Identification of Models for Nonlinearity and Hysteresis in Piezoelectricity

DFG Research Group (Emmy Noether Program)

Inverse Problems in Piezoelectricity

Barbara Kaltenbacher,
Tom Lahmer,
Marcus Mohr

Department of Sensor Technology,
University of Erlangen:
Fred Hofer,
Manfred Kaltenbacher,
Erich Leder,
Reinhard Simkovics
Overview

• Piezoelectric effect: PDE model, material tensors

• Determination of constant coefficients: PDE based fit

• Identification of nonlinear material parameter curves: multiharmonic formulation

• Hysteresis identification iterative methods
Piezoelectric Transducers

Direct effect: apply mechanical force $\rightarrow$ measure electric voltage
Indirect effect: impress electric voltage $\rightarrow$ observe mechanical displacement

Application Areas:

- Ultrasound (medical imaging & therapy)
- Force- and acceleration Sensors
- Actor injection valves (common–rail Diesel engines)
- SAW (surface-acoustic-wave) sensors
- . . .
Piezoelectric Effect

\[
\begin{align*}
\vec{\sigma} &= c^E \vec{S} - \mathbf{e}^T \vec{E} \\
\vec{D} &= \mathbf{e} \vec{S} + \varepsilon^S \vec{E}
\end{align*}
\]

\[
\begin{align*}
\vec{\sigma} &= (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy})^T \ldots \text{stress} \\
\vec{S} &= (S_{xx}, S_{yy}, S_{zz}, S_{yz}, S_{xz}, S_{xy})^T = \text{DIV}^T \vec{d} \ldots \text{strain} \\
\vec{E} &= (E_x, E_y, E_z) = -\text{grad}\phi \ldots \text{electr. field} \\
\vec{D} &= (D_x, D_y, D_z)^T \ldots \text{electr. displacement} \\
\vec{d} &= (d_x, d_y, d_z) \ldots \text{mech. displacement} \\
\phi &= \ldots \text{electr. potential}
\end{align*}
\]
Piezoelectric PDEs:

\[ \rho \frac{\partial^2 \vec{d}}{\partial t^2} - \text{DIV} \left( c^E \text{DIV}^T \vec{d} + e^T \text{grad} \phi \right) = 0 \text{ in } \Omega \]

\[ -\text{div} \left( e \text{DIV}^T \vec{d} - \varepsilon^S \text{grad} \phi \right) = 0 \text{ in } \Omega \]

Boundary conditions:

\[ N^T \sigma = 0 \text{ on } \partial \Omega \]
\[ \phi = 0 \text{ on } \Gamma_g \]
\[ \phi = \phi^e \text{ on } \Gamma_e \]
\[ \vec{D} \cdot \vec{n} = 0 \text{ on } \Gamma \]

\[ \Gamma_e \ldots \text{loaded electrode} \quad \Gamma_g \ldots \text{grounded electrode} \]
\[ \Gamma = \partial \Omega \setminus (\Gamma_g \cup \Gamma_e) \]
\[ \phi^e \ldots \text{impressend voltage} \]

Fast forward solution → Marcus Mohr [B.K. & Lahmer & Mohr, EJAM, to appear]

Simulation of piezoelectric transducers requires knowledge of material tensors $c^E, e, \varepsilon^S$
# Material Tensors

\[
\begin{array}{cccccc}
  c_{11}^E & c_{12}^E & c_{13}^E & \cdots & \cdots & e_{31} \\
  c_{12}^E & c_{11}^E & c_{13}^E & \cdots & \cdots & e_{31} \\
  c_{13}^E & c_{13}^E & c_{33}^E & \cdots & \cdots & e_{33} \\
  \cdots & \cdots & c_{44}^E & \cdots & \cdots & e_{15} \\
  \cdots & \cdots & \cdots & \frac{1}{2}(c_{11}^E - c_{12}^E) & \cdots & \cdots \\
  e_{31} & e_{31} & e_{33} & \cdots & \cdots & e_{15} \\
\end{array}
\]

- **elasticity**
- **piezoelectric coupling**
- **dielectric**

→ **10 different scalar coefficients**
State of the Art:

1-d model simplification via Test sample scheme

→ explicit relation between material parameters and resonance frequencies

- costly: probes, measurements
- imprecise
- restricted to constant coefficients

Direct and Inverse Problems in Piezoelectricity, Linz, 2005
Find material tensors $c^E$, $e$, $\varepsilon^S$ from impedance measurements for different frequencies $\omega$

$$Z(\omega) = \frac{\hat{\phi}^e(\omega)}{j\omega \hat{q}(\omega)|_{\Gamma_e}}$$

$\hat{\phi}^e$... impressed voltage
$\hat{q}$... surface charge
Inverse Problem: Identification by Simulation of Piezo PDEs (II)

\[
\begin{align*}
-\rho \omega^2 \ddot{\mathbf{d}} - \text{DIV} \left( c^E \text{DIV}^T \mathbf{d} + \mathbf{e}^T \text{grad} \hat{\phi} \right) &= 0 \quad \text{in } \Omega \\
-\text{div} \left( \mathbf{e} \text{DIV}^T \mathbf{d} - \varepsilon^S \text{grad} \hat{\phi} \right) &= 0 \quad \text{in } \Omega \\
\hat{\phi} &= \hat{\phi}^e \quad \text{inhom. BC} \\
\int_{\Gamma_e} \mathbf{D} \cdot \mathbf{n} &= \hat{q} \quad \text{measurements}
\end{align*}
\]

\[\begin{align*}
\text{...hom. harmonic PDE} \\
\text{...inhom. BC} \\
\text{...measurements}
\end{align*}\]

- Nonlinear operator equation \( F(c^E, \mathbf{e}, \varepsilon^S) = \hat{q} \) where \( F \) contains PDE solution

- Solve by Newton’s method: \( F'(c^E, \mathbf{e}, \varepsilon^S)[d\mathbf{c}^E, d\mathbf{e}, d\varepsilon^S] \) via solution \( (d\mathbf{d}, d\hat{\phi}) \) of

\[
\begin{align*}
-\rho \omega^2 d\ddot{\mathbf{d}} - \text{DIV} \left( c^E \text{DIV}^T d\mathbf{d} + \mathbf{e}^T \text{grad} d\hat{\phi} \right) &= -\text{DIV} \left( d\mathbf{c}^E \text{DIV}^T \mathbf{d} + d\mathbf{e}^T \text{grad} d\hat{\phi} \right) \\
-\text{div} \left( \mathbf{e} \text{DIV}^T d\mathbf{d} - \varepsilon^S \text{grad} d\hat{\phi} \right) &= \text{div} \left( d\mathbf{e} \text{DIV}^T \mathbf{d} - d\varepsilon^S \text{grad} d\hat{\phi} \right) \\
d\hat{\phi} &= 0 \quad \text{on } \Gamma_e
\end{align*}\]

- optimum experiment design: choice of measurement frequencies
- sensitivity matrix \( F'(c^E, \mathbf{e}, \varepsilon^S) \) yields confidence intervals
Nonlinear dependence

Large excitations (actuator applications):

\[ \rho \frac{\partial^2 \vec{d}}{\partial t^2} - \text{DIV} \left( c^E(S) \text{DIV}^T \vec{d} + e(S, E)^T \text{grad} \phi \right) = 0 \]

\[ -\text{div} \left( e(S, E) \text{DIV}^T \vec{d} - \varepsilon^S(E) \text{grad} \phi \right) = 0 \]

\[ S = |\text{DIV} \vec{d}| \quad E = |\text{grad} \phi| \]

→ infinite dimensional problem, instability
**Multiharmonic Formulation (I)**

**Linear case:**

\[-\rho \omega^2 \vec{d} - DIV \left( c^E DIV^T \vec{d} + e^T \text{grad} \hat{\phi} \right) = 0\]

\[-\text{div} \left( e DIV^T \vec{d} - \varepsilon^S \text{grad} \hat{\phi} \right) = 0\]

Excitation at frequency $\omega$

$\rightarrow$ spectrum of $\vec{d}$, $\phi$ concentrated to $\omega$.

**Nonlinear case:**

$c^E(|DIV^T \vec{d}|)$, $e(|DIV^T \vec{d}|, |\text{grad} \phi|)$, $\varepsilon^S(|\text{grad} \phi|)$

Higher harmonics appear

$\rightarrow$ multiharmonic Ansatz:

\[\vec{d}(\vec{x}, t) \approx \sum_{n=-N}^{N} e^{jn\omega t} \hat{d}_n(\vec{x})\]

\[\phi(\vec{x}, t) \approx \sum_{n=-N}^{N} e^{jn\omega t} \hat{\phi}_n(\vec{x})\]

[Bachinger, Schöberl, Langer], nonlinear magnetics
Multiharmonic Formulation (II)

Model problem
\[ d_{tt} - \left( c(d_x) \ d_x \right)_x = f \]

Multiharmonic Ansatz
\[ d(x, t) \approx \sum_{n=-N}^{N} e^{jn\omega t} \hat{d}_n(x) \]

Insert into nonlinear PDE and test with \( \frac{\omega}{2\pi} e^{-jk\omega t} \)

Orthogonality
\[ \frac{\omega}{2\pi} \int_{0}^{2\pi/\omega} e^{jn\omega t} e^{-jk\omega t} \ dt = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else} \end{cases} \]

\[ -\omega^2 k^2 \hat{d}_k(x) - \left( \frac{\omega}{2\pi} \sum_{n=-N}^{N} \int_{0}^{2\pi/\omega} c(d_x(x, t)) e^{jn\omega t} e^{-jk\omega t} \ dt \right) \hat{d}_n(x) = 0 \]
Multiharmonic formulation (III)

\[-\omega^2 k^2 \hat{d}_k(x) - \left( \frac{\omega}{2\pi} \sum_{n=-N}^{N} \int_{0}^{2\pi/\omega} c(d_x(x, t)) e^{jn\omega t} e^{-jk\omega t} \ dt \right) \hat{d}_n(x) = 0 \]

Polynomial Approx. \[ c(d_x(x, t)) \approx \sum_{p=0}^{P} a_p \left( \sum_{n=-N}^{N} e^{jn\omega t} \hat{d}_n(x) \right)^p \]

Multinomial Thrm. \[ \left( \sum_{n=-N}^{N} e^{jn\omega t} \hat{s}_n \right)^p = \sum_{p=(p_0, \ldots, p_N)} \binom{p}{p_0, \ldots, p_N} e^{j(\sum n p_n)\omega t} \cdot \hat{s}_{-N}^{p_0} \ldots \hat{s}_N^{p_N} \]

by factoring out \( e^{jm\omega t} \) and using orthogonality.
Identification in Multiharmonic Formulation

Determine \( c(\lambda) \approx \sum_{p=0}^{P} a_p \lambda^p \) from boundary measurements of \( \hat{d}_{-N}, \ldots, \hat{d}_N \) solving the PDE system

\[
-\omega^2 k^2 \hat{d}_k - \left( \sum_{n=-N}^{N} \sum_{p=0}^{P} a_p \bar{c}_{p, n-k}^p (\hat{d}_0, \ldots, \hat{d}_N) \hat{d}_n x \right) = 0.
\]

Compute polynomial coefficients \( a_0, \ldots, a_P \) of \( c \) by fitting measurements and simulations of \( \hat{d}_{-N}, \ldots, \hat{d}_N \) using Newton’s method

- Arbitrary good approximation of \( c \) for \( N, P \rightarrow \infty \);
- Regularization by finite dimensionality \( N, P < \infty \);
- Multilevel strategy applicable: successively solve problems with increasing \( N, P \);
- Applicable to 3D complex valued piezoelectric PDEs.
Multiharmonic Formulation: Data smoothing

— ... measurement
— ... approximation with $N = 6$ higher harmonics

→ Motivation for regularization by discretization

[Natterer 77], [Vainikko, Hämari 85], [Engl, Neubauer 87], [Groetsch, Neubauer 88], [Kirsch 96], [Pereverzev 00], nonlinear case: [BK 00-]
Numerical Test Example

1D piezoelectric PDEs

\[ \rho d_{tt} - \left( c^E d_x + e(|\phi_x|)^T \phi_x \right)_x = 0 \]
\[ -\left( e(|\phi_x|) d_x - \varepsilon^S |\phi_x| \phi_x \right)_x = 0 \]

• Pz27 material (Ferroperm Piezoceramics A/S);

• Synthetic data from [Andersen et al., 2000] on large signal behaviour of Pz27

• Starting values: material constants for small-signal behaviour from Ferroperm datasheet
Numerical Results

Simultaneous identification of $c_{33}^E$, $e_{33}$ and $\varepsilon_{33}^S$ by Newton’s method, applied to the multiharmonic formulation of the piezoelectric PDEs with exact data:

![Graphs showing numerical results for $c_E$, $e$, and $\varepsilon$.]
with 1% noise:

Direct and Inverse Problems in Piezoelectricity, Linz, 2005
Tests for Layered Media

PDEs:
\[
\rho d_{tt} - (c_A(d_x)d_x)_x = 0 \quad x \in [0, L_1] \cup [L_1 + L_2, L_1 + L_2 + L_3], \quad t \in [0, T]
\]
\[
\rho d_{tt} - (c_B(d_x)d_x)_x = 0 \quad x \in [L_1, L_1 + L_2], \quad t \in [0, T]
\]

boundary conditions:
\[
(c_A(d_x)d_x)(0, t) = g(t), \quad (c_A(d_x)d_x)(L_1 + L_2 + L_3, t) = g(t),
\]

measurements:
\[
d(L, t) = f(t).
\]
Convergence, simultaneously for $c_A$ (top) and $c_B$ (bottom)
Hysteresis Identification

\[
\vec{D} = \varepsilon \vec{S} + \varepsilon_0^S \vec{E} + \vec{P}
\]

Preisach Model:

\[
p(t) = \int \int_{-1 \leq \beta \leq \alpha \leq 1} w(\alpha, \beta) R_{\alpha, \beta}[e](t) d\alpha d\beta
\]

\[
p(t) = \frac{P(t)}{P_{\text{max}}} \quad e(t) = \frac{E(t)}{E_{\text{max}}}
\]
Rate independence,  
Volterra property,  
Congruency of hysteresis loops,  
Wipe out (memory deletion),  
⇒ Shape (Everett) function \( s \) exists such that

\[
p(t) = \int_{-1}^{1} \int_{-1}^{1} w(\alpha, \beta) R_{\alpha,\beta}[e](t) \, d\alpha \, d\beta = \frac{1}{2} s(-e_0, e_0) + \sum_{k=1}^{N} s(e_i, e_{i+1})
\]

where \( e_0, e_1, \ldots e_N \) are the preceding “dominant” extremal input values, \( e_{N+1} := e(t) \)

**Forward problem:**

*Input-output model:* In each time step, evaluate Preisach operator in all space points:

- apply deletion rules
- compute Everett sum

*Hysteresis in PDEs:* involves implicit iteration of hysteretic input-output model

**Inverse Problem:**

*Determine weight function \( w \) or shape function \( s \) in Preisach operator \( P \).*
Iterative Hysteresis Identification

Model problem:

\[
\begin{cases}
\text{PDE : } & \rho d_{tt} - (\mathcal{P}[d_x])_x = 0 \quad x \in [0, L], \ t \in [0, T] \\
\text{boundary conditions: } & d(0, t) = f_0(t) \quad d(L, t) = f_L(t) \\
\text{initial conditions: } & d(x, 0) = d_0(x) \quad d_t(x, 0) = d_1(x) \\
\text{measurements: } & (\mathcal{P}[d_x])(L, t) = g(t)
\end{cases}
\]

a) Identify by alternating iterations: Each step consists of
- solve measurements (**) for \( \mathcal{P} \).
- solve PDE (*) for \( d \).

b) Apply Newton’s method to (**) as an equation \( F(\mathcal{P}) = g \)
with \( F \) containing solution of (*).
Alternating iteration:

\( n=0 \): choose starting values for \( d \) and \( P \)
for \( n = 0, 1, 2, 3, \ldots \)
- fit measurements \((P[d^{(n)}])(L, t) = g(t) \rightarrow P^{n+1}\)
- solve PDE with hysteresis \( P^{n+1} \rightarrow d^{n+1} \)

Newton iteration for \( F(P) = g \):

\( n=0 \): choose starting values for \( P \)
for \( n = 0, 1, 2, 3, \ldots \)
- evaluate \( F \):
  - solve PDE with hysteresis \( P^n \)
  - compute Newton step \( \Delta P^n = F'(P^n)^{-1}(g - F(P^n)) \):
  - solve linearized PDE and fit measurements (iterative coupling):
    \( P^{n+1} = P^n + \Delta P^n \).
Test Example: Hysteresis with Saturation

Main hysteresis loop:

weight function $w$: 

shape function $s$: 

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Numerical Results: Hysteresis with Saturation, Weight fcn.

$w_0$:  

$w_1$:  

$w_2$:  

$w_3$:  

Direct and Inverse Problems in Piezoelectricity, Linz, 2005
Numerical Results: Hysteresis with Saturation, Everett fcn.

$s_0 - s$:

$s_1 - s$:

$s_2 - s$:

$s_3 - s$:
Test example: Narrow Hysteresis

Main hysteresis loop:

weight function $w$:  

shape function $s$:  

Direct and Inverse Problems in Piezoelectricity, Linz, 2005
Numerical Results: Narrow Hysteresis

\[ w_0: \]
\[ w_1: \]
\[ w_2: \]
\[ w_3: \]
Test example: Broad Hysteresis

Main hysteresis loop:

weight function \( w \):

shape function \( s \):
Numerical Results: Broad Hysteresis

$w_0$: 

$w_1$: 

$w_2$: 

$w_3$: 

Direct and Inverse Problems in Piezoelectricity, Linz, 2005
## Comparison of Alternating Iteration versus Newton

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<tr>
<td>CPU sec.</td>
<td>1.6e-4</td>
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<td>residual</td>
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**Direct and Inverse Problems in Piezoelectricity, Linz, 2005 33**
Conclusions and Outlook

• PDE based identification of material parameters

• Nonlinearity

• Hysteresis

→ 3D implementation of multiharmonic parameter id.

← Hysteresis modelling in piezoelectricity

→ Temperature dependence
Full PDE Model versus Input-Output Simplification (I)

Estimated $d_{est}^x(L, t) := \frac{d(0, t) - d(L, t)}{L} = \frac{f_L(t) - f_0(t)}{L}$

versus correct values $d_x(L, t)$ - 

Hysteresis curve $P[d_x](L, t)$:
Full PDE Model versus Input-Output Simplification (II)

Difference between exact and identified $w$ (left) and $s$ (right)

a) in PDE model:

b) with simplification $u_x(L, t) \approx u_{est}^x(L, t)$:

→ Depending on the ratio $\rho/c \left(\frac{L}{T}\right)^2$, it can be necessary to identify within the PDE.
→ Shape function identification is less ill-posed than weight function identification