Eigenvalue Problems in Surface Acoustic Wave Filter Simulations

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"3D hp-Finite Elements: Fast Solvers and Adaptivity"
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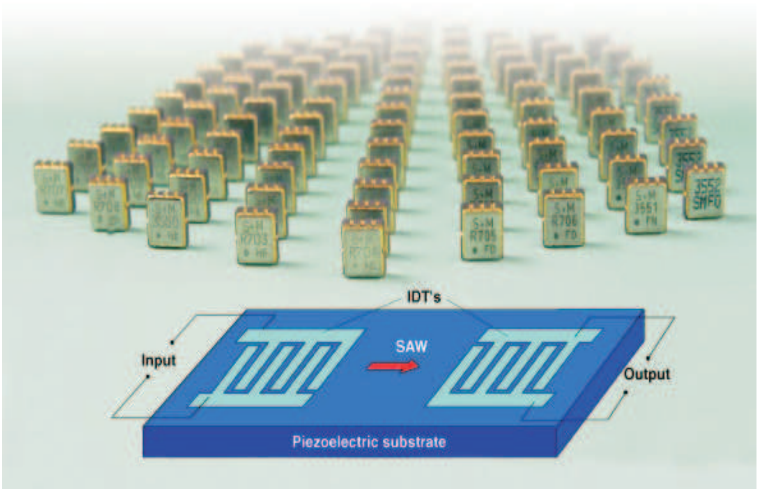
Outline

1. Introduction to SAW filters an piezoelectricity
2. Mathematical modeling
3. Extraction and solution of parameter-dependent Eigenvalue problem
4. Some numerical aspects and results

Surface Acoustive Wave (SAW) Filters

are piezoelectric devices used for frequency filtering in cell phones, tv-sets, ...



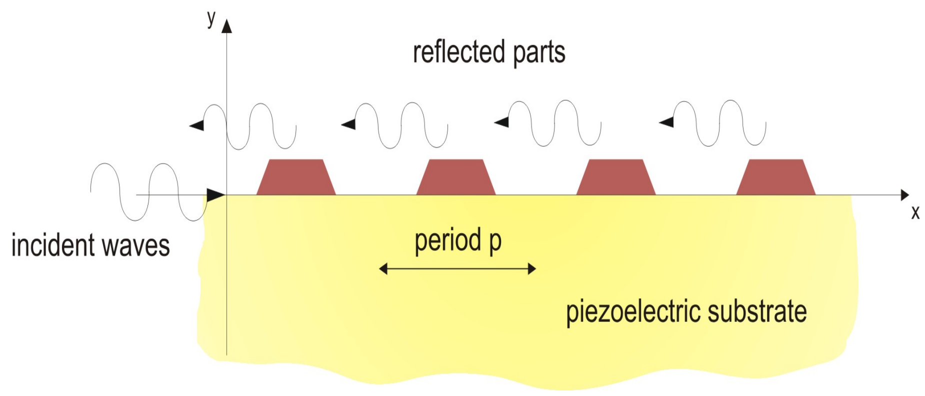
Basically the device consists of

- piezoelectric substrate
- sender and receiver comb of electrodes on substrate surface

Sender and receiver consist in general of $>$ 1000 periodically arranged electrodes

Surface wave propagation and periodically arranged electrodes

Surface acoustic waves:
propagating along surface, amplitude negliable within depth of a few wavelengths.



Periodically arranged electrodes: partial reflection of propagating surface waves.

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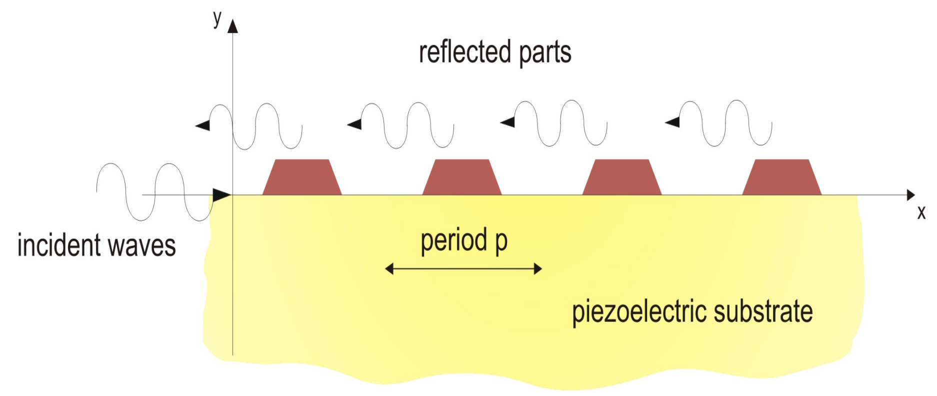

Periodically arranged electrodes: partial reflection of propagating surface waves.

If period p of cell is half the wavelength, i.e. $\lambda = 2.p$, the reflected parts are in phase.

$\longrightarrow$ Constructive interference of reflected waves

$\longrightarrow$ Impedes the propagation of the surface wave

$\longrightarrow$ Excitation-frequency not in output signal

Frequency domain splitted in **stop-bands** and **pass-bands** $\rightarrow$ **frequency-filtering**

Wave propagation in periodic geometries

We assume time-harmonic wave excitation:

$$u(x,t) = e^{i\omega t} u(x)$$

Due to Floquet-Bloch Theorem (later) waves propagating in infinite 1D-periodically pertubed geometries are *quasi-periodic*, i.e.

$$\boxed{u(x,t) = u_p(x)\, e^{(\alpha + i\beta)x}\, e^{i\omega t}}$$

with $u_p(.,.)$ and

$\alpha \cdot p$	...	attenuation of wave per cell of length p,
$\beta \cdot p$	...	phase shift of wave per cell,
$u_p(.,.)$	...	p-periodic field in x_1.

The Diagram of Dispersion

$$u(x,t) = u_p(x)\, e^{(\alpha+i\beta)x}\, e^{i\omega t}$$

Connection between frequency ω and complex propagations constant $\alpha + i\beta$?

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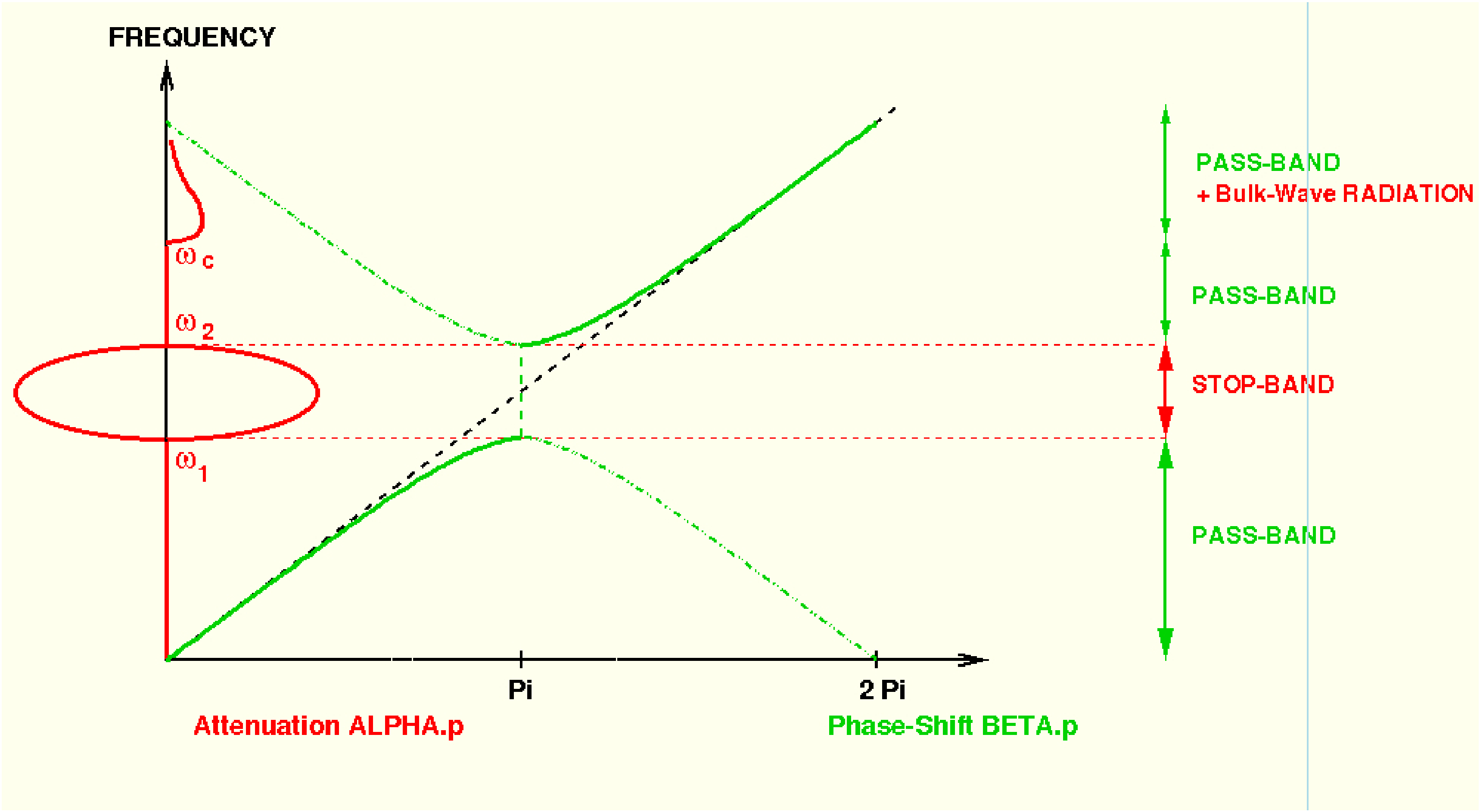

TASK: Compute all modes of wave propagation i.e. get the complete dispersion context.

Mathematical Modeling

Three main steps in mathemtical modeling:

- **piezoelectric coupled field equations**
- **infinite structure with periodic pertubations**
  $\longrightarrow$ Floquet-Bloch theory
- **truncation of the computation domain in y-direction (depth of piezoelectric substrate)**
  volume wave radiation
  $\longrightarrow$ non-reflecting boundary conditions for piezoelectric equations

Piezoelectric Equations (1)

Linear coupling of mechanical and electrostatic field describing direct and indirect piezoelectric effect.

1. **Mechanical field equations** with displacement u in $\mathbb{R}^d, d = 2, 3$

   Newton's law: $\text{div}_x\, T = \rho \frac{\partial^2 u}{\partial t^2}$

   geometric property: $Bu = S = \frac{1}{2}(\nabla_x^T u + \nabla_x u)$

2. **Electrical field equations**

   Maxwell's equations:

   electric potential: $\nabla \times E = 0 \rightarrow E = -\nabla\, \Phi$

   insulating material: $\text{div}_x D = q_{free} = 0$

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3. **Piezoelectric coupling**

   Linear coupling of mechanical strain S and electric field E

$$\begin{aligned} T &= cS - e^T E \\ D &= eS + \varepsilon E \end{aligned}$$

<u>Remarks:</u>

Piezoelectric materials are **anisitropic** (45 material coefficients: in general 10 independent)

for $d = 3$ we assume plane strain ($\partial_z = 0$)

Piezoelectric Equations (2)

Variational formulation of **piezoelectric saddle point problem**

$$\begin{array}{rcrcl} \int_\Omega (Bv)^T : c\, Bu - \omega^2 \rho\, v^T u\, dx & + & \int_\Omega (Bv)^T : e^T \nabla\Phi\, dx & = & f_1 \quad \forall\, v \in (H^1(\Omega))^3 \\ \int_\Omega (\nabla\Psi)^T : e\, Bu\, dx & - & \int_\Omega (\nabla\Psi)^T \varepsilon\, \nabla\Phi\, dx & = & f_2 \quad \forall\, \Psi \in H^1(\Omega) \end{array}$$

with strains $Bu := \frac{1}{2}((\nabla u)^T + \nabla u)$

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with strains $Bu := \frac{1}{2}((\nabla u)^T + \nabla u)$

The according FE-system of the piezoelectric problem is of the form

$$\left[\begin{pmatrix} K_{uu} & K_{u\Phi} \\ K_{u\Phi}^T & -K_{\Phi\Phi} \end{pmatrix} - \omega^2 \begin{pmatrix} M_{uu} & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} u \\ \Phi \end{pmatrix} = f_h$$

i.e. indefinite sitffness matrix K and positive semidefine mass matrix M.

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i.e. indefinite sitffness matrix K and positive semidefine mass matrix M.

Remark: Scaling properties of stiffness matrix due to material coefficients:

$$\begin{array}{rcl} K_{uu} & \approx & \mathcal{O}(10^{10}) \\ K_{\Phi\Phi} & \approx & \mathcal{O}(10^{-10}) \\ K_{u\Phi} & \approx & \mathcal{O}(1). \end{array}$$

The Periodic Problem - Floquet-Bloch Theory

= Spectral theory of periodic partial differential operators

Let L be a partial differential operator, which is p-periodic in x_1 and view the eigenvalue problem

$$Lu = \lambda u \text{ in } \Omega_{strip}$$
$$+ \text{ BC on bottom and top}$$
$$\text{no radiation conditios as } x_1 \to \pm\infty$$

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Floquet-Bloch theorem:

The eigenfunctions are *quasi-periodic Bloch waves*, i.e. of the form

$$\boxed{u(x_1, x_2) = u_p(x_1, x_2) e^{(\alpha + i\beta)x_1},}$$

with in x_1 p-periodic function $u_p(x_1 + p, x_2) = u_p(x_1, x_2)$

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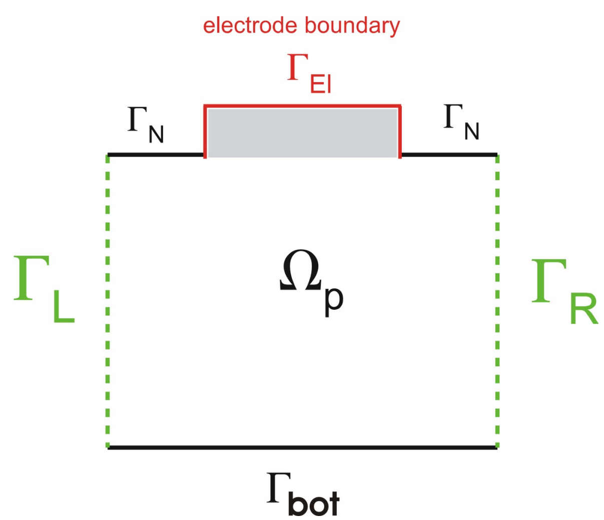
Implies that solutions of infinite periodic strip are governed by

- solutions $u_p(x)$ on a unit-cell Ω_p,
- complex propagation constants $\alpha + i\beta$.

The unit cell problem



$$
\begin{aligned}
-\operatorname{div}(c\ \tfrac{1}{2}((\nabla u)^T + \nabla u) + e^T\ \nabla\Phi) &= \omega^2 \rho u \text{ in } \Omega_P \\
-\operatorname{div}(e\ \tfrac{1}{2}((\nabla u)^T + \nabla u) - \epsilon\ \Phi) &= 0 \text{ in } \Omega_P \\
T.n = 0, D.n &= 0 \text{ on } \Gamma_N \\
\Phi &= 0 \text{ on } \Gamma_{El}
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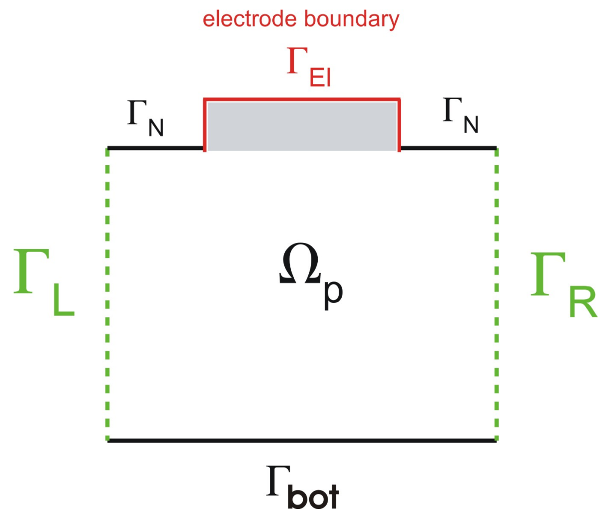

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Quasi-periodic boundary conditions on Γ_L, Γ_R

$$
\begin{aligned}
\tilde{u}(p, x_2) &= \gamma\, \tilde{u}(0, x_2) \\
\frac{\partial \tilde{u}}{\partial N_r}(p, x_2) &= -\gamma \frac{\partial \tilde{u}}{\partial N_l}(0, x_2)
\end{aligned}
\quad \text{with } \tilde{u} = (u_1, u_2, u_3, \Phi)
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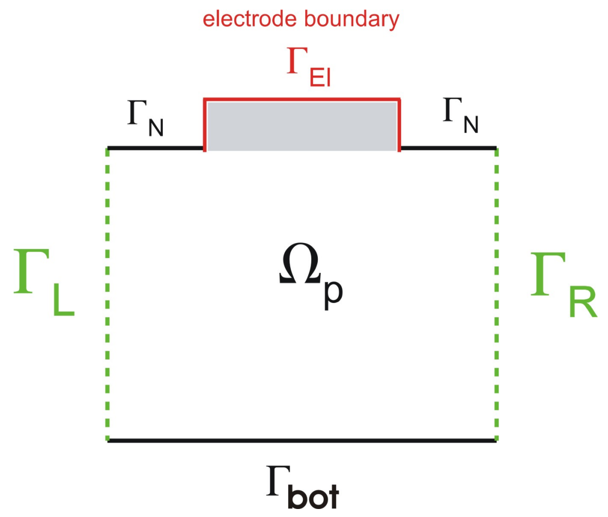
with propagation parameter $\gamma := e^{(\alpha + i\beta)p}$.

The unit cell problem



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with propagation parameter $\gamma := e^{(\alpha+i\beta)p}$.

Absorbing BCs on bottom

$$\frac{\partial \tilde{u}}{\partial N}(x_1, y) = iR\tilde{u}(x_1, y) \quad \text{on } \Gamma_{bot}$$

The effect of absorbing BCs on Γ_{bot}

Source problem $(K - \omega^2 M)u = q$ with 4 electrodes with given alternating potential

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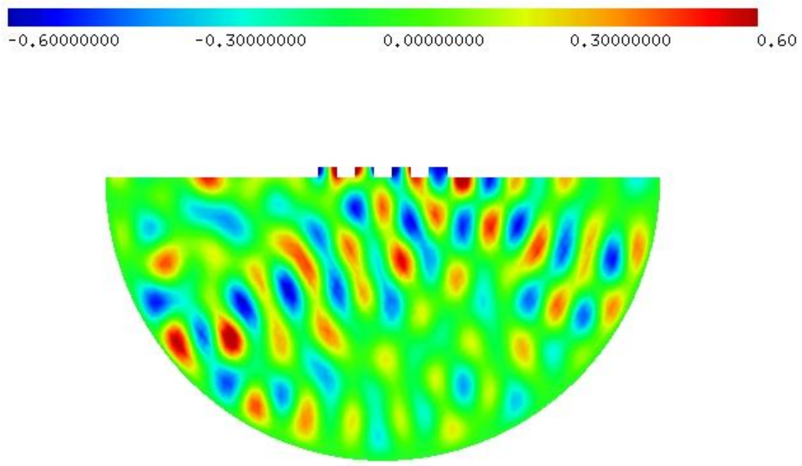

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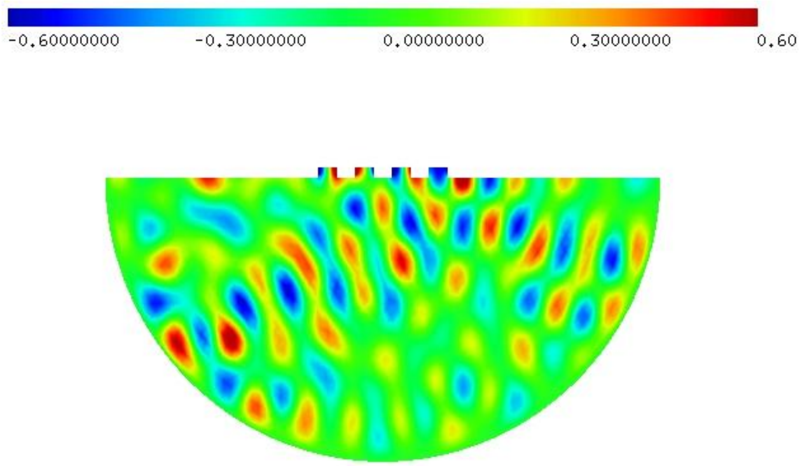

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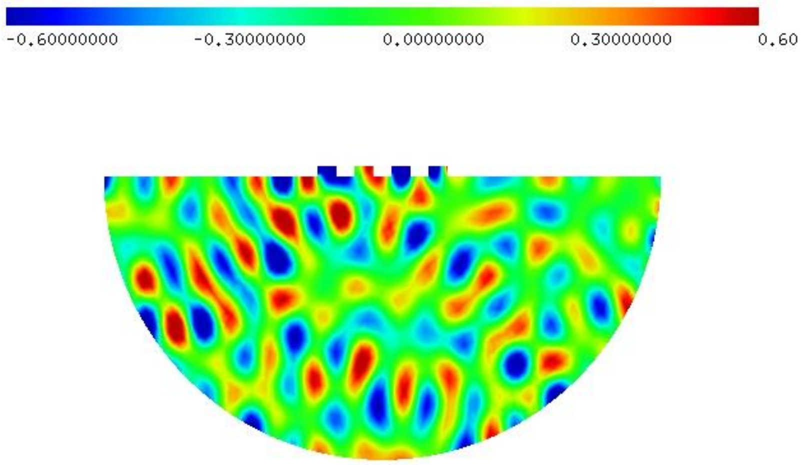

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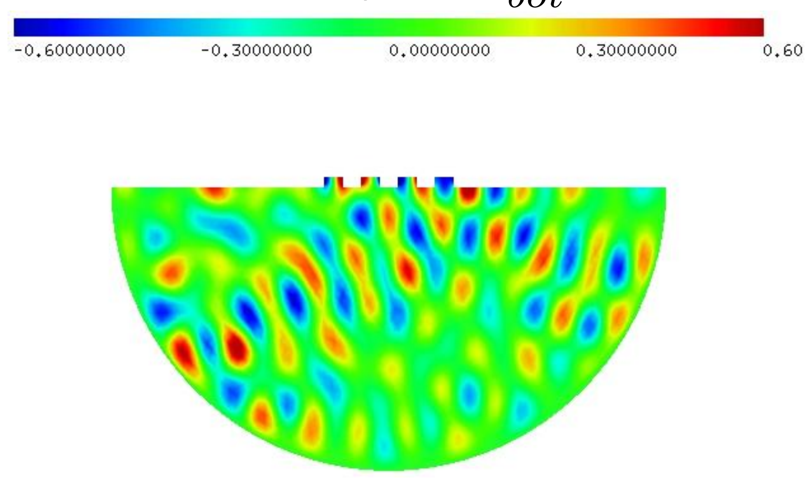

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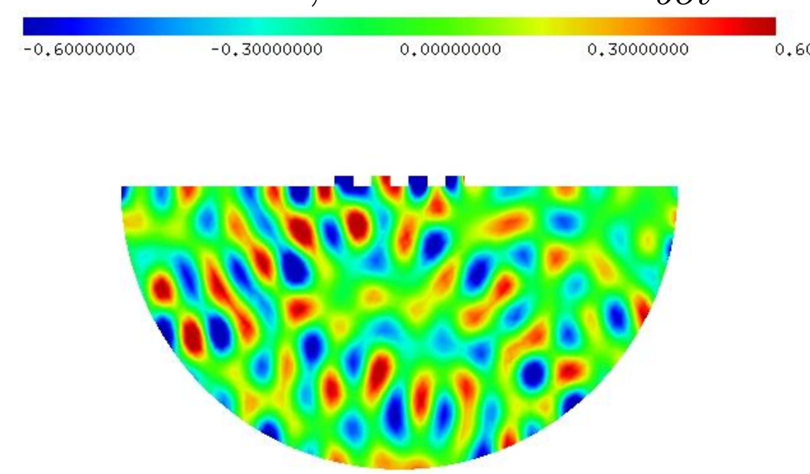
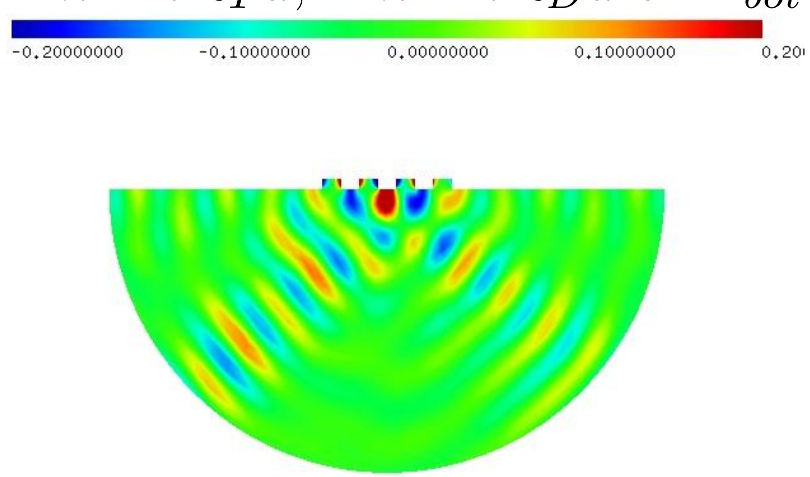


$T.n = iR_T u, D.n = iR_D u$ on Γ_{bot}



Mathematical Modeling

Three main points in mathemtical modelling:

- **piezoelectric coupled field equations**
  $\longrightarrow$ indefinte system matrix (saddle point problem)

- **infinite structure with periodic pertubations**
  $\longrightarrow$ Floquet-Bloch theory
  $\longrightarrow$ parameter-dependent eigenproblem on unit-cell

- **truncation of the computation domain in y-direction (depth of piezoelectric substrate)**
  volume wave radiation
  $\longrightarrow$ non-reflecting boundary conditions for piezoelectric equations
  $\longrightarrow$ complex-valued system matrices

$\Rightarrow$ **Piezoelectric quasi-periodic unit-cell problem ($\gamma = e^{(\alpha+i\beta).p}$, ω)**

Lagrange-parameter formulation of unit-cell problem

For the quasi-periodic problem

$$a(u,v) + i\,\omega\,c(u,v) - \omega^2\,m(u,v) + \int_{\Gamma_L} \frac{\partial u}{\partial N} \cdot v\,ds + \int_{\Gamma_R} \frac{\partial u}{\partial N} \cdot v\,ds = 0 \quad \forall\,v\,\in\,\left[H^1(\Omega)\right]^4$$

$$u_r = \gamma u_l \text{ and } \frac{\partial u_r}{\partial N_r} = -\gamma\,\frac{\partial u_l}{\partial N_l}$$

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we introduce

- **Lagrange-parameter** $\lambda = \frac{\partial u}{\partial N_l} \in \left[H^{-\frac{1}{2}}(\Gamma)\right]^4$

- **Trace-Operators** for left/right boundary $\begin{array}{ccccc} tr_l & : & \left[H^1(\Omega)\right]^4 & \to & \left[H^{\frac{1}{2}}(\Gamma)\right]^4 \\ & & u & \to & u_l \end{array}$, tr_r

  and the adjoints tr_l^*, tr_r^*.

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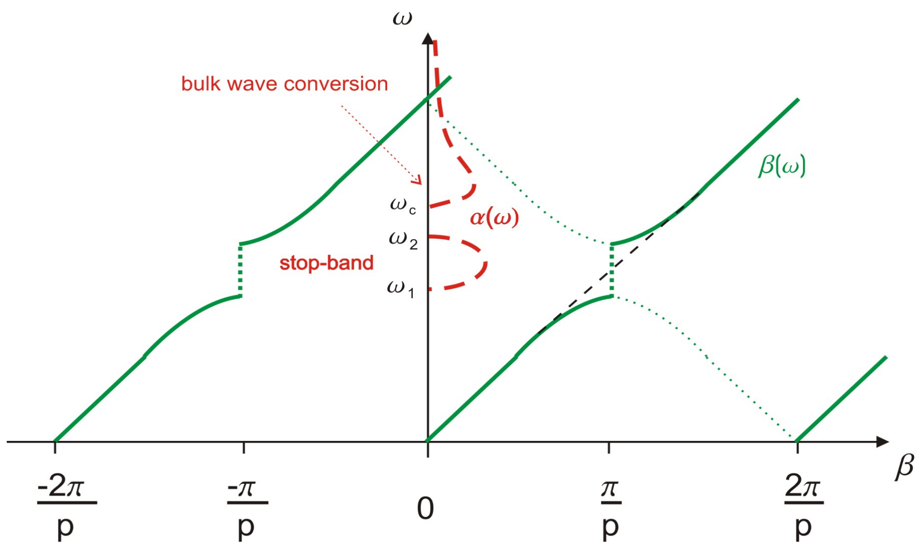
and get a **mixed variational formulation on the unit cell**

Find (u, λ) in $(H^1(\Omega), H^{-\frac{1}{2}}(\Gamma))$ such that

$$\begin{array}{lclcll} a(u,v) + i\,\omega\,c(u,v) - \omega^2\,m(u,v) & + & \int_\Gamma (tr_l v - \gamma\, tr_r v)\,\lambda ds & = & 0 & \forall\, v\, \in\, \left[H^1(\Omega)\right]^4 \\ \int_\Gamma (tr_r u - \gamma\, tr_l u)\mu\, ds & & & = & 0 & \forall\, \mu\, \in\, \left[H^{-\frac{1}{2}}(\Gamma)\right]^4 \end{array}$$

How to formulate the parameter-dependent eigenvalue problem ?

for the computation of the dispersion diagram



1. The more natural approach

Computation of the eigenfrequencies ω according to the given parameter γ

Problem: How to choose the input parameter γ since it depends on two parameters $\gamma = e^{\alpha+i\beta}$?

$\longrightarrow$ useful method for computing only pass-bands, $\gamma = e^{i\beta}$ (unit-circle)

2. Alternative: Frequency-dependent eigenvalue problem

Compute for a given frequency ω the according complex propagation-constants γ.

The frequency-dependent eigenvalue problem

For given parameters ω^2 search for $(\gamma, (u, \lambda)^T)$ such that

$$\begin{array}{rcll} \overbrace{a(u,v) + i\,\omega\, c(u,v) - \omega^2 m(u,v)}^{k(\omega)(u,v)} \; + \; <(tr_l^* - \gamma tr_r^*)\lambda, v> & = & 0 & \forall\, v \in \left[H^1(\Omega)\right]^4 \\ <(tr_r - \gamma tr_l)u, \mu> & = & 0 & \forall\, \mu \in \left[H^{-0.5}(\Gamma)\right]^4 \end{array}$$

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The FE-discretiszed problem is of the form

$$\begin{pmatrix} \overline{K}(\omega) & {Tr_l}^T \\ Tr_r & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix} = \gamma \begin{pmatrix} 0 & {Tr_r}^T \\ Tr_l & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \lambda_h \end{pmatrix}$$

with $\overline{K} := K + i\omega C - \omega^2 M$ complex-symmetric and indefinite (of saddle point structure).

General choice pf FE-space for dual space $H^{-0.5}(\Gamma)$:
$\longrightarrow$ Mortar Finite Elements [Wohlmuth,Maday].

The frequency-dependent eigenvalue problem (2)

If we use **periodic FE-meshes** (adaption of mesh-generator), we achieve $Tr_l = I_l, \quad Tr_r = I_r$.
Splitting in inner, left and right boundary nodes, we get

$$\left(\begin{array}{ccc|c} \overline{K}_{ii} & \overline{K}_{li}^T & \overline{K}_{ri}^T & 0 \\ \overline{K}_{li} & \overline{K}_{ll} & 0 & I \\ \overline{K}_{ri} & 0 & \overline{K}_{rr} & 0 \\ \hline 0 & 0 & I & 0 \end{array}\right) \begin{pmatrix} u_i \\ u_l \\ u_r \\ \lambda \end{pmatrix} = \gamma \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \hline 0 & I & 0 & 0 \end{array}\right) \begin{pmatrix} u_i \\ u_l \\ u_r \\ \lambda \end{pmatrix}.$$

$$\begin{array}{rcl} \text{dim}(\ A\ ) & = & n_i + 3.n_l \\ \text{dim}(\ \text{ker}\ B\ ) & = & n_i + n_l \end{array} \quad \Rightarrow \quad \begin{array}{l} 2.n_l \text{ finite} \\ = n_i - n_l \text{ infinite} \end{array} \quad \text{eigenvalues}$$

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What eigenvalues are we interested in ?

pass-band $\alpha = 0, \beta.p \in [0, 2\,\pi]$
stop-band $|\alpha| \neq 0$ small $, \beta.p = \pi$
volume-wave interaction $|\alpha|$ small $, \beta.p \in [0, 2\,\pi]$

We are interested in eigenvalues $\gamma = e^{(\alpha + i\beta)p}$ on and near the unit-circle, i.e. $|\gamma| \approx 1$.

Inner-Node-Matrix Method(1)

Reduce the problem by eliminating some infinite eigenvalues

- Eliminate u_r, since last line tells $u_r = \gamma u_l$
- Eliminate λ by $\lambda = -\overline{K}_{li} u_i - \overline{K}_{ll} u_l$

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- Eliminate λ by $\lambda = -\overline{K}_{li} u_i - \overline{K}_{ll} u_l$

we get the reduced generalized linear (non-hermitian) eigenvalue problem

$$\begin{pmatrix} \overline{K}_{ii} & \overline{K}_{il} \\ \overline{K}_{ir}^T & 0 \end{pmatrix} \begin{pmatrix} u_i \\ u_l \end{pmatrix} = \gamma \begin{pmatrix} 0 & -\overline{K}_{ir} \\ -\overline{K}_{il}^T & -\overline{K}_{ll} - \overline{K}_{rr} \end{pmatrix} \begin{pmatrix} u_i \\ u_l \end{pmatrix}$$

Inner-Node-Matrix Method(1)

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By spectral-transformation we achieve an eigenvalue problem with pencil $(A - \tilde{\lambda} B)$ where B is regular and complex symmetric

$$\begin{pmatrix} 0 & -\overline{K}_{ir} \\ -\overline{K}_{il}^T & -\overline{K}_{ll} - \overline{K}_{rr} \end{pmatrix} \begin{pmatrix} u_i \\ u_l \end{pmatrix} = \frac{1}{\gamma - 1} \begin{pmatrix} \overline{K}_{ii} & \overline{K}_{il} + \overline{K}_{ir} \\ \overline{K}_{ir}^T + \overline{K}_{il}^T & \overline{K}_{ll} + \overline{K}_{rr} \end{pmatrix} \begin{pmatrix} u_i \\ u_l \end{pmatrix}.$$

This EVP is solved by non-hermitian Arnoldi solver (ARPACK) requiring sparse Cholesky-factorization for $B^{-1}v$.

Schur-Complement Method

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Quadratic eigenvalue problem in u_l (nodes on Γ_L)

Find eigenvalues $\gamma \in \mathbb{C}$ with $|\gamma| \approx 1$

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- dense, but small-dimensioned ($\sharp${dofs on left boundary})
- linearization to a generalized non-hermitian EVP (small-dimensioned)
- Application of direct QZ-Solver (LAPACK)

On the special structure of the Eigenvalue problem

Since the Schur-complement is symmetric the reduced Schur-Complement eigenvalue problem

$$\gamma^2 S_{lr} u_l + \gamma(S_{ll} + S_{rr}) u_l + S_{lr}^T u_l = 0$$

is of the form

$$\gamma^2 A v + \gamma B v + A^T v = 0$$

with $B = B^T$ complex-symmetric.

If (γ, v) **is an eigen-pair then** $(1/\gamma, v^T)$ **is a left eigenpair.**
I.e. the reduced problem is symplectic.

$\Rightarrow$ one can apply structure-preserving methods [Mehrmann]

Requirements in each frequency step (ω)

for the Inner-Node-Matrix method

1. Large-dimensioned eigenvalue problem

2. Generalized eigenvalue problem" $Ax = \tilde{\lambda} Bx$"
   with Cholesky factorization of sparse matrix B

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for the Schur-Complement method

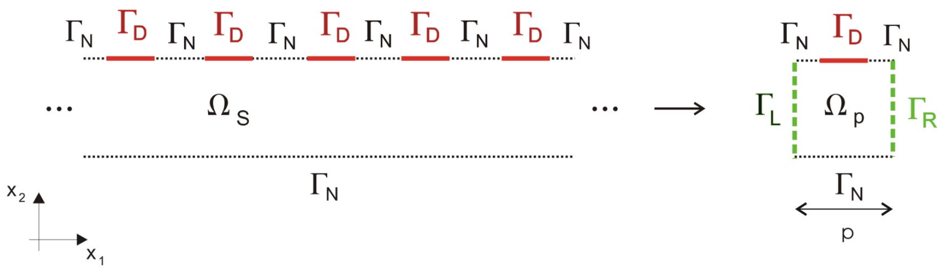
1. Evaluation of the Schur-Complement blocks via sparse Cholesky factorization

2. Small-dimensioned linear eigenvalue problems ($2.\sharp\{$dofs on left bd. $\}$))

3. Direct solver computes all eigenvalues

Results - Effect of periodic pertubations (1)

Elastic plane strain problem without/with periodic pertubation



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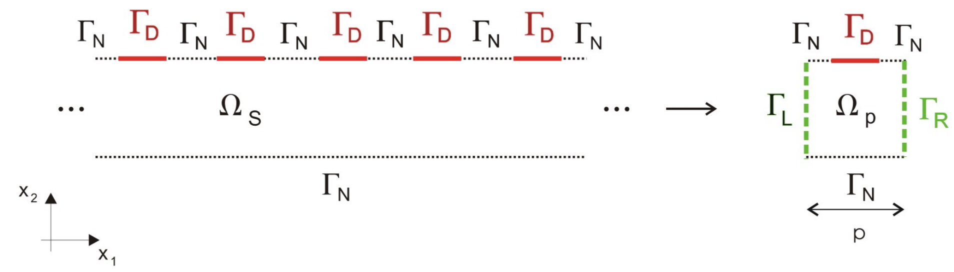
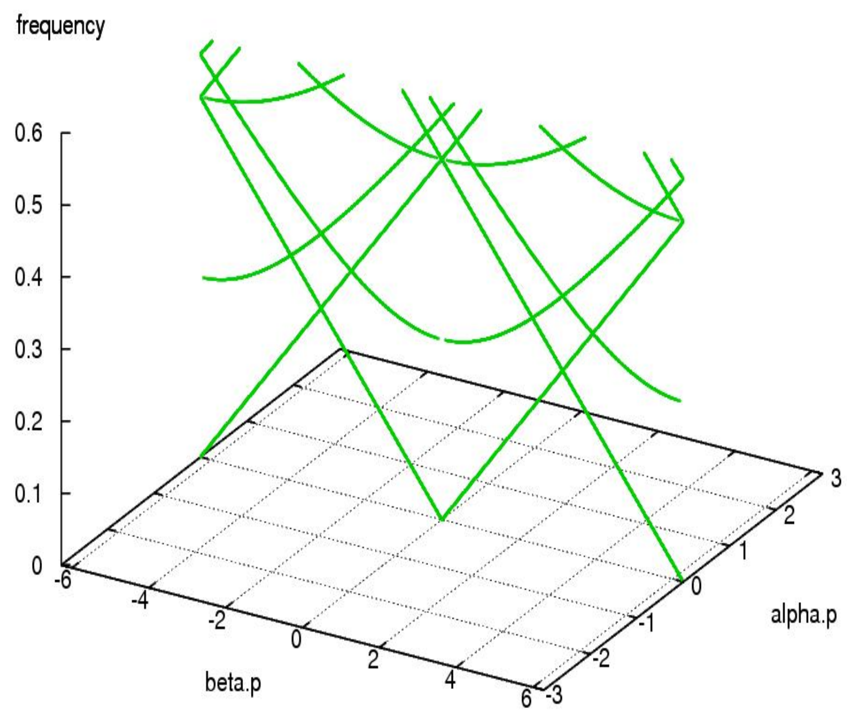
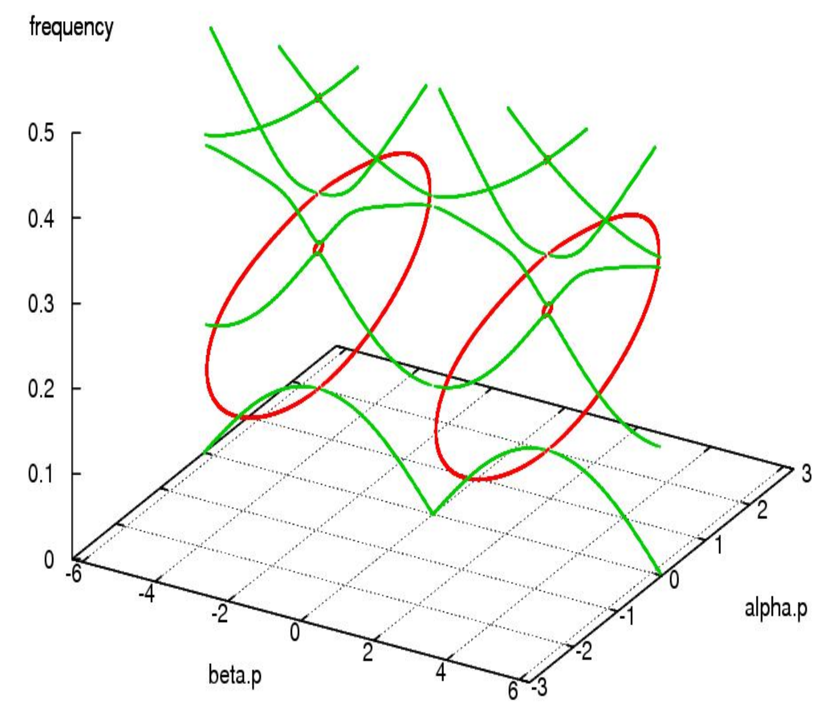
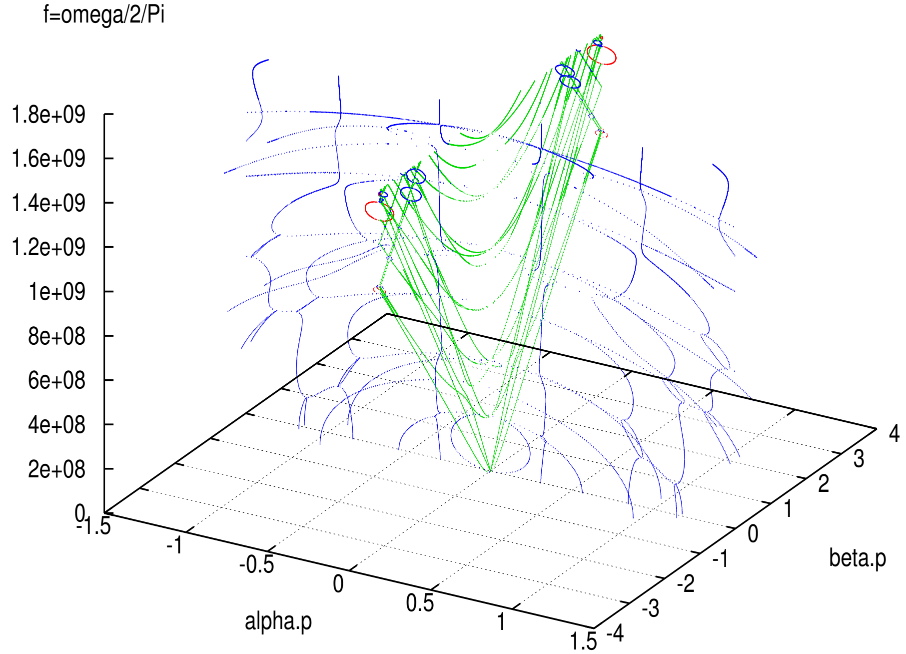


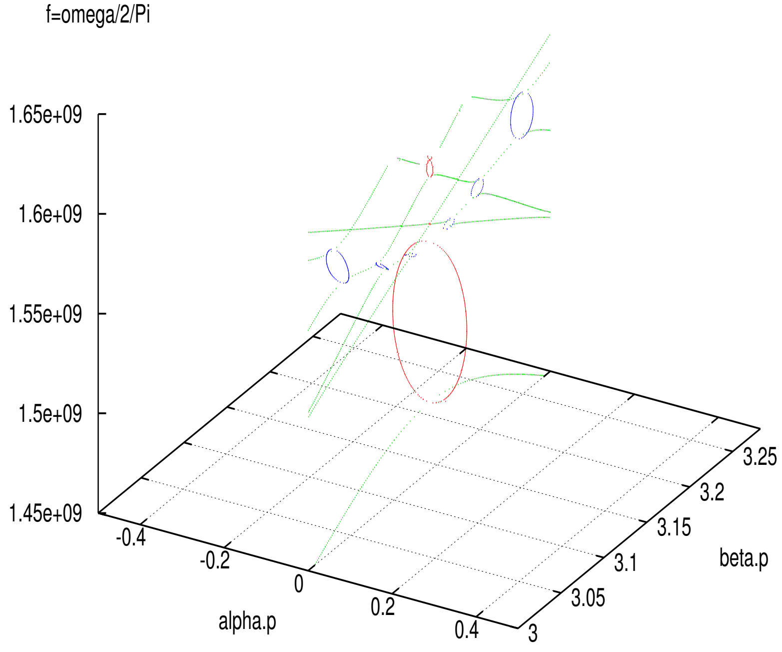




Real life problem - GSM-filter structure

Simulated diagram of dispersion for GSM-filter structure:

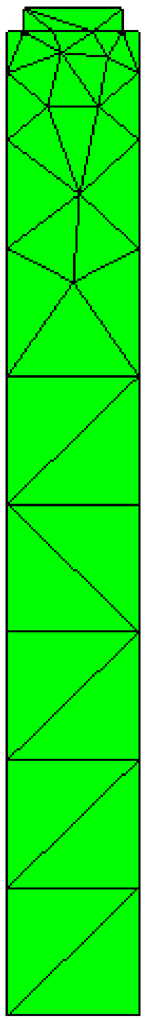




Computational times for simulating piezoelectric problem:
500 frequency steps by 6972 unknowns last $\approx 8.5h$.

Using higher-order finite elements

hp-methods: In domains, where function is smooth: coarse elements of higher order.
Resolve singularities by h-refinement.



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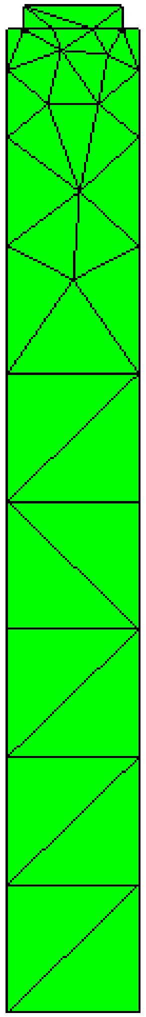
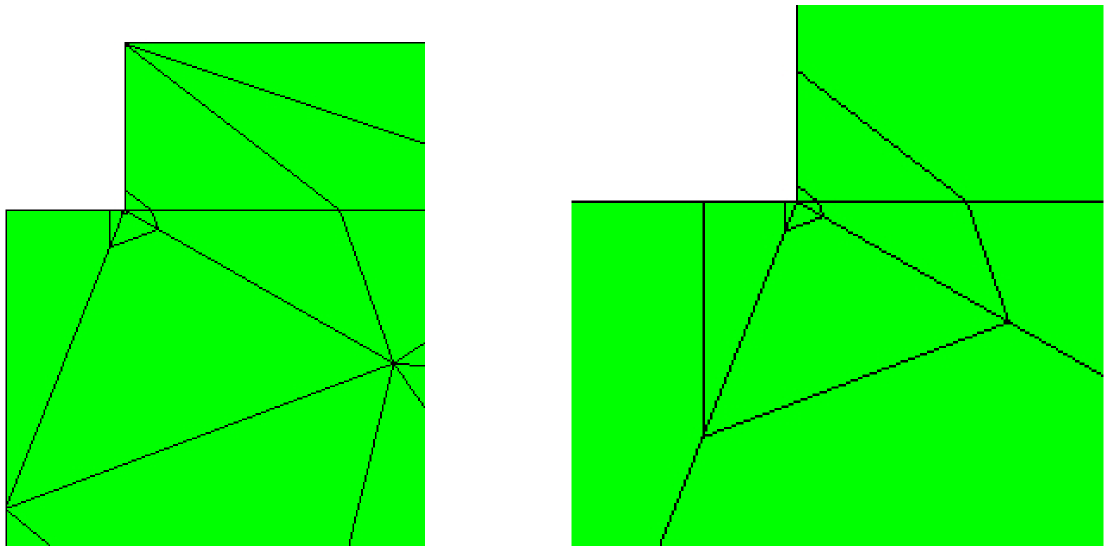
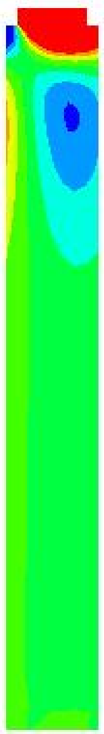


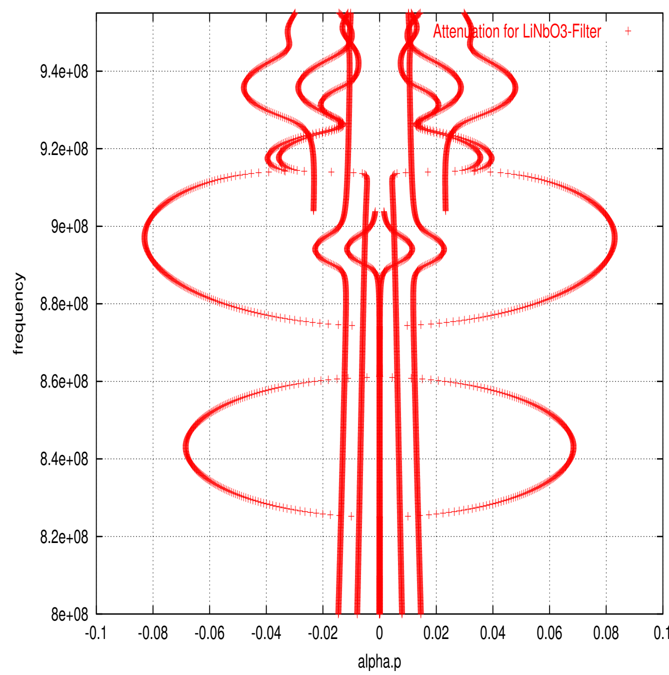
- Singularities at cornerpoints: geometric h-refinement ($\rightarrow$ trigs,quads)

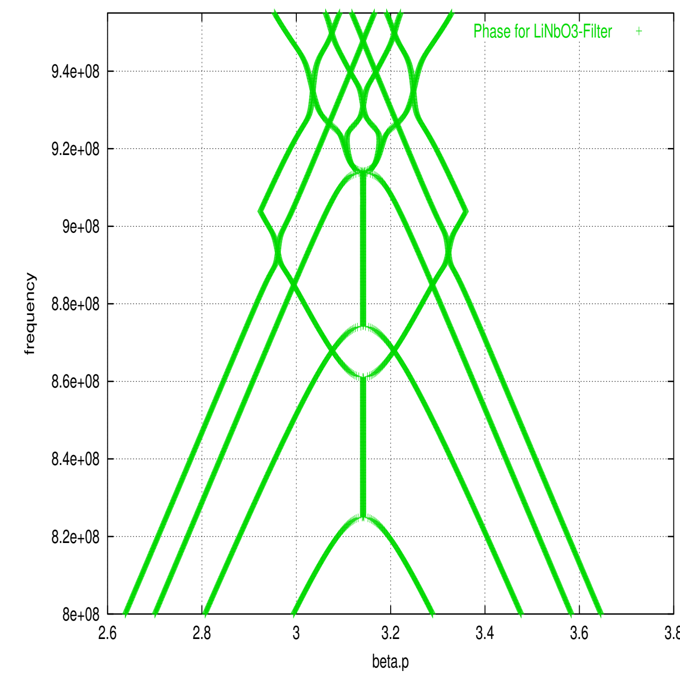


- Adopt periodic BC's to hierarchical high-order elements ($\rightarrow$ identify periodic vertex and edge dofs)

Higher-order finite elements: TV-Filter Structure







Compute complex propagation parameters (dispersion context) around 1st stop-band of LiNb-structure:

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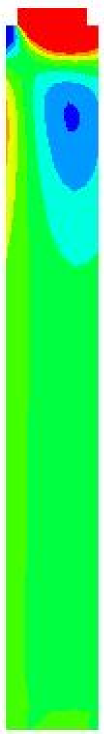

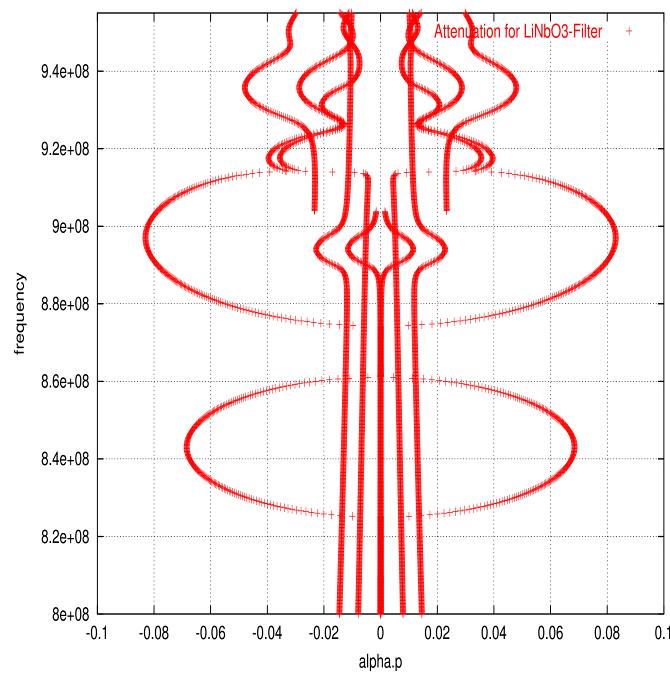

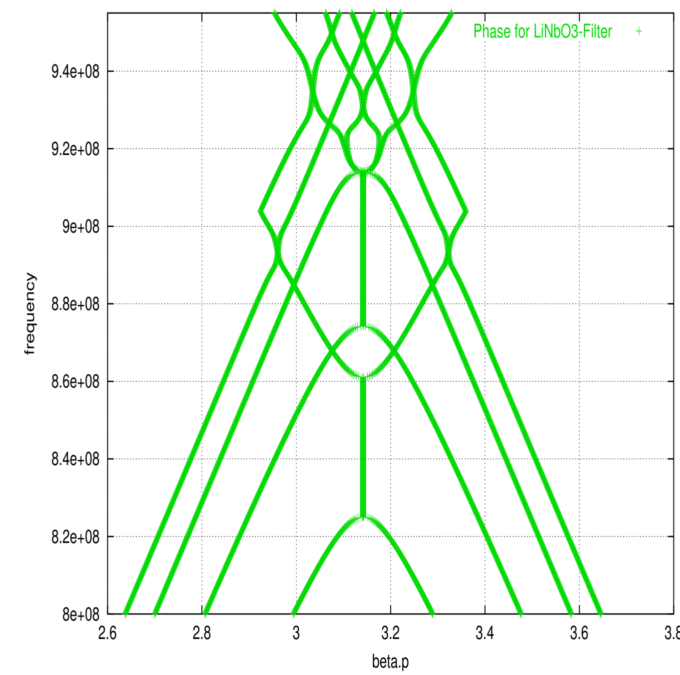

Compute complex propagation parameters (dispersion context) around 1st stop-band of LiNb-structure:

p	hp-level	elements	dofs	time per frequ	500 steps
1	0	3450	4×1817	42 s	$\approx$ 6 h
1,..,3	2	59	4×355	2.8 s	$\approx$ 23 min
1,..,4	3	69	4×667	9.8 s	$\approx$ 1.4 h

The accuracy by using 2 hp-levels is competitive with h-version with 3450 elements !

Conclusions

- Mathematical model for periodic piezoelectric structures
- Frequency-dependent eigenvalue problem (2 solution strategies)
- Acceleration by using hp-methods

Ongoing Work

1. Improve non-reflecting boundary condition $\rightarrow$ Perfectly Matched Layers for Piezo
2. Improve Eigenvalue solver:
   e.g. exploit "symplectic" structure of EVP: structure preserving methods
   for 2D simulation: Schur-Complement Method is efficient