Workshop: Direct and Inverse Problems in Piezoelectricity
Linz, October 06-07, 2005

Discussion of the Griffith fracture criterion in piezoelectricity

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Outline

1. Introduction
2. The Griffith fracture criterion
3. 3D-Field equations
4. 2D-Field equations
5. Griffith formula
6. J-integral
7. Stress intensity factors
8. Conclusion

Introduction

The fracture behaviour in piezoelectric materials under mechanical and electrical loading is not fully understood. In particular, it is not clear, whether the electric field impedes or enhances crack propagation. Different fracture criterions for linear models are discussed in the literature:

- Total energy release rate. (Pak 90)
- Mechanical strain energy release rate. (Park/Sun 95)
- Stress intensity factors. (Suo/Kuo/Barnett/Willis 92)
- Local energy release rate. (Gao et al. 97)

Introduction

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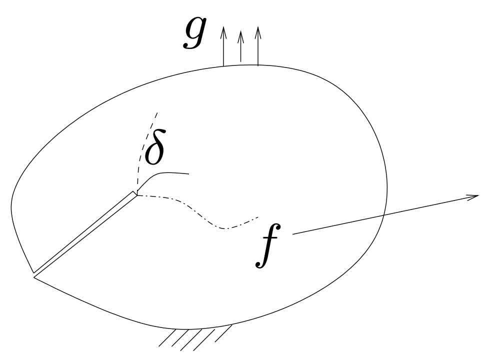
- Local energy release rate. (Gao et al. 97)

We will discuss the Griffith energy criterion which is based on the total energy release rate, in particular:

- What is the total energy?
- Which formulas for the energy release rate are available?
- How are the Maz'ya-Plamenevski coefficient formulas related to the stress intensity factors?

Griffith fracture criterion

The Griffith criterion is based on energy balance: The body is in an equilibrium state, if the total energy is minimal, that means, the crack grows if there is a neighboured configuration with smaller energy as the actual configuration.



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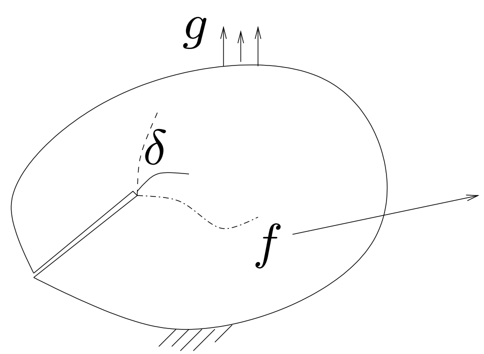

Total energy: $\Pi = \hat{U} + \hat{D} = \hat{E} - F + \hat{D}$

$\hat{U}$- **potential energy,** $\hat{D}$- **dissipative energy,**
$\hat{E}$- **elasto-electrical energy (electric enthalpy),**
$\hat{E} = \frac{1}{2} \int\limits_{\Omega} AB(U) \cdot B(U)\, dy - \int_{\Omega} MB(U) \cdot (-\nabla\varphi)$,
F- external energy, $F = \int\limits_{\Omega} fU\ dy + \int\limits_{\Gamma_N} gU\ da$,

$U = (u_1, u_2, u_3, \varphi)^\top$

Field equations:

$-B^\top ABU = f$ in Ω,
$U = 0$ on Γ,
$ABU = (\sigma, D)^\top = 0$ on $\Gamma_\pm$,
$g = 0.$
$MB(U) = D$

3D Field equations

Hooke's law in 3D:

$$(\sigma)_{ij} = \sum_{k,l=1}^{3} (C)_{ijkl} \gamma_{C_{kl}} - \sum_{k=1}^{3} e_{kij} (E)_k,$$
$$(D)_i = \sum_{j=1}^{3} (\varepsilon)_{ij} (E)_j + \sum_{j,k=1}^{3} e_{ijk} (\gamma)_{jk}$$

Voigt's notation:

$$\begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{D}} \end{pmatrix} = \underline{\underline{\boldsymbol{A}}} \begin{pmatrix} \underline{\gamma} \\ \underline{\boldsymbol{E}} \end{pmatrix} = \underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{B}}} \begin{pmatrix} \underline{\boldsymbol{u}} \\ \varphi \end{pmatrix},$$

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transversal isotrop, poling in y_3-direction

$$\underline{\underline{\boldsymbol{A}}} = \begin{pmatrix} \underline{\underline{\boldsymbol{C}}} & -\underline{\underline{\boldsymbol{e}}}^\top \\ \underline{\underline{\boldsymbol{e}}} & \underline{\underline{\boldsymbol{\varepsilon}}} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\ 0 & 0 & 0 & c_{44} & 0 & 0 & 0 & -e_{15} & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 & -e_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{c_{11}-c_{12}}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \varepsilon_{11} & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix},$$

$$\underline{\underline{\boldsymbol{Div}}} = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}, \qquad \underline{\underline{\boldsymbol{B}}} = \begin{pmatrix} \underline{\underline{\boldsymbol{Div}}}^\top & \underline{\boldsymbol{0}} \\ \underline{\underline{\boldsymbol{0}}} & -\nabla \end{pmatrix}.$$

3D field equations: $-\underline{\underline{\boldsymbol{B}}}^\top \underline{\underline{\boldsymbol{A}}}\, \underline{\underline{\boldsymbol{B}}}\, \underline{\boldsymbol{U}} = \underline{\boldsymbol{f}}\,.$

Reduction to 2D-Problems: $U = U(x_1, x_3) \Longrightarrow$

Decoupling of the 3D-problem into a plane strain-state problem for (u_1, u_3, φ) and anti-plane strain-state problem for u_2.

Hookes law for the in-plane state $(\gamma_{21} = \gamma_{22} = \gamma_{23} = 0, E_2 = 0, U = (u_1, u_3, \varphi)^\top)$:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \\ D_1 \\ D_3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{13} & 0 & 0 & -e_{31} \\ c_{13} & c_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & c_{44} & -e_{15} & 0 \\ 0 & 0 & e_{15} & \varepsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \varepsilon_{33} \end{pmatrix} \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_3 & 0 \\ \partial_3 & \partial_1 & 0 \\ 0 & 0 & -\partial_1 \\ 0 & 0 & -\partial_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ \varphi \end{pmatrix} = ABU.$$

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Hooke's law for the out-of-plane strain state $(\gamma_{11} = \gamma_{33} = \gamma_{13} = 0, E_1 = E_3 = 0)$:

$$\begin{pmatrix} \sigma_{23} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} c_{44} & 0 \\ 0 & \frac{c_{11} - c_{12}}{2} \end{pmatrix} \nabla u_2 = A_2 B_2 u_2.$$

2D Field equations: $-B^\top ABU = f, U = (u_1, u_3, \varphi), \; -B_2^T A_2 B_2 u_2 = f_2.$

2D Total energy

Elasto-electrical energy (electric enthalpy):

$$\boxed{\hat{E} = \frac{1}{2}\int_\Omega ABU \cdot BU dy - \int_\Omega MBU \cdot (-\nabla\varphi) dy + \frac{1}{2}\int_\Omega A_2 B_2 u_2 \cdot B_2 u_2 dy}$$

Exterior energy:

$$\boxed{F = \int_\Omega fU dy + \int_\Omega f_2 u_2 dy}$$

Total energy:

$$\boxed{\Pi = \hat{E} - F + \hat{D}}$$

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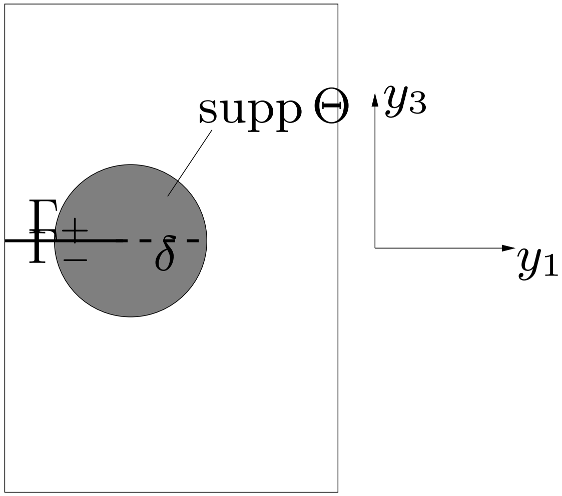
Straight crack propagation in y_1-direction,

Ω_δ= **domain with crack of the length** $l+\delta$,

Ω= **domain with crack of the length** l,

Π_δ=**total energy in** Ω_δ.



Energy release rate

Griffith criterion:

$$\boxed{\frac{d\Pi_\delta}{d\delta}|_{\delta=0} = 0}$$

Assume $\hat{D}_\delta = 2\delta\hat{\gamma}, \hat{\gamma} = \hat{\gamma}(\sigma_c, D_c)$, **is known, then we have to calculate the**

Energy Release Rate $\boxed{ERR = \frac{d(E_\delta - F_\delta)}{d\delta}|_{\delta=0} = \lim_{\delta\to 0} \frac{(E_\delta - F_\delta) - (E_0 - F_0)}{\delta}}$

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Energy Release Rate

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Derivation of formulas for the ERR

Assume: Ω_δ **(x-coordinates) and** Ω **(y-coordinates) are connected by a mapping:**

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \delta \begin{pmatrix} \Theta(x) \\ 0 \end{pmatrix},$$

where $\Theta \in C_0^\infty(\Omega_\delta)$, **supp**$\Theta \subset$ **a neighborhood of the crack tip,** $\Theta \equiv 1$ **near the crack tip. Note, that these mappings are diffeomorphisms for** $0 < \delta < \delta_0$, **since** $\frac{\partial(y_1, y_2)}{(\partial x_1, x_2)} = 1 - \delta\partial_1\Theta > 0.$

Matrix-vector notation for the energy

$$\hat{E} = \int_\Omega \frac{1}{2} C\gamma \cdot \gamma - e\gamma \cdot E - \frac{1}{2}\varepsilon E \cdot E dy, \qquad e = \begin{pmatrix} 0 & 0 & e_{15} \\ e_{31} & e_{33} & 0 \end{pmatrix}, \qquad \gamma = Div^\top u$$

$=$ **elastic e. + piezo e. - electric e.**

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Theorem (Griffith formula)

$$\mathbf{ERR} = \int_\Omega (-C\gamma \cdot \begin{pmatrix} \partial_1 \Theta & 0 \\ 0 & \partial_3 \Theta \\ \partial_3 \Theta & \partial_1 \Theta \end{pmatrix} \partial_1 u + \tfrac{1}{2} C\gamma \cdot \gamma \partial_1 \theta)\, dy$$ **elastic part**

$$+ \int_\Omega (\varepsilon E \cdot \nabla \Theta E_1 - \tfrac{1}{2} \varepsilon E \cdot E \partial_1 \Theta)\, dy$$ **electric part**

$$+ \int_\Omega e\gamma E_1 \cdot \nabla \Theta + e \begin{pmatrix} \partial_1 \Theta & 0 \\ 0 & \partial_3 \Theta \\ \partial_3 \Theta & \partial_1 \Theta \end{pmatrix} \partial_1 u \cdot E - e\gamma \cdot E \partial_1 \Theta\, dy$$ **coupled part**

$$- \int_\Omega U \partial_1 (f\Theta) dy$$ **exterior forces**

$$- \int_\Omega A_2 \nabla u_2 \partial_1 u_2 \cdot \nabla \Theta dy + \int_\Omega \tfrac{1}{2} A_2 \nabla u_2 \cdot \nabla u_2 \partial_1 \Theta dy - \int_\Omega u_2 \partial_1 (f_2 \Theta) dy$$ **antiplane part**

Proof:

- **Ideas from Khludnev/Sokolowski 99/00.**
- **Transformation of the energy integrals** $\int_{\Omega_\delta} \cdots dx$ **into** $\int_{\Omega} \cdots dy$.
- **convergence:** $\|U_\delta - U_0\|_{H^1(\Omega)} \to 0, \|f_\delta - f_0\|_{L_2(\Omega)} \to 0.$

Proof:

- Ideas from Khludnev/Sokolowski 99/00.
- Transformation of the energy integrals $\int_{\Omega_\delta} \cdots dx$ into $\int_\Omega \cdots dy$.
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Remarks:

Formula is independent of the choice of Θ.

Integration on supp Θ only!

Theorem (J-integral)

$$\textbf{ERR} = \int_\Gamma W n_1 ds - \int_\Gamma N\sigma \cdot \partial_1 u ds + \int_\Gamma D E_1 \cdot n ds - \int_\Gamma U \cdot f n_1 ds$$

$$+ \int_\Gamma \frac{1}{2} A_2 \nabla u_2 \cdot \nabla u_2 n_1 ds - \int_\Gamma A_2 \nabla u_2 \partial_1 u_2 \cdot n ds - \int_\Gamma u_2 f_2 n_1 ds,$$

provided $\underline{\boldsymbol{f}} \equiv \underline{\boldsymbol{const}}$ near the crack tip.

Theorem (J-integral)

ERR $= \int_\Gamma W n_1 ds - \int_\Gamma N\sigma \cdot \partial_1 u ds + \int_\Gamma DE_1 \cdot n ds - \int_\Gamma U \cdot f n_1 ds$

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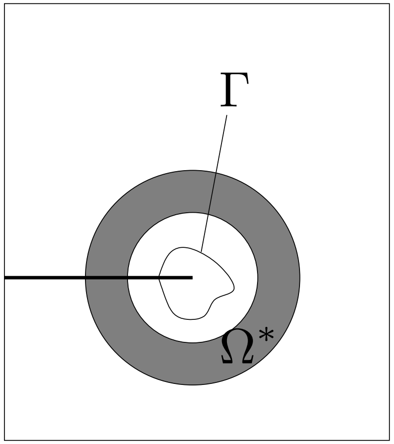
provided $\underline{\boldsymbol{f}} \equiv \underline{\boldsymbol{const}}$ near the crack tip.

$$\sigma = C\gamma - e^\top E$$

$$D = e\gamma + \varepsilon E$$

$$N = \begin{pmatrix} n_1 & 0 & n_3 \\ 0 & n_3 & n_1 \end{pmatrix}$$

$$W = \frac{1}{2} C\gamma \cdot \gamma - e\gamma \cdot E - \frac{1}{2}\varepsilon E \cdot E$$



Proof: Partial integration

The Griffith criterion yields

$$\boxed{\frac{d\Pi_\delta}{d\delta}\Big|_{\delta=0} = \frac{d(\hat{E} - F + \hat{D})_\delta}{d\delta}\Big|_{\delta=0} = 0}$$

Assume, $\hat{D}_\delta = 2\delta\hat{\gamma}$. **Therefore** $\frac{d\hat{D}_\delta}{d\delta}|_{\delta=0} = 2\hat{\gamma}(\sigma_c, D_c)$.

The Griffith fracture criterion reads:

The J-integral

The Griffith criterion yields

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Assume, $\hat{D}_\delta = 2\delta\hat{\gamma}$. **Therefore** $\frac{d\hat{D}_\delta}{d\delta}|_{\delta=0} \;= 2\hat{\gamma}(\sigma_c, D_c)$.

The Griffith fracture criterion reads:

The crack propagates, if

$$\boxed{-\ \textbf{ERR} \geq \textbf{critical value} = 2\hat{\gamma}(\sigma_c, D_c).}$$

From the general theory of elliptic boundary value problems follows:

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \varphi \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \\ u_3 \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \\ 0 \\ 0 \end{pmatrix} =$$

$$= c_1 r^{\frac{1}{2}} \begin{pmatrix} s_1^1(\omega) \\ 0 \\ s_3^1(\omega) \\ \varphi^1(\omega) \end{pmatrix} + c_2 r^{\frac{1}{2}} \begin{pmatrix} s_1^2(\omega) \\ 0 \\ s_3^2(\omega) \\ \varphi^2(\omega) \end{pmatrix} + c_4 r^{\frac{1}{2}} \begin{pmatrix} s_1^4(\omega) \\ 0 \\ s_3^4(\omega) \\ \varphi^4(\omega) \end{pmatrix} + c_3 r^{\frac{1}{2}} \begin{pmatrix} 0 \\ s_2^3(\omega) \\ 0 \\ 0 \end{pmatrix} + U_{reg},$$

where $U_{reg} = O(r^{\frac{3}{2}})$.

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where $U_{reg} = O(r^{\frac{3}{2}})$. **Inserting this decomposition of** U **into the J-integral we get:**

$$\boxed{\textbf{ERR} = \sum_{i,j=1,2,4} a_{ij} c_i c_j + a_3 c_3^2}$$

Stress intensity factors

The coefficients a_{ij}, c_i can be calculated, if the singular vectors $r^{\frac{1}{2}} S^i(\omega)$

$$r^{\frac{1}{2}} \begin{pmatrix} s_1^1(\omega) \\ 0 \\ s_3^1(\omega) \\ \varphi^1(\omega) \end{pmatrix}, \quad r^{\frac{1}{2}} \begin{pmatrix} s_1^2(\omega) \\ 0 \\ s_3^2(\omega) \\ \varphi^2(\omega) \end{pmatrix}, \quad r^{\frac{1}{2}} \begin{pmatrix} s_1^4(\omega) \\ 0 \\ s_3^4(\omega) \\ \varphi^4(\omega) \end{pmatrix}, \quad r^{\frac{1}{2}} \begin{pmatrix} 0 \\ s_2^3(\omega) \\ 0 \\ 0 \end{pmatrix}$$

and the so called dual singular vectors $r^{-\frac{1}{2}} S^{-i}(\omega), i = 1, 2, 3, 4,$

$$r^{-\frac{1}{2}} \begin{pmatrix} s_1^{-1}(\omega) \\ 0 \\ s_3^{-1}(\omega) \\ \varphi^-1(\omega) \end{pmatrix}, \quad r^{-\frac{1}{2}} \begin{pmatrix} s_1^{-2}(\omega) \\ 0 \\ s_3^{-2}(\omega) \\ \varphi^{-2}(\omega) \end{pmatrix}, \quad r^{-\frac{1}{2}} \begin{pmatrix} s_1^{-4}(\omega) \\ 0 \\ s_3^{-4}(\omega) \\ \varphi^{-4}(\omega) \end{pmatrix}, \quad r^{-\frac{1}{2}} \begin{pmatrix} 0 \\ s_2^{-3}(\omega) \\ 0 \\ 0 \end{pmatrix}$$

are known.

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are known.

Stroh's formalism is used by Lothe/Barnett(76), Deeg(80) to calculate the singular terms. Explicit closed form ??
In preparation: Computation of the singular terms by numerical solution of a quadratic eigenvalue problem (W.Geis)

Calculation of the coefficients c_1, c_2, c_3, c_4 by Maz'ya-Plamenevski coefficient formulas (76):

$$\boxed{c_i = b_i \int_\Omega f \cdot r^{-\frac{1}{2}} S^{-i}\, dy + \int_{\partial\Omega} N^\top ABU \cdot r^{-\frac{1}{2}} S^{-i} - U \cdot N^\top ABr^{-\frac{1}{2}} S^{-i}\, ds}$$

$b_i =$ **scaling constants. Shortly**

$$\boxed{c_i = b_i \int_\Omega f \cdot S_{-i}\, dy + \int_{\partial\Omega} g \cdot S_{-i} - U \cdot N^\top ABS_{-i}\, ds.}$$

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We have assumed, that the normal mechanical and electric stresses

$$g = N^\top ABU = N^\top \begin{pmatrix} \sigma \\ D \end{pmatrix} = 0.$$

Using FEM and BEM the coefficients c_1 can be computed by postprocessing at least for linear elasticity. Maybe, it can be done for piezoelectric ceramics in a similar way.

Note, the coefficients c_i are fixed numbers, depending on the volume and surface loads.

Stress intensity factors

We remind of the K-factors in linear isotropic elasticity (plane strain):

$$r^{\frac{1}{2}} S^1 = \frac{K_I \sqrt{r}}{2\mu\sqrt{2\pi}}(\kappa - \cos\omega)\begin{pmatrix} \cos(\frac{\omega}{2}) \\ \sin(\frac{\omega}{2}) \end{pmatrix}, \quad r^{\frac{1}{2}} S^2 = \frac{K_{II}\sqrt{r}}{2\mu\sqrt{2\pi}}\begin{pmatrix} \sin(\frac{\omega}{2})(\kappa + 2 + \cos\omega) \\ \cos(\frac{\omega}{2})(\kappa - 2 + \cos\omega \end{pmatrix}$$

$$\begin{pmatrix} \sigma^1_{11} \\ \sigma^1_{22} \\ \sigma^1_{12} \end{pmatrix} = \frac{K_I \cos(\frac{\omega}{2})}{\sqrt{2\pi r}}\begin{pmatrix} 1 - \sin(\frac{\omega}{2})\sin(\frac{3\omega}{2}) \\ 1 + \sin(\frac{\omega}{2})\sin(\frac{3\omega}{2}) \\ \sin(\frac{\omega}{2})\cos(\frac{3\omega}{2}) \end{pmatrix}, \quad \begin{pmatrix} \sigma^2_{11} \\ \sigma^2_{22} \\ \sigma^2_{12} \end{pmatrix} = \frac{K_{II}}{\sqrt{2\pi r}}\begin{pmatrix} -\sin(\frac{\omega}{2})(2 + \cos(\frac{\omega}{2})\cos(\frac{3\omega}{2})) \\ \sin(\frac{\omega}{2})\cos(\frac{\omega}{2})\cos(\frac{3\omega}{2}) \\ \cos(\frac{\omega}{2})(1 - \sin(\frac{\omega}{2})\sin(\frac{3\omega}{2})) \end{pmatrix}$$

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We remind of the K-factors in linear isotropic elasticity (plane strain):

$$r^{\frac{1}{2}} S^1 = \frac{K_I \sqrt{r}}{2\mu\sqrt{2\pi}} (\kappa - \cos\omega) \begin{pmatrix} \cos(\frac{\omega}{2}) \\ \sin(\frac{\omega}{2}) \end{pmatrix}, \quad r^{\frac{1}{2}} S^2 = \frac{K_{II} \sqrt{r}}{2\mu\sqrt{2\pi}} \begin{pmatrix} \sin(\frac{\omega}{2})(\kappa + 2 + \cos\omega) \\ \cos(\frac{\omega}{2})(\kappa - 2 + \cos\omega \end{pmatrix}$$

$$\begin{pmatrix} \sigma^1_{11} \\ \sigma^1_{22} \\ \sigma^1_{12} \end{pmatrix} = \frac{K_I \cos(\frac{\omega}{2})}{\sqrt{2\pi r}} \begin{pmatrix} 1 - \sin(\frac{\omega}{2}) \sin(\frac{3\omega}{2}) \\ 1 + \sin(\frac{\omega}{2}) \sin(\frac{3\omega}{2}) \\ \sin(\frac{\omega}{2}) \cos(\frac{3\omega}{2}) \end{pmatrix}, \quad \begin{pmatrix} \sigma^2_{11} \\ \sigma^2_{22} \\ \sigma^2_{12} \end{pmatrix} = \frac{K_{II}}{\sqrt{2\pi r}} \begin{pmatrix} -\sin(\frac{\omega}{2})(2 + \cos(\frac{\omega}{2}) \cos(\frac{3\omega}{2})) \\ \sin(\frac{\omega}{2}) \cos(\frac{\omega}{2}) \cos(\frac{3\omega}{2}) \\ \cos(\frac{\omega}{2})(1 - \sin(\frac{\omega}{2}) \sin(\frac{3\omega}{2})) \end{pmatrix}$$

$$\boxed{\begin{aligned} K_I &= \lim_{r\to 0} \sqrt{2\pi r}\, \sigma^1_{22}(\omega = 0) = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{22}(\omega = 0) \\ K_{II} &= \lim_{r\to 0} \sqrt{2\pi r}\, \sigma^2_{12}(\omega = 0) = \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{12}(\omega = 0) \end{aligned}}$$

Break down of this method in piezoelectricity (Pak 92, Park/Sun 95) or anisotropic elastic case:

$$\boxed{\lim_{r\to 0} \lim_{\omega\to 0} \sqrt{2\pi r}\, \sigma_{..}(r,\omega) \neq \lim_{\omega\to 0} \lim_{r\to 0} \sqrt{2\pi r}\, \sigma_{..}(r,\omega)}$$

These authors have investigated an infinite piezoelectric medium containing a center crack with far-field loading. They observed that for $\omega = 0$ the stress and electrial fields are not coupled.

Citation Park/Sun 95: This leads us to conclude that stress intensity factor is not suitable as fracture criterion for piecielectric materials.

They used the mechanical part of the energy release rate (crack closure integral) instead.

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Mathematical point of view:

To overcome these difficulties use the correct formulas for the computation of the stress intensity factors!

Conclusion

- Crack growth in piezoceramics can be enhanced or retarded by an electric field, depending on its magnitude and direction. In particular, if the crack is perpendicular to the poling direction, the resulting field equations for the mechanical and electric fields are strongly coupled.
- The Griffith criterion, based on the total energy balance, is discussed. The energy release rate is given by a Griffith formula from which the J-integral follows. The influence of volume forces (e.g. temperature) is regarded. Surface forces (tractions, charges) can be considered too.
- It is possible to express the total energy release rate by stress intensity factors in a correct way.
- Since the computation of the stress intensity factors demands the knowledge of the dual singular terms, it seems to be more efficient to compute the J-integral.
- Numerical experiments should be done in future.