d}\bar{X} \ , \qquad \gamma_i \in C^\infty \left(\partial \mathcal{D}\right)$$

one gets the second map

$$\zeta : \pi_0^{1,*} \left(\wedge_p^0 \left(\mathcal{T}^* \left(\mathcal{X}\right)\right)\right) \to \pi_0^{1,*} \left(\wedge_{p-1}^1 \mathcal{T}^* \left(i^* \left(\mathcal{X}\right)\right)\right) \ ,$$

given in coordinates by

$$\dot{\hat{r}}_\alpha = i^* \left(\zeta_\alpha \left(H\right)\right) \ , \qquad \zeta_\alpha = \gamma_i \partial^i_\alpha \ .$$

Fact: The exterior derivative of the Hamiltonian density splits into the two maps δ, ζ in a natural manner.

PDE (15)

Given a skew symmetric map

$$J : \pi_0^{2,*}\left(\wedge_p^1\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)\right) \to \pi_0^{2,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$$

or in coordinates

$$\dot{x}^\alpha = J^{\alpha\beta}\dot{r}_\beta\,, \qquad J^{\alpha\beta} \in C^\infty\left(J^2\left(\mathcal{X}\right)\right)$$

one derives the evolutionary Hamiltonian equations as

$$\dot{x}^\alpha = v^\alpha\,, \qquad v^\alpha = J^{\alpha\beta}\delta_\beta\left(H\right)$$

with the Hamiltonian operator $v \in \pi_0^{2,*}\left(\mathcal{V}\left(\mathcal{E}\right)\right)$.
The change of $\int_\mathcal{D} H\mathrm{d}X$ along solutions of the system is given by

$$\int_\mathcal{D} j\left(v\right)\left(H\mathrm{d}X\right) = \int_\mathcal{D} \underbrace{\delta_\alpha\left(H\right)J^{\alpha\beta}\delta_\beta\left(H\right)}_{=0}\mathrm{d}X + \int_{\partial\mathcal{D}} v^\alpha\left(\partial_\alpha^i H\right)\gamma_i\mathrm{d}\bar{X}\,.$$

PDE (16)

Remark 1: If suitable boundary condition are choosen, such that

$$v^{\alpha}\left(\partial_{\alpha}^{i} H\right)\gamma_i = 0$$

is met, then $\int_{\mathcal{D}} H \mathrm{d}X$ is an invariant.

Remark 2: The interior product

$$\mathcal{V}\left(i^*\left(\mathcal{X}\right)\right) \times \wedge_{p-1}^{1} \mathcal{T}^*\left(i^*\left(\mathcal{X}\right)\right) \rightarrow \wedge_{p-1}^{0} \mathcal{T}^*\left(i^*\left(\mathcal{X}\right)\right)$$

measure the flow of power over the boundary.

Remark 3: J can be replaced by skew symmetric differential operator,

$$J^{\alpha\beta} = J^{\alpha\beta I} d_I \ , \quad \#I \leq r \ , \quad d_I = \left(d_1\right)^{i_1} \circ \cdots \circ \left(d_1\right)^{i_1} \ .$$

PDE (17)

The choice of a positive semi definite map

$$R : \pi_0^{2,*}\left(\wedge_p^1\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)\right) \to \pi_0^{2,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$$

allows us to take dissipative effects into accout. One gets

$$\dot{x}^\alpha = v^\alpha\,, \quad v^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta\left(H\right)\,,$$

as well as

$$\int_{\mathcal{D}} j\left(v\right)\left(H\mathrm{d}X\right) = -\int_{\mathcal{D}} \underbrace{\delta_\alpha\left(H\right)R^{\alpha\beta}\delta_\beta\left(H\right)}_{\geq 0}\mathrm{d}X + \int_{\partial\mathcal{D}} v^\alpha\left(\partial_\alpha^i H\right)\gamma_i \mathrm{d}\bar{X}\,.$$

Remark 1: Provided that suitable boundary conditions are choosen, then $\int_{\mathcal{D}} H\mathrm{d}X$ is non-increasing along solutions.
Remark 2: One can also replace R by a differential operator.

PDE (18)

To introduce ports we choose the vector bundle $\mathcal{U}$ as as input space

$$(\mathcal{U}, \rho, \mathcal{D}) \quad \text{with coordiantes} \quad \left(X^i, u^\varsigma\right)\ ,\ \varsigma = 1, \ldots, m\ .$$

The output space is given by the vector bunde $(\mathcal{Y}, \rho^*, \mathcal{D})$, which is dual with repect to the product

$$\rho \times \rho^* \to \wedge_p \left(\mathcal{T}^* \left(\mathcal{D}\right)\right)$$

and in coordinates given by $r = u^\varsigma y_\varsigma$, with coordinates $\left(X^i, y_\varsigma\right)$ for $\mathcal{Y}$.

Let us choose a linear map $B : \mathcal{U} \to \pi^{2,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$, then one derives the system

$$\begin{aligned} \dot{x}^\alpha &= v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta H + B_\varsigma^\alpha u^\varsigma \\ y_\varsigma &= B_\varsigma^\alpha \delta_\alpha H\ , \end{aligned}$$

PDE (19)

as well as

$$\int_{\mathcal{D}} j(v)(H\mathrm{d}X) = -\int_{\mathcal{D}} \underbrace{\delta_\alpha(H) R^{\alpha\beta}\delta_\beta(H)}_{\geq 0} \mathrm{d}X + \int_{\mathcal{D}} y_\varsigma u^\varsigma \mathrm{d}X$$
$$+ \int_{\partial\mathcal{D}} v^\alpha \left(\partial_\alpha^i H\right)\gamma_i \mathrm{d}\bar{X} \leq \int_{\mathcal{D}} y_\varsigma u^\varsigma \mathrm{d}X\ .$$

Remark 1: One can replace B by a differential operator.

Remark 2: Ports on the boundary can be introduced in a similar manner.

H. Ennsbrunner, K. Schlacher: *On the geometrical representation and interconnection of infinite dimensional port controlled Hamiltonian systems.* To appear in the proceedings of CDC-ECC, Sevilla, Spain 2005.

PDE (20)

An alternative approach: One chooses finite dimensional vector spaces $\mathcal{U}$, $\mathcal{Y} = \mathcal{U}^*$.

Given a linear map

$$B : \mathcal{U} \rightarrow \pi^{2,*}\left(\mathcal{V}\left(\mathcal{X}\right)\right)$$

and its dual

$$B^* : \pi_0^{2,*}\left(\wedge_p^1\left(\mathcal{T}^*\left(\mathcal{X}\right)\right)\right) \rightarrow \mathcal{Y}\ ,$$

then one derives the system

$$\begin{aligned} \dot{x}^\alpha &= v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta H + B_\varsigma^\alpha u^\varsigma \\ y_\varsigma &= \int_{\mathcal{D}} B_\varsigma^\alpha \delta_\alpha H \mathrm{d}X\ . \end{aligned}$$

PDE (21)

An alternative: If there exists functions $H_\varsigma \in C^\infty(\mathcal{X})$, such that

$$B_\sigma^\alpha = -J^{\alpha\beta}\delta_\beta H_\varsigma \ , \quad R^{\alpha\beta}\delta_\beta H_\varsigma = 0 \ , \quad \delta_\beta(H_\varsigma)\, J^{\alpha\beta}\delta_\beta H_\varsigma = 0$$

is met, then one can rewrite the system as

$$\begin{aligned} \dot{x}^\alpha &= v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta\left(H - H_\varsigma u^\varsigma\right) \ , \\ y_\varsigma &= v_H(H_\varsigma) \ . \end{aligned}$$

Remark: Often one chooses the output $Y_\varsigma = H_\varsigma$ instead of y_ς. Furthermore,

$$y_\varsigma = \frac{\mathrm{d}}{\mathrm{d}t} Y_\varsigma$$

is met along solutions of the system.

PDE (22)

Remark: If $H_\varsigma \in C^\infty(J(\mathcal{X}))$ is met, then the input map B is already a differential operator because of

$$J^{\alpha\beta}\delta_\beta\left(H_\varsigma u^\varsigma\right) = J^{\alpha\beta}\left(\partial_\alpha\left(H_\varsigma\right)u^\varsigma + \sum_I (-1)^{\#I} d_I\left(\partial_\alpha^I\left(H_\varsigma\right)u^\varsigma\right)\right) ,$$

where d_i is extended by $u^\varsigma_{I+1_i}\partial^I_\sigma$.

Coordinaten transformations, PDE

The structure of a PCHD system is invariant with respect to bundle isomorphism (ψ, φ) of the type

$$\begin{aligned} \bar{X}^{\bar{i}} &= \psi^{\bar{i}}(x) \\ \bar{x}^{\bar{\alpha}} &= \varphi^{\bar{\alpha}}(X, x) \ . \end{aligned}$$

One gets

$$\begin{aligned} \bar{H} &= H_0 \circ j(\psi, \varphi)^{-1} \\ \bar{J}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} J^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ j^2(\psi, \varphi)^{-1} \\ \bar{R}^{\bar{\alpha}\bar{\beta}} &= \left(\partial_\alpha \varphi^{\bar{\alpha}} R^{\alpha\beta} \partial_\beta \varphi^{\bar{\beta}}\right) \circ j^2(\psi, \varphi)^{-1} \ , \end{aligned}$$

where $j(\psi, \varphi)$, $j^2(\psi, \varphi)$ denote the prolongations of (ψ, φ) to $J(\pi)$, $J^2(\pi)$.

K.trans. PDE (2)

Remark: E.g. $j(\psi, \varphi)$ is given by

$$\bar{x}^{\bar{\alpha}}_{1_{\bar{i}}} \partial_i \psi^{\bar{i}} = d_i \varphi^{\bar{\alpha}} \ .$$

Input and output transformation according to

$$\begin{aligned} \bar{u}^{\bar{\varsigma}} &= M^{\bar{\varsigma}}_{\varsigma} \left(u^{\varsigma} + f^{\varsigma} \right) \\ \bar{y}_{\bar{\varsigma}} &= \bar{M}^{\varsigma}_{\bar{\varsigma}} \left(y_{\varsigma} + \delta_{\alpha} \left(H_f \right) B^{\alpha}_{\varsigma} \right) \ , \qquad M^{\bar{\varsigma}}_{\sigma} \bar{M}^{\sigma}_{\varsigma} = \delta^{\bar{\varsigma}}_{\varsigma} \end{aligned}$$

with $M^{\bar{\varsigma}}_{\varsigma}, f^{\varsigma} \in C^{\infty}\left(J^2 \left(\mathcal{X} \right) \right)$, provided that there exist a function $H_f \in C^{\infty}\left(J^2 \left(\mathcal{X} \right) \right)$, which meets

$$\left(J^{\alpha\beta} - R^{\alpha\beta} \right) \delta_{\beta} H_f + B^{\alpha}_{\varsigma} f^{\varsigma} = 0 \ .$$

K.trans. PDE (3)

Remark: Take into account, that the kernel of δ is non-trivial.

The input matrix B transforms as

$$\bar{B}_{\bar{\varsigma}}^{\bar{\alpha}} = \left(\partial_\alpha \varphi^{\bar{\alpha}} B_\varsigma^\alpha \bar{M}_{\bar{\varsigma}}^\varsigma\right) \circ j^2 \left(\psi, \varphi\right)^{-1} .$$

Remark 1: Inputs and outputs are transformed in an affine manner.

Remark 2: Given a solution H_f for $f = 0$, then the choice $H + H_f$ leads to the same evolutionary equations, but may change the boundary conditions.

D-Control, PDE

Damping injection: One connects the system

$$\begin{aligned} \dot{x}^\alpha &= v_H^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta H + B_\varsigma^\alpha u^\varsigma \\ y_\varsigma &= B_\varsigma^\alpha \delta_\alpha H \ , \end{aligned}$$

with

$$u^\varsigma = -\tilde{y}^\varsigma + v^\varsigma \ , \quad \tilde{u}_\varsigma = y_\varsigma \ , \quad \tilde{y}^\varsigma = D^{\varsigma\tau}\tilde{u}_\tau \ ,$$

$C^\infty(\mathcal{D})^{m\times m} \ni D \geq 0$ and derives

$$\begin{aligned} \dot{x}^\alpha &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta H + B_\varsigma^\alpha\left(-D^{\varsigma\tau}y_\tau + \bar{u}^\varsigma\right) \\ &= \left(J^{\alpha\beta} - R^{\alpha\beta} - B_\varsigma^\alpha D^{\varsigma\tau} B_\tau^\beta\right)\delta_\beta H + B_\varsigma^\alpha \bar{u}^\varsigma \ . \end{aligned}$$

D, PDE (2)

Damping injection on the boundary: Let us introduce ports on $\partial\mathcal{D}$ such that

$$v^{\alpha}\left(\partial^{i}_{\alpha}H\right)\gamma_i = \hat{y}_{\hat{\varsigma}}\hat{u}^{\hat{\varsigma}} \ ,$$

$\hat{\varsigma} = 1, \ldots, \hat{m}$ is met, then the choice

$$\hat{u}^{\hat{\varsigma}} = -\hat{D}^{\hat{\varsigma}\hat{\tau}}\hat{y}_{\hat{\tau}}$$

with $C^{\infty}\left(\mathcal{D}\right)^{\hat{m}\times\hat{m}} \ni D \geq 0$ leads to

$$\int_{\partial\mathcal{D}} v^{\alpha}\left(\partial^{i}_{\alpha}H\right)\gamma_i \mathrm{d}\bar{X} = -\int_{\partial\mathcal{D}} \hat{y}_{\hat{\varsigma}}\hat{D}^{\hat{\varsigma}\hat{\tau}}\hat{y}_{\hat{\tau}}\mathrm{d}\bar{X} \leq 0 \ .$$

P/I-Control, PDE

Connect the system

$$\dot{x}^{\alpha} = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_{\beta}H + B^{\alpha}_{\varsigma}u^{\varsigma}$$
$$y_{\varsigma} = B^{\alpha}_{\varsigma}\partial_{\alpha}H$$

by

$$u^{\varsigma} = -\tilde{y}^{\varsigma} + v^{\varsigma}\ , \qquad \tilde{u}_{\varsigma} = y_{\varsigma}$$

with the controller

$$\dot{\tilde{x}}^{\varsigma} = \tilde{u}^{\varsigma}$$
$$\tilde{y}_{\varsigma} = \partial_{\varsigma}\tilde{H} = \delta_{\varsigma}\tilde{H} = P_{\varsigma\tau}\tilde{x}^{\tau}$$

with $\tilde{J} = 0$, $\tilde{R} = 0$, $\tilde{H}\left(\tilde{x}\right) = \frac{1}{2}\tilde{x}^{\varsigma}P_{\varsigma\tau}\tilde{x}^{\tau}$, $C^{\infty}\left(\mathcal{D}\right)^{m\times m} \ni P \geq 0$, then one gets

P/I, PDE (2)

$$\begin{aligned} \dot{x}^\alpha &= \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\delta_\beta H - B^\alpha_\varsigma \partial^\varsigma \tilde{H} + B^\alpha_\varsigma v^\varsigma \\ \dot{\tilde{x}}_\varsigma &= B^\alpha_\varsigma \delta_\alpha H \; . \end{aligned}$$

Provided that $J^{\alpha\beta}\delta_\beta\left(H\right)\partial_\alpha Y_\varsigma = y_\varsigma = B^\alpha_\varsigma \delta_\alpha H$ and

$$R^{\alpha\beta}\delta_\beta\left(H\right)\partial_\alpha Y_\varsigma = 0 \; , \quad B^\alpha_\upsilon \partial_\alpha Y_\varsigma = 0 \; , \quad J^{\alpha\beta}\delta_\beta\left(H_\tau\right)\partial_\alpha Y_\varsigma = 0$$

is met, then the transformation

$$\begin{aligned} \bar{x}^\alpha &= x^\alpha \\ \bar{\tilde{x}}_\varsigma &= \tilde{x}_\varsigma - Y_\varsigma\left(X, x\right) \\ \bar{\tilde{H}} &= \tilde{H}\left(\bar{\tilde{x}} + Y\right) \end{aligned}$$

P/I, PDE (3)

leads to

$$\dot{x}^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta\left(H+\right) - B^\alpha_\varsigma P^{\varsigma\tau}\left(\bar{\tilde{x}}_\tau + Y_\tau\right) + B^\alpha_\varsigma v^\varsigma$$

$$\dot{\bar{\tilde{x}}}_\varsigma = 0$$

or

$$\dot{x}^\alpha = \left(J^{\alpha\beta} - R^{\alpha\beta}\right)\partial_\beta\left(H + \left(\bar{\tilde{x}}_\varsigma + Y_\varsigma\right)\frac{P^{\varsigma\tau}}{2}\left(\bar{\tilde{x}}_\tau + Y_\tau\right)\right) + B^\alpha_\varsigma v^\varsigma .$$

Remark: P/I-design for a boundary control is analogous.

Etc. PDE

H_2-Control and H_∞-control are extensible to the PDE case.

Controller design with Casimirs:

A. Macchelli, A. van der Schaft, C. Melchiorri: *Control by interconnection for distributed port Hamiltonian systems.* Proceedings: IFAC World Congress, Prague, 2005.

State feedback, simple or IDA-PBC, can be extended to the PDE case.

In certain cases the controller can be any passive system.

Open problem: A simple and straightforward stability proof is missing.

Summary PDE

The approach is based on

lumped parameter	distributed parameter
$H \in C(\mathcal{X})$	$H\,\mathrm{dX} \in \wedge_p^0(\mathcal{T}^*(\mathcal{D}))$
$\mathrm{d} : C(\mathcal{X}) \to \mathcal{T}^*(\mathcal{X})$	$\delta : \wedge_p^0(\mathcal{T}^*(\mathcal{X})) \to \wedge_p^1(\mathcal{T}^*(\mathcal{X}))$ $\zeta : \wedge_p^0(\mathcal{T}^*(\mathcal{X})) \to \wedge_{p-1}^1 \mathcal{T}^*(i^*(\mathcal{X}))$
$J, R : \mathcal{T}^*(\mathcal{X}) \to \mathcal{T}(\mathcal{X})$	$J, R : \wedge_p^0(\mathcal{T}^*(\mathcal{X})) \to \mathcal{V}(\mathcal{X})$
$v_H \in \Gamma(\mathcal{T}(\mathcal{X}))$	$v_H \in \Gamma(\mathcal{V}(\mathcal{X}))$
$B : \mathcal{U} \to \mathcal{T}(\mathcal{X})$	$B : (\mathcal{U}, \rho, \mathcal{D}) \to \mathcal{V}(\mathcal{X})$
$B^* : \mathcal{T}^*(\mathcal{X}) \to \mathcal{Y} = \mathcal{U}^*$	$B^* : \wedge_p^1(\mathcal{T}^*(\mathcal{X})) \to \mathcal{Y} = \mathcal{U}^*$

Piezoelectric Systems

We consider the linear elastic and time invariant case only.

$\mathcal{D}$ manifold of the independent spatial coordinates $\left(X^i\right)$, $i = 1, 2, 3$, which is the standard 3-dimensional Euclidean space.

The total manifold $(\mathcal{X}, \pi, \mathcal{D})$ is equipped with the local coordinates $\left(X^i, x^\chi\right)$, $\chi = 1 \ldots 6$ with $x = (u^\alpha, p_\alpha)$, $\alpha = 1, 2, 3$, u^α are the displacements , p_α are the generalized momenta.

We assume, that there exists an energy function e_E,

$$
\begin{aligned}
\mathrm{d}\left(e_E + E_\alpha D^\alpha\right) \wedge \mathrm{d}X &= \left(\sigma^{\alpha\beta} \mathrm{d}\varepsilon_{\alpha\beta} + E_\alpha \mathrm{d}\mathrm{D}^\alpha\right) \wedge \mathrm{d}X \\
&= \left(\sigma^{\alpha\beta} \mathrm{d}\varepsilon_{\alpha\beta} + \mathrm{d}\left(\mathrm{E}_\alpha \mathrm{D}^\alpha\right) - D^\alpha \mathrm{d}\mathrm{E}_\alpha\right) \wedge \mathrm{d}X
\end{aligned}
$$

with the stress

$$
\sigma = \sigma^{\alpha\beta} \partial_\alpha \otimes \partial_\beta \ , \qquad \sigma^{\alpha\beta} = \sigma^{\beta\alpha} \ ,
$$

Piezo (2)

the strain

$$\varepsilon = \varepsilon_{\alpha\beta} \mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta} \,, \qquad 2\varepsilon_{\alpha\beta} = u^{\alpha}_{1_{\beta}} + u^{\beta}_{1_{\alpha}} \,,$$

the electrical field strength $E = E_{\alpha} \mathrm{d}X^{\alpha}$ and the electrical displacement $D = D^{\alpha} \partial_{\alpha} \rfloor \mathrm{d}X$.

The linearized constitutive equations are

$$\begin{aligned} \sigma^{\alpha\beta} &= C^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} - G^{\alpha\beta\gamma} E_{\gamma} \\ D^{\alpha} &= G^{\beta\gamma\alpha} \varepsilon_{\beta\gamma} + F^{\alpha\beta} E_{\beta}, \end{aligned}$$

$\beta, \gamma = 1, 2, 3$ with $C^{\alpha\beta\gamma\delta}, G^{\alpha\beta\gamma}, F^{\alpha} \in C^{\infty}(\mathcal{D})$.

Piezo (3)

If the integrability conditions are $C^{\alpha\beta\gamma\delta} = C^{\beta\alpha\gamma\delta} = C^{\alpha\beta\delta\gamma} = C^{\gamma\delta\alpha\beta}$, $G^{\alpha\beta\gamma} = G^{\beta\alpha\gamma}$, $F^{\alpha\beta} = F^{\beta\alpha}$ are met, we get the density $e_E \mathrm{d}X$,

$$e_E \mathrm{d}X = \left(\frac{1}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} - G^{\alpha\beta\gamma} E_\gamma \varepsilon_{\alpha\beta} - \frac{1}{2} F^{\alpha\beta} E_\alpha E_\beta \right) \mathrm{d}X \ .$$

With the kinetic energy density $e_K \mathrm{d}X$,

$$e_K \mathrm{d}X = \frac{1}{2\rho} p_\alpha \delta^{\alpha\beta} p_\beta \mathrm{d}X \ ,$$

with $\rho \in C^\infty(\mathcal{D})$, we derive the Hamiltonian density $H\mathrm{d}X$ of the free system,

$$H\mathrm{d}X = \left(e_k + \frac{1}{2} \varepsilon_{\alpha\beta} C^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta} \right) \mathrm{d}X \ .$$

Piezo (4)

If one chooses E_γ as the control input, then the Hamiltonian density of the plant is

$$\left(H - G^{\alpha\beta\gamma}\varepsilon_{\alpha\beta}E_\gamma\right) \mathrm{d}X \ ,$$

and we derive the evolutionary equations as

$$\begin{aligned} \dot{u}^\alpha &= \delta^\alpha H = \frac{1}{\rho} p_\alpha \\ \dot{p}_\alpha &= -\delta_\alpha H = d_i \left(\varepsilon_{\gamma\delta} C^{\alpha i\gamma\delta} - G^{\alpha i\gamma} E_\gamma\right) \end{aligned}$$

with J, R,

$$J = \begin{bmatrix} 0 & I_{3x3} \\ -I_{3x3} & 0 \end{bmatrix}, \qquad R = 0 \ .$$

Piezo (5)

The collocated output is

$$y^\gamma = d_i \left(\frac{1}{\rho} p_\alpha G^{\alpha i \gamma} \right) \ .$$

Of special interest is the case, where the electrical field strength E has a potential $U^\varsigma \Phi_\varsigma$, or

$$E = U^\varsigma \mathrm{d}_h \Phi_\varsigma \ , \quad \Phi_\varsigma \in C^\infty \left(J^n \left(\mathcal{X} \right) \right)$$

is met.

Let us choose the voltages U^ς, $\varsigma = 1, \ldots, m$ as inputs, then the Hamiltonian density is given by

$$\left(H - G^{\alpha\beta\gamma} \varepsilon_{\alpha\beta} d_\gamma \left(\Phi_\varsigma \right) U^\varsigma \right) \mathrm{d}X = \left(H - H_\varsigma U^\varsigma \right) \mathrm{d}X \ .$$

Piezo (5)

Let the the Hamiltonian density be is given by

$$\left(H - H_\varsigma U^\varsigma - \hat{H}_\varsigma \hat{d}^\varsigma\right) \mathrm{d}X$$

with the disturbance $\hat{d}^\varsigma$.

Actuator shaping: How must we design the actuator, such that it acts in the same manner on the structure like the disturbances?

Answer: The relation

$$\delta\left(\left(H_\tau a_\varsigma^\tau - \hat{H}_\varsigma\right) \mathrm{d}X\right) = 0$$

must be met for $a_\varsigma^\tau \in \mathbb{R}$. Furthermore, one has to add suitable boundary condition.

Conclusions

- PCHD-systems have turned out to be a very useful tool for the analysis and design in the ODE case.
- The extension of the approach to the PDE case is not unique at all.
- The presented approach works well for piezoelectric systems.
- Temperature effects lie outside the presented framework, but they can be taken into account by a nonlinear approach.
- Several control schemes like D- or PD-control can be adopted in a more or less straightforward manner, others like H_2- or H_∞-control can be extended under certain circumstances. Also state feedback design approaches like IDA-PBC need further investigations concerning the implementation.

• • •

Thank you for attending.