The Parametric Solution of Underdetermined linear ODEs

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Outline

Problem

Given

We consider a single linear ODE

$$0 = \omega := \sum_{i=1}^{r} \sum_{j=0}^{n_i} a_{ij}(x)\, f_i^{(j)}(x) \;+\; a_{00}(x)$$

for functions $f_1 \ldots f_r$ of the independent variable x.

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Task
Find the general solution

$$f_i = F_i(x, h_1(x), \ldots, h_{r-1}(x)), \quad i = 1, \ldots, r$$

in terms of parametric functions $h_1(x), \ldots, h_r(x)$ which are either all free, or $h_1(x)$ having to satisfy an ODE and the other $h_i(x)$ being free.

Example 1

$$f' + \sin(x)g' = 0 \text{ for } f(x), g(x)$$

Find a parametric solution

$$f = D_1(x, h), \quad g = D_2(x, h)$$

D_1, D_2 are differential expressions
$h(x)$ is a free parametric function

Example 2

The ODE

$$0 = f''x^2 + g''x - g'x^2 + f + 3x$$

for $f = f(x),\ g = g(x)$ has the general solution

$$\begin{aligned} f &= \frac{x}{x^8 - 2x^6 + 7x^4 - 6x^2 + 9} \cdot \Big(\\ &\quad (x^5 - x^3 + 3x)h'' - (x^6 + x^4 + 3x^2 - 6)h' \\ &\quad + (3x^5 + 3x^3 + 17x)h - 3x^8 + 3x^6 - 16x^4 + 9x^2 \Big), \\ g &= \frac{x}{2(x^8 - 2x^6 + 7x^4 - 6x^2 + 9)} \cdot \Big(\\ &\quad (-2x^6 + 2x^4 - 6x^2)h'' + (8x^5 - 4x^3)h' \\ &\quad - (14x^4 + 14x^2 + 6)h + 4x^7 + x^5 + 3x^3 - 27x \Big) \end{aligned}$$

where $h = h(x)$ is an arbitrary function.

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(differential orders $n_i > 0$ $\forall f_i$) **do**

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• Factor out $\frac{d}{dx}$ from ω once as far as possible
by introducing a new function $f_{s+1}(x)$ and
by computing expressions b_i:

$$0 = \omega = f_{s+1}{}' + \sum_{i=1}^{s} f_i b_i + a_{00} \tag{1}$$

where f_{s+1}, b_i are given through:

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where f_{s+1}, b_i are given through:

$$f_{s+1} = \sum_{i=1}^{s} \sum_{j=0}^{n_i-1} f_i^{(n_i-1-j)} \sum_{k=0}^{j} (-1)^{(j+k)} a_{i,n_i-k}^{(j-k)} \tag{2}$$

$$b_i = \sum_{k=0}^{n_i} (-1)^{(n_i+k)} a_{i,n_i-k}^{(n_i-k)} \tag{3}$$

Algorithm continued

- **if** $b_i = 0$ for all i **then**
  the ODE $0 = \omega$ is exact (apart from a_{00}), i.e.
  consider new ODE $0 = \hat{\omega} := f_{s+1} + \int a_{00}\, dx$
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  - regard f_{s+1} as a new unknown function,
  - solve (1) for one function f_j of $f_1, .., f_s$
    that has a non-vanishing coefficient b_i in (1)
    and is of lowest possible order in ω, i.e. in (2),

$$f_j = -\frac{1}{b_j}\left(f_{s+1}^{'} + \sum_{i \neq j} f_i b_i + a_{00}\right) \tag{4}$$

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  - use (4) to substitute f_j in the definition of f_{s+1} (2) to get
    $0 = \hat{\omega} := -f_{s+1} + \sum_{i=1}^{s} \sum_{j=0}^{n_i - 1} \cdots$
    for functions $f_1, .., f_{j-1}, f_{j+1}, .., f_s, f_{s+1}$
  - update
    $\omega := \hat{\omega}, \quad s := s + 1, \quad I := \{f_j\} \cup I, \quad L := \{f_j = ..\} \cup L$

end % of while

Algorithm continued

if the ODE involves a function f_j purely algebraically **then**
solve for f_j and add substitution to L : $L := \{f_j = ..\} \cup L$
add f_j to I : $I := \{f_j\} \cup I$

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% next: substitutions each as stored in the first
% element of P are performed in the rest of P
$P := L$ % to remember the complete list of substitutions L
while $s > r$ **do**
$P := rest(P)|_{f_s=\ldots}$ % as given in $first(P)$
$s := s - 1$
end
end % of Body

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Output

- I % list of new functions
- L % the complete list of substitutions
- P % All initial functions $f_1, .., f_r$ are either
  % parametric, or are given in P in terms of one function $h(x)$
- $0 = \omega(x, h(x))$ % only if the first while loop terminates due
  % to ω not being underdetermined anymore (=GCD)

Termination

After substitution of f_j the new differential orders $\hat{n}_i$ of functions f_i that occur in the new ODE are:

$$\hat{n}_i \quad \begin{cases} = & n_j & \text{for } i = s+1 \\ \leq & \max(n_j - 1, n_i - 1) = n_i - 1 & \text{for } i \leq s,\ b_i \neq 0 \\ = & n_i - 1 & \text{for } i \leq s,\ b_i = 0. \end{cases}$$

$\rightarrow$ total sum $\sum_{i=1}^{s} n_i$ of differential orders n_i decreases and is bounded by 0

$\rightarrow$ algorithm must terminate

Two representations of solutions

- 1^{st} (fast) while loop: substitution list

- 2^{nd} (slow) while loop: explicit solution

Example 3

$$0 = (x-1)^3 f_1^{(5)} + 3f_1^{(3)} + xf_1'' + (1-x^2)f_1' + f_1 - (x-2)(x-3)f_2'' - xf_2'$$

Solution as substitution list:

```
f6=(7289/9*df(f8,x)*x^6 - 39400/3*df(f8,x)*x^5 + 725945/9*df(f8,x)*x^4 -
    691667/3*df(f8,x)*x^3 + 2665016/9*df(f8,x)*x^2 - 463541/3*df(f8,x)*x +
    40582*df(f8,x) - 82543/9*f8*x^5 + 393212/3*f8*x^4 - 1997777/3*f8*x^3 +
    1404610*f8*x^2 - 9064364/9*f8*x + 863648/3*f8)/(x^12 - 27*x^11 +
    339*x^10 - 2551*x^9 + 12566*x^8 - 43294*x^7 + 111667*x^6 -
    221121*x^5 + 332143*x^4 - 447477*x^3 + 625912*x^2 - 422746*x +
    117948)

f7=( - 197/419*df(f8,x) - 2227/6704*f6*x^6 + 40655/6704*f6*x^5 -
    282747/6704*f6*x^4 + 937545/6704*f6*x^3 - 737203/3352*f6*x^2 +
    236215/1676*f6*x - 15636/419*f6)/(x^5 - 5988/419*x^4 + 30423/419*x^3 -
    64170/419*x^2 + 46012/419*x - 13152/419)

f1=( - 288/197*df(f7,x) - 450/197*f6)/(x^5 - 2988/197*x^4 + 15683/197*x^3 -
    32571/197*x^2 + 18956/197*x - 6093/197)

f5= - 9/16*df(f6,x) + 37/32*f1*x^4 - 415/32*f1*x^3 + 347/8*f1*x^2 -
    815/32*f1*x + 261/32*f1

f4= - 4/9*df(f5,x) - 17/9*f1*x^3 + 124/9*f1*x^2 - 176/9*f1*x + 11*f1

f3= - 1/4*df(f4,x) + 5/4*f1*x^2 - 17/4*f1*x - 11/4*f1

f2=df(f3,x) + 2*f1*x + f1
```

Example 3 continued

Explicit solution:

```
f2=(
394*df(f8,x,5)*x^51 - 43734*df(f8,x,5)*x^50 + 2386458
*df(f8,x,5)*x^49 - 85217078*df(f8,x,5)*x^48 + 2236714754
*df(f8,x,5)*x^47 - 45961973470*df(f8,x,5)*x^46 + 769193693258
*df(f8,x,5)*x^45 - 10770363205862*df(f8,x,5)*x^44 + 128669233122374
*df(f8,x,5)*x^43 - 1331106348581066*df(f8,x,5)*x^42 + 12064628513749222
*df(f8,x,5)*x^41 - 96713773700247866*df(f8,x,5)*x^40 + 691121867571101914
*df(f8,x,5)*x^39 - 4432204569048039414*df(f8,x,5)*x^38 + 25656785149920297770
*df(f8,x,5)*x^37 - 134746911973057250854*df(f8,x,5)*x^36 + 644982858617800645668
*df(f8,x,5)*x^35 - 2825392156503814951228*df(f8,x,5)*x^34 + 11369643079009559994140
*df(f8,x,5)*x^33 - 42176203771707281241220*df(f8,x,5)*x^32 + 144696765103049531202292
*df(f8,x,5)*x^31 - 460531661206362465612348*df(f8,x,5)*x^30 + 1363760226604280734316668
*df(f8,x,5)*x^29 - 3767844144041916629995028*df(f8,x,5)*x^28 + 9737412336389614799932246
*df(f8,x,5)*x^27 - 23594628081627883976974538*df(f8,x,5)*x^26 + 53716508075775640025450550
*df(f8,x,5)*x^25 - 115105601305785527625158826*df(f8,x,5)*x^24 + 232485854899853866206117466
*df(f8,x,5)*x^23 - 443064024580252931546602966*df(f8,x,5)*x^22 + 797266596729901358938082010
*df(f8,x,5)*x^21 - 1355003701297989539400462678*df(f8,x,5)*x^20 + 217488318476796662342669199
*df(f8,x,5)*x^19 - 3295021547601096746504064490*df(f8,x,5)*x^18 + 470722905585093377244110739
*df(f8,x,5)*x^17 - 6331152782069854852101415314*df(f8,x,5)*x^16 + 799969715692345468619068591
*df(f8,x,5)*x^15 - 9468482031027385107120409754*df(f8,x,5)*x^14 + 104576196091631953424007914
*df(f8,x,5)*x^13 - 10723730904687055310839898442*df(f8,x,5)*x^12 + 10144225013060477363285699
*df(f8,x,5)*x^11 - 8778557400004544868724328464*df(f8,x,5)*x^10 + 687156234712016053218468385
*df(f8,x,5)*x^9 - 47908233899812207838548028000*df(f8,x,5)*x^8 + 2915664754419720732126751392
*df(f8,x,5)*x^7 - 151141079282886527173068796 8*df(f8,x,5)*x^6 + 648467283510334125798172768
*df(f8,x,5)*x^5 - 22254534182158876098235 1776*df(f8,x,5)*x^4 + 584491186925197701491 34336
*df(f8,x,5)*x^3 - 11004457864911651373771008*df(f8,x,5)*x^2 + 13219795850636551029 94944
*df(f8,x,5)*x - 76253282965162836232704*df(f8,x,5) - 21276*df(f8,x,4)*x^50 + 2316720
*df(f8,x,4)*x^49 - 123965796*df(f8,x,4)*x^48 + 4339041624*df(f8,x,4)*x^47 -
111587115820*df(f8,x,4)*x^46 + 2245675177648*df(f8,x,4)*x^45 - 36789872878356
*df(f8,x,4)*x^44 + 504030913993768*df(f8,x,4)*x^43 - 5888649790693364*df(f8
```

Example 3 continued

```
,x,4)*x^42 + 59543912919716688*df(f8,x,4)*x^41 - 527206208325653852*df(f8,x,
4)*x^40 + 41261358298487224 88*df(f8,x,4)*x^39 - 287695335561474725 40*df(f8,x
,4)*x^38 + 17990460661009511 8080*df(f8,x,4)*x^37 - 10147905730105422 90452*df(
f8,x,4)*x^36 + 51896400444857415 10344*df(f8,x,4)*x^35 - 24170597414020744055784
*df(f8,x,4)*x^34 + 10294331242824343173 8032*df(f8,x,4)*x^33 - 40242717173723043639 2760
*df(f8,x,4)*x^32 + 14489381818798398868 56560*df(f8,x,4)*x^31 - 48203903523184771737 40408
*df(f8,x,4)*x^30 + 14862754185686019658 738128*df(f8,x,4)*x^29 - 42593522901841837807765688
*df(f8,x,4)*x^28 + 11375869213498171901 7263552*df(f8,x,4)*x^27 - 28386331398988039062 0564612
*df(f8,x,4)*x^26 + 66329103345984784430 9867632*df(f8,x,4)*x^25 - 14542315227841950598 04085756
*df(f8,x,4)*x^24 + 29965324143664051867 11466296*df(f8,x,4)*x^23 - 58106342443686479976 5139964
*df(f8,x,4)*x^22 + 10613141032834314213 754376224*df(f8,x,4)*x^21 - 18268665829448882894 7848572
*df(f8,x,4)*x^20 + 29639017831732805049 125506536*df(f8,x,4)*x^19 - 45308486259122251454 2709708
*df(f8,x,4)*x^18 + 65209746751092146067 421531296*df(f8,x,4)*x^17 - 88245222222746270818 4737773
*df(f8,x,4)*x^16 + 11206957458604684550 2951939272*df(f8,x,4)*x^15 - 13322341875169301648 21030
*df(f8,x,4)*x^14 + 14773623312962435063 1804202336*df(f8,x,4)*x^13 - 15214442905137016634 62941
*df(f8,x,4)*x^12 + 14466058943805720276 7514568168*df(f8,x,4)*x^11 - 12604333160103414404 85640
*df(f8,x,4)*x^10 + 99664389742690190240 994957760*df(f8,x,4)*x^9 - 70579932422797184832 4645331
*df(f8,x,4)*x^8 + 43963185991077688579 600452352*df(f8,x,4)*x^7 - 23520048672969285012 50548396
*df(f8,x,4)*x^6 + 10492301743428547557 957967488*df(f8,x,4)*x^5 - 37639012816273170788 11542464
*df(f8,x,4)*x^4 + 10362525286754748770 55783808*df(f8,x,4)*x^3 - 20462355076599878776 8429312
*df(f8,x,4)*x^2 + 25746094636118779547 828736*df(f8,x,4)*x - 15507394339591436116 93056
*df(f8,x,4) + 499986*df(f8,x,3)*x^49 - 53375968*df(f8,x,3)*x^48 + 2798983092
```

Example 3 continued

```
*df(f8,x,3)*x^47 - 95970109058
*df(f8,x,3)*x^46 + 2416642042530*df(f8,x,3)*x^45 - 47599980366934
*df(f8,x,3)*x^44 + 762862553333542*df(f8,x,3)*x^43 - 10219249089585086
*df(f8,x,3)*x^42 + 116679295151709736*df(f8,x,3)*x^41 - 1152361883176794824
*df(f8,x,3)*x^40 + 9959681157046152038*df(f8,x,3)*x^39 - 76040127927429145908
*df(f8,x,3)*x^38 + 516852938902638323114*df(f8,x,3)*x^37 - 3148398532662330726432
*df(f8,x,3)*x^36 + 1728588994429895892l456*df(f8,x,3)*x^35 - 859704491838l6100213726
*df(f8,x,3)*x^34 + 389041389388213550552928*df(f8,x,3)*x^33 - 160831027773299904593419 8
*df(f8,x,3)*x^32 + 609608207911890153846647 4*df(f8,x,3)*x^31 - 212563745792951653954823 2
*df(f8,x,3)*x^30 + 683959232805116001919611 8*df(f8,x,3)*x^29 - 203672056967841282507291250
*df(f8,x,3)*x^28 + 562825004684843051457761192*df(f8,x,3)*x^27 - 144697757453987309169389754 2
*df(f8,x,3)*x^26 + 346913713286474579360077538 4*df(f8,x,3)*x^25 - 777287968794512982317634478 8
*df(f8,x,3)*x^24 + 163064005907951570182947165 70*df(f8,x,3)*x^23 - 320796419056073757549060448
*df(f8,x,3)*x^22 + 592547385602901727306311117 50*df(f8,x,3)*x^21 - 102849439474869956481054071
*df(f8,x,3)*x^20 + 167826112098904312610412176296*df(f8,x,3)*x^19 - 257463056927443518538916 91
*df(f8,x,3)*x^18 + 371195583963660472603411478326*df(f8,x,3)*x^17 - 502522222654498071020525 80
*df(f8,x,3)*x^16 + 637934164783133724977366276210*df(f8,x,3)*x^15 - 757893879628159078854407 63
*df(f8,x,3)*x^14 + 840418496403475313453832505236*df(f8,x,3)*x^13 - 866773030830727237082443 68
*df(f8,x,3)*x^12 + 827571128442886705187574905978*df(f8,x,3)*x^11 - 726940704287966793745700 14
*df(f8,x,3)*x^10 + 582639787849091340364049171160*df(f8,x,3)*x^9 - 421280770152076359336472 73
*df(f8,x,3)*x^8 + 270348463842711706187183477552*df(f8,x,3)*x^7 - 150449163131727627626904717 1
*df(f8,x,3)*x^6 + 70389756005201454582933672576*df(f8,x,3)*x^5 - 266187298858642235136885473 28
*df(f8,x,3)*x^4 + 7735319094444542153517178912*df(f8,x,3)*x^3 - 160824575746964472481227 9360
*df(f8,x,3)*x^2 + 211804175637174935958311808*df(f8,x,3)*x - 13229921318644288867495680
*df(f8,x,3) + 394*df(f8,x,2)*x^49 - 6398166*df(f8,x,2)*x^48 + 667081794
*df(f8,x,2)*x^47 - 34223827758
```

Example 3 continued

```
*df(f8,x,2)*x^46 + 1148173784434*df(f8,x,2)*x^45 - 28283376981410
*df(f8,x,2)*x^44 + 544784060518710*df(f8,x,2)*x^43 - 8534627931799072
*df(f8,x,2)*x^42 + 111705984218121602*df(f8,x,2)*x^41 - 1245505411554091892
*df(f8,x,2)*x^40 + 1200563859116806844 2*df(f8,x,2)*x^39 - 10120669408372548427 6
*df(f8,x,2)*x^38 + 753125399656271663334*df(f8,x,2)*x^37 - 4985554922423755804238
*df(f8,x,2)*x^36 + 295519084205909399330 54*df(f8,x,2)*x^35 - 1577345966139261151878 58
*df(f8,x,2)*x^34 + 761854622421259849416144*df(f8,x,2)*x^33 - 3344305751665684529330868
*df(f8,x,2)*x^32 + 133939287140370938797 16492*df(f8,x,2)*x^31 - 491117551752653066237 20942
*df(f8,x,2)*x^30 + 165386759236379520483720268*df(f8,x,2)*x^29 - 5129724746780885909539 21566
*df(f8,x,2)*x^28 + 146927470901916657397336 9112*df(f8,x,2)*x^27 - 3895635913623255710796119388
*df(f8,x,2)*x^26 + 958287153927304282311782 3218*df(f8,x,2)*x^25 - 2191598941053404270831667067
*df(f8,x,2)*x^24 + 466885703422221986682949 91130*df(f8,x,2)*x^23 - 9281154957886029915548730 41
*df(f8,x,2)*x^22 + 172425488689485373315168227558*df(f8,x,2)*x^21 - 2997551993281816683292004
*df(f8,x,2)*x^20 + 488127220163614626998572580102*df(f8,x,2)*x^19 - 7450827683523403938758249
*df(f8,x,2)*x^18 + 106647782696756587883265560 9402*df(f8,x,2)*x^17 - 1431561081348861491979460
*df(f8,x,2)*x^16 + 180162726289599815194506831 8894*df(f8,x,2)*x^15 - 2124313720586023567473108
*df(f8,x,2)*x^14 + 234380052490641705955486011 8038*df(f8,x,2)*x^13 - 2414697284531595041842890
*df(f8,x,2)*x^12 + 231536478925198342030189109 1982*df(f8,x,2)*x^11 - 2056030500979253163389190
*df(f8,x,2)*x^10 + 167856854127918713386902069 8552*df(f8,x,2)*x^9 - 1246916252763246137257153
*df(f8,x,2)*x^8 + 830078871112933708990229207504*df(f8,x,2)*x^7 - 484089500128740363097467028
*df(f8,x,2)*x^6 + 239398038419113602983395022368*df(f8,x,2)*x^5 - 961218972780024762597758062
*df(f8,x,2)*x^4 + 296187245166241226621411992 00*df(f8,x,2)*x^3 - 648018503365173207904188748 8
*df(f8,x,2)*x^2 + 885603459896403893575338624*df(f8,x,2)*x - 5620796247044796449330073 6
*df(f8,x,2) - 394*df(f8,x)*x^50 + 42552*df(f8,x)*x^49 - 2265106
```

Example 3 continued

```
*df(f8,x)*x^48 + 122589160*df(f8,x)*x^47 - 6490338754
*df(f8,x)*x^46 + 265073849986*df(f8,x)*x^45 - 8033574340460
*df(f8,x)*x^44 + 186641984569590*df(f8,x)*x^43 - 3449436332754766
*df(f8,x)*x^42 + 52250202900718722*df(f8,x)*x^41 - 663657119464018048
*df(f8,x)*x^40 + 719329119890002336 8*df(f8,x)*x^39 - 67451412138848868588
*df(f8,x)*x^38 + 553207956906984664564*df(f8,x)*x^37 - 4004061802889904880084
*df(f8,x)*x^36 + 257668114794700672336 16*df(f8,x)*x^35 - 14835730898709196604 7602
*df(f8,x)*x^34 + 768428261708315971473402*df(f8,x)*x^33 - 35974367855225995869 88180
*df(f8,x)*x^32 + 152851614767829454113385 42*df(f8,x)*x^31 - 59156448893527381384089454
*df(f8,x)*x^30 + 209198572105448083693341672*df(f8,x)*x^29 - 67784462103867212599 0260758
*df(f8,x)*x^28 + 201718836562745645623135 0690*df(f8,x)*x^27 - 55245661910320967039 11409006
*df(f8,x)*x^26 + 139495657941958835423586 11030*df(f8,x)*x^25 - 32525395767214738232656967520
*df(f8,x)*x^24 + 701322961967581364834615 69340*df(f8,x)*x^23 - 14004306585715568526046061876 0
*df(f8,x)*x^22 + 259347990155234155751534 905500*df(f8,x)*x^21 - 44612666111931921871268117608
*df(f8,x)*x^20 + 714060540512919625473996 369580*df(f8,x)*x^19 - 10654832949057633197338128143
*df(f8,x)*x^18 + 148529674836035867721797 1383462*df(f8,x)*x^17 - 19387741694206031715121136750
*df(f8,x)*x^16 + 237527772318652569734330 4325662*df(f8,x)*x^15 - 27374786725415640075821530083
*df(f8,x)*x^14 + 297303189424879966513632 3445910*df(f8,x)*x^13 - 30446299763178392737045780705
*df(f8,x)*x^12 + 293598322155250291589900 6134156*df(f8,x)*x^11 - 26543672754340855286385743222
*df(f8,x)*x^10 + 223182444325209445596728 4105616*df(f8,x)*x^9 - 17243322455598688763702521783
*df(f8,x)*x^8 + 120397027883300715262816 6892552*df(f8,x)*x^7 - 74207598443343070982759064339 2
*df(f8,x)*x^6 + 390262501062605051404043 199600*df(f8,x)*x^5 - 16697753399480086398396203184 0
*df(f8,x)*x^4 + 5452757739655176882648 1983264*df(f8,x)*x^3 - 12447240634350449377778741824
*df(f8,x)*x^2 + 172647426677843295456992 1216*df(f8,x)*x - 10657943012135237792 8898688
*df(f8,x) + 4728*f8*x^49 - 502350*f8*x^48 + 26259312*f8*x^47 - 1026278198
```

Example 3 continued

```
*f8*x^46 + 35405643760*f8*x^45 - 1074679186706
*f8*x^44 + 27353928873116*f8*x^43 - 571882742221768
*f8*x^42 + 9855508823736796*f8*x^41 - 141802591644977454
*f8*x^40 + 1727883049412361474*f8*x^39 - 18064958749710517994
*f8*x^38 + 163880518703618083250*f8*x^37 - 1302280602665519910966
*f8*x^36 + 9137929056664363023540*f8*x^35 - 57005324509606650361216
*f8*x^34 + 318012773649568620741158*f8*x^33 - 1594500456662765418767456
*f8*x^32 + 7216837195133201818433916*f8*x^31 - 29596008330678669174568952
*f8*x^30 + 110318614839380009238509886*f8*x^29 - 374719468453254315763380284
*f8*x^28 + 1162155408940384146323693110*f8*x^27 - 3295554784416353752001919016
*f8*x^26 + 8551821530742107617978457428*f8*x^25 - 20313370352853703932372268286
*f8*x^24 + 44159096176998561983836947770*f8*x^23 - 878075932918773407060977675l4
*f8*x^22 + 159583096981367558459205965446*f8*x^21 - 2649053259002943043329373374 30
*f8*x^20 + 401612206567596326179361649328*f8*x^19 - 556841225764242432593713111176
*f8*x^18 + 709069243354132892933098530606*f8*x^17 - 836648606902844203142958897118
*f8*x^16 + 928787720328002921686574382040*f8*x^15 - 990514962151399778688024764158
*f8*x^14 + 1035547558311234056996703694442*f8*x^13 - 10708245030286816770076797979 90
*f8*x^12 + 1085456887377026043488710595178*f8*x^11 - 1054803439258813697696958154232
*f8*x^10 + 957758349187925664443478135124*f8*x^9 - 7942033892276521993083750573 84
*f8*x^8 + 589468709358815060134683600640*f8*x^7 - 382991619204916329588856793376
*f8*x^6 + 211015672918699725497952683808*f8*x^5 - 940281970235685222860124216 64
*f8*x^4 + 317158180238599227163317233 60*f8*x^3 - 73871640799006297552426174 72
*f8*x^2 + 10271198918904568850810858 88*f8*x - 6186928266804090375231014 4*f8)
/(x^60 - 135*x^59 + 8985*x^58 -
```

Example 3 continued

```
392645*x^57 + 12660715*x^56 -
320973917*x^55 + 6657910800*x^54 -
116118424090*x^53 + 1736799086810*x^52 -
22614779155570*x^51 + 259376240997994*x^50 -
2645278173986010*x^49 + 24176985607056420*x^48 -
199328959260619820*x^47 + 1490780148002270290*x^46 -
10163678419544882366*x^45 + 63438134590671371375*x^44 -
363897562361385150425*x^43 + 1925046927221404643765*x^42 -
9421233841873594682565*x^41 + 42779883419354229913951*x^40 -
180720379128236614093665*x^39 + 712042118223636717188940*x^38 -
2622817616726229226683750*x^37 + 9052620635162924688483005*x^36 -
29339850904766533756060359*x^35 + 89476491604258600675968005*x^34 -
257260800607478987433208305*x^33 + 698634882951332255544876225*x^32 -
1795076180471379689139725535*x^31 + 4370719397039142370454861082*x^30 -
10098825488549468603329146420*x^29 + 22170155526714447361317150080*x^28 -
46290766381192291978057558740*x^27 + 92004131646934623496374698450*x^26 -
174170992056261525334263977246*x^25 + 314176353462667191664302897770*x^24 -
540101353842933037646529208110*x^23 + 884813869134902605856501910640*x^22 -
1380903278292250617765287652140*x^21 + 2051867452660334064884538540293*x^20 -
2900134533019638006280859465755*x^19 + 3894294736225229150504296109665*x^18 -
4959845401753907215119486119725*x^17 + 5978898629031521450582133122775*x^16 -
6803518200767511672781224470137*x^15 + 7283799405888927975155373169000*x^14 -
7305907248595632517409952648610*x^13 + 6829394458724952820514515002740*x^12 -
5909887009644947961718030853560*x^11 + 4694300049259647273973875977712*x^10 -
3384683119827637359097255348480*x^9 + 2182605770363284169661896610000*x^8 -
1234754466516880717147911038800*x^7 + 598446764403706874425488780800*x^6 -
241562689362887036925876767776*x^5 + 78478022233446042795237919680*x^4 -
19626920032290655716945953280*x^3 + 3538134265740671049345031680*x^2 -
409083380208004167081607680*x + 22827213754251335553235968),
```

Example 3 continued

```
f1=(394*df(f8,x,2)*x^14 - 12608*df(f8,x,2)*x^13 + 189120*df(f8,x,2)*x^12 -
1736752*df(f8,x,2)*x^11 + 10777870*df(f8,x,2)*x^10 - 47843420*df(f8,x,2)*x
^9 + 158992002*df(f8,x,2)*x^8 - 409452680*df(f8,x,2)*x^7 + 830453500*df(f8
,x,2)*x^6 - 1353357692*df(f8,x,2)*x^5 + 1913325070*df(f8,x,2)*x^4 - 2457444192
*df(f8,x,2)*x^3 + 2358937100*df(f8,x,2)*x^2 - 1231729104*df(f8,x,2)*x +
278829072*df(f8,x,2) - 8274*df(f8,x)*x^13 + 246250*df(f8,x)*x^12 - 3417162*
df(f8,x)*x^11 + 28811250*df(f8,x)*x^10 - 162537608*df(f8,x)*x^9 +
649015712*df(f8,x)*x^8 - 1921070322*df(f8,x)*x^7 + 4355956438*df(f8,x)*x^6
 - 7641204874*df(f8,x)*x^5 + 10618216866*df(f8,x)*x^4 - 12905684336*df(f8,x
)*x^3 + 13470452604*df(f8,x)*x^2 - 8374413264*df(f8,x)*x + 2161159344*df(f8
,x) + 47674*f8*x^12 - 1312414*f8*x^11 + 16741454*f8*x^10 - 128592144*f8*x
^9 + 653203538*f8*x^8 - 2317756614*f8*x^7 + 6009086666*f8*x^6 - 11710976260
*f8*x^5 + 17124493708*f8*x^4 - 18805586904*f8*x^3 + 16174027808*f8*x^2 -
 8863751808*f8*x + 2140252128*f8)/(x^24 - 54*x^23 + 1407*x^22 - 23408*x^21
 + 277807*x^20 - 2494730*x^19 + 17588559*x^18 - 99937324*x^17 + 467105390*x
^16 - 1826540556*x^15 + 6059553358*x^14 - 17258862900*x^13 + 42693822083*x^
12 - 92875821466*x^11 + 179786699099*x^10 - 312247104348*x^9 + 487568526275*x
^8 - 678683717714*x^7 + 829318079525*x^6 - 893148256820*x^5 + 848455260556*x
^4 - 634761623096*x^3 + 326364317668*x^2 - 99724090416*x + 13911730704) .
```

Outline

An appropriate representation of the ODE

Let $D = \frac{d}{dx}$ be maximally factored out $\rightarrow$ negligible effort to substitute one function by an algebraic combination of other functions

$$
\begin{aligned}
0 &= \sum_{i}^{r} A_i f_i + c_{00}(x) \\
A_i &= D\tilde{A}_i + c_{i0}(x) \\
\tilde{A}_i &= D^{n_i - 1} c_{in_i}(x) + \ldots + D c_{i2}(x) + c_{i1}(x)
\end{aligned}
$$

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One iteration step is performed by

- introducing a new function f_{r+1} through

$$f_{r+1} = \sum_{i} \tilde{A}_i f_i \tag{5}$$

giving the ODE the form

$$0 = D f_{r+1} + \sum_{i} c_{i0}(x) f_i + c_{00}, \tag{6}$$

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$$0 = D f_{r+1} + \sum_{i} c_{i0}(x) f_i + c_{00}, \tag{6}$$

- solving (6) for one f_j and substituting this in (5).

Vector notation of new algorithm

In vector notation the algorithm changes the differential operators according to (assuming for simplicity of notation f_1 is replaced):

$$\begin{pmatrix} A_1 \\ \\ A_2 \\ \vdots \\ A_r \end{pmatrix} \rightarrow \begin{pmatrix} A_{r+1} \\ \\ A_2 \\ \vdots \\ A_r \end{pmatrix} = \begin{pmatrix} -1 - \tilde{A}_1 \frac{1}{c_{10}} D \\ \tilde{A}_2 - \tilde{A}_1 \frac{c_{20}}{c_{10}} \\ \vdots \\ \tilde{A}_r - \tilde{A}_1 \frac{c_{r0}}{c_{10}} \end{pmatrix}$$

$$c_{00} \rightarrow -\tilde{A}_1 \left(\frac{c_{00}}{c_{10}} \right)$$

The Euclid version of the algorithm

Also for applying the 'right' Euclid algorithm the ODE is given in a form with D being completely factored out:

$$\begin{aligned} 0 &= \sum_{i}^{r} A_i f_i + c_{00}(x) \qquad (7) \\ A_i &= D^{n_i} c_{in_i}(x) + \ldots + D c_{i1}(x) + c_{i0}(x) \end{aligned}$$

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One iteration step is performed by

- choosing two functions f_i, f_j (w.l.o.g. $n_i \geq n_j$) and introducing a new function f_{r+1} through

$$f_j = f_{r+1} - D^{n_i - n_j} \left(\frac{c_{in_i}}{c_{jn_j}} f_i \right)$$

The Euclid version of the algorithm

Also for applying the 'right' Euclid algorithm the ODE is given in a form with D being completely factored out:

$$\begin{aligned} 0 &= \sum_{i}^{r} A_i f_i + c_{00}(x) \qquad (7) \\ A_i &= D^{n_i} c_{in_i}(x) + \ldots + D c_{i1}(x) + c_{i0}(x) \end{aligned}$$

One iteration step is performed by

- choosing two functions f_i, f_j (w.l.o.g. $n_i \geq n_j$) and introducing a new function f_{r+1} through

$$f_j = f_{r+1} - D^{n_i - n_j} \left(\frac{c_{in_i}}{c_{jn_j}} f_i \right)$$

- and performing this substitution in the ODE (7).

Vector notation of Euclid's algorithm

Assuming A_1 is used to lower the order of A_2 the vector notation gives

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_r \end{pmatrix} \rightarrow \begin{pmatrix} A_{r+1} \\ A_2 \\ A_3 \\ \vdots \\ A_r \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 - A_1 D^{n_2 - n_1} \frac{c_{2n_2}}{c_{1n_1}} \\ A_3 \\ \vdots \\ A_r \end{pmatrix}$$

$$c_{00} \rightarrow c_{00}.$$

Comparison between both algorithms

- structural difference:

  Euclid: $A_i, A_j \longrightarrow A_i, A_{r+1}$
  new: ODE $\longrightarrow$ substitution rule for old f_j
  definition of new $f_{r+1} \longrightarrow$ new ODE

Comparison between both algorithms

- structural difference:

  Euclid: $A_i, A_j \longrightarrow A_i, A_{r+1}$
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  definition of new $f_{r+1} \longrightarrow$ new ODE

- Euclid step lowers the order of *one* A_i.
  New step lowers the order of *all but one* A_j.
  $\longrightarrow$ either Euclid's method gives solution with higher derivatives of parametric functions, or most often a solution with many more terms, both are disadvantageous.

Example 4

$$0 = x^3 f^{(a)} + (x-1) g^{(b)} + h^{(5)}, \quad \text{for } f, g, h \text{ of } x$$

where a, b are the differential orders of f and g.

Solutions involve 2 parametric functions $c_1(x), c_2(x)$.

Example 4, observations

$a = b = 1$

- differential order of parametric functions:
  Euclid: 5 new: 4
- size of expressions: both algorithms give solutions of comparable size

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- differential order of parametric functions:
  Euclid: 5 new: 4
- size of expressions: both algorithms give solutions of comparable size

$a = 3, b = 3$

- differential order of parametric functions:
  Euclid: 7
  new: 3 in $g = \ldots$, 2 in $f = \ldots$, 1 in $h = \ldots$.
- size of expressions: see below

Example 4, solution 1

$a = 3, b = 3$, i.e.

$$0 = x^3 f^{(3)} + (x-1) g^{(3)} + h^{(5)}, \quad \text{for } f, g, h \text{ of } x$$

Example 4, solution 1

$a = 3, b = 3$, i.e.

$$0 = x^3 f^{(3)} + (x-1) g^{(3)} + h^{(5)}, \quad \text{for } f, g, h \text{ of } x$$

New algorithm:

```
h = - df(c1,x)*x + df(c1,x) - c2 + 5*c1

g = - df(c2,x,2)*x^2 - df(c2,x,2)*x - df(c2,x,2) + 6*df(c2,x)*x
    + 3*df(c2,x) + df(c1,x,3) - 12*c2

f = df(c2,x,2)
```

Example 4, solution 2

Euclid's algorithm:

```
h=61440*c1*x^18 - 1474560*c1*x^17 + 16865280*c1*x^16 - 122142720*c1*x
^15 + 627667200*c1*x^14 - 2430351360*c1*x^13 + 7348419840*c1*x^12 - 17740523520
*c1*x^11 + 34667761680*c1*x^10 - 55249008000*c1*x^9 + 71980769880*c1*
x^8 - 76489629120*c1*x^7 + 65809123935*c1*x^6 - 45225759240*c1*x^5 + 24282925200
*c1*x^4 - 9830833920*c1*x^3 + 2826420480*c1*x^2 - 515082240*c1*x +
44789760*c1

g=(4096*df(c1,x,7)*x^26 - 118784*df(c1,x,7)*x^25 + 1656832*df(c1,x,7)*x
^24 - 14772224*df(c1,x,7)*x^23 + 94392064*df(c1,x,7)*x^22 - 459420416*df(
c1,x,7)*x^21 + 1767313664*df(c1,x,7)*x^20 - 5503475968*df(c1,x,7)*x^19
+ 14095779056*df(c1,x,7)*x^18 - 30005571376*df(c1,x,7)*x^17 + 53423104520*
df(c1,x,7)*x^16 - 79792675240*df(c1,x,7)*x^15 + 99970601089*df(c1,x,7)*x
^14 - 104736819293*df(c1,x,7)*x^13 + 91162715698*df(c1,x,7)*x^12 - 65223853062
*df(c1,x,7)*x^11 + 37744293168*df(c1,x,7)*x^10 - 17243538336*df(c1,x,7)*
x^9 + 5991293952*df(c1,x,7)*x^8 - 1488886272*df(c1,x,7)*x^7 + 235892736*
df(c1,x,7)*x^6 - 17915904*df(c1,x,7)*x^5 + 532480*df(c1,x,6)*x^25 -
14700544*df(c1,x,6)*x^24 + 194715648*df(c1,x,6)*x^23 - 1643915264*df(c1,
x,6)*x^22 + 9914628608*df(c1,x,6)*x^21 - 45379907328*df(c1,x,6)*x^20 + 163478859776
*df(c1,x,6)*x^19 - 474457766912*df(c1,x,6)*x^18 + 1126313852256*df(c1,x,
6)*x^17 - 2207917905712*df(c1,x,6)*x^16 + 3592697387872*df(c1,x,6)*x^15 -
```

Example 4, solution 2 continued

```
 4859784114888*df(c1,x,6)*x^14 + 5453563869166*df(c1,x,6)*x^13 - 5047483623673
*df(c1,x,6)*x^12 + 3813258058536*df(c1,x,6)*x^11 - 2313228656541*df(c1,x
,6)*x^10 + 1098585064872*df(c1,x,6)*x^9 - 392339530224*df(c1,x,6)*x^8 + 98321133312
*df(c1,x,6)*x^7 - 15014087424*df(c1,x,6)*x^6 + 891316224*df(c1,x,6)*x^5
 + 44789760*df(c1,x,6)*x^4 + 28098560*df(c1,x,5)*x^24 - 736505856*df(c1,
x,5)*x^23 + 9236152320*df(c1,x,5)*x^22 - 73591179264*df(c1,x,5)*x^21 + 417337989120
*df(c1,x,5)*x^20 - 1788622414080*df(c1,x,5)*x^19 + 6004354068224*df(c1,x
,5)*x^18 - 16148370650880*df(c1,x,5)*x^17 + 35292872442816*df(c1,x,5)*x^
16 - 63206592574800*df(c1,x,5)*x^15 + 93097987010400*df(c1,x,5)*x^14 - 112713385277832
*df(c1,x,5)*x^13 + 111622036853920*df(c1,x,5)*x^12 - 89529496487943*df(
c1,x,5)*x^11 + 57207624669075*df(c1,x,5)*x^10 - 28363492072820*df(c1,x,5
)*x^9 + 10449881224128*df(c1,x,5)*x^8 - 2647088130240*df(c1,x,5)*x^7 + 387804540672
*df(c1,x,5)*x^6 - 15369108480*df(c1,x,5)*x^5 - 2358927360*df(c1,x,5)*x^
4 - 59719680*df(c1,x,5)*x^3 + 777830400*df(c1,x,4)*x^23 - 19298795520*df(
c1,x,4)*x^22 + 228364738560*df(c1,x,4)*x^21 - 1710680432640*df(c1,x,4)*x
^20 + 9082748236800*df(c1,x,4)*x^19 - 36268829107200*df(c1,x,4)*x^18 + 112804660999680
*df(c1,x,4)*x^17 - 279240187952640*df(c1,x,4)*x^16 + 557364164823360*df(
c1,x,4)*x^15 - 903120384120000*df(c1,x,4)*x^14 + 1189776305076000*df(c1,
x,4)*x^13 - 1269932501233440*df(c1,x,4)*x^12 + 1088259402554220*df(c1,x,4)
*x^11 - 736556470195980*df(c1,x,4)*x^10 + 383162783065650*df(c1,x,4)*x^9
- 146305198525650*df(c1,x,4)*x^8 + 37614731506560*df(c1,x,4)*x^7 - 5285900652480
*df(c1,x,4)*x^6 + 97546705920*df(c1,x,4)*x^5 + 44531596800*df(c1,x,4)*x
^4 + 3433881600*df(c1,x,4)*x^3 + 12157747200*df(c1,x,3)*x^22 - 284562309120
*df(c1,x,3)*x^21 + 3165186539520*df(c1,x,3)*x^20 - 22194983731200*df(c1,
```

Example 4, solution 2 continued

```
x,3)*x^19 + 109778830540800*df(c1,x,3)*x^18 - 4060690963200000*df(c1,x,3)*x
^17 + 11621937970636800*df(c1,x,3)*x^16 - 26265903385804800*df(c1,x,3)*x^15
 + 47412174587040000*df(c1,x,3)*x^14 - 68669193396336000*df(c1,x,3)*x^13 + 79682326616496000
*df(c1,x,3)*x^12 - 73495285143782400*df(c1,x,3)*x^11 + 53032606192682400*df(
c1,x,3)*x^10 - 29118372643221000*df(c1,x,3)*x^9 + 11589716915145000*df(c1,
x,3)*x^8 - 3043693230648000*df(c1,x,3)*x^7 + 813964065721600*df(c1,x,3)*x^6
 - 233111001609600*df(c1,x,3)*x^5 - 33670494720000*df(c1,x,3)*x^4 - 7537121280000*
df(c1,x,3)*x^3 + 1068879052800*df(c1,x,2)*x^21 - 23510325657600*df(c1,x,2)
*x^20 + 244738503475200*df(c1,x,2)*x^19 - 1598414890598400*df(c1,x,2)*x^18
+ 7322039003750400*df(c1,x,2)*x^17 - 24917030499225600*df(c1,x,2)*x^16 + 65088626586624000
*df(c1,x,2)*x^15 - 132976266091315200*df(c1,x,2)*x^14 + 214428692489702400*
df(c1,x,2)*x^13 - 273319676210342400*df(c1,x,2)*x^12 + 273726650495232000*df
(c1,x,2)*x^11 - 212191587445996800*df(c1,x,2)*x^10 + 123834588631648800*df(
c1,x,2)*x^9 - 51745429611795600*df(c1,x,2)*x^8 + 14012260052637600*df(c1,x
,2)*x^7 - 1889211460628400*df(c1,x,2)*x^6 - 92174434272000*df(c1,x,2)*x^5 + 3814669728000
*df(c1,x,2)*x^4 + 786466713600*df(c1,x,2)*x^3 + 487312588800*df(c1,x)*x
^20 - 100283115110400*df(c1,x)*x^19 + 97206368993280000*df(c1,x)*x^18 - 58784176963584000
*df(c1,x)*x^17 + 24767306558668800*df(c1,x)*x^16 - 7690304998471680000*df(c1
,x)*x^15 + 1815254325679104000*df(c1,x)*x^14 - 331119329718988800000*df(c1,x)*x
^13 + 46954951633290240000*df(c1,x)*x^12 - 5159944235426688000000*df(c1,x)*x^11 +
 433624186673740800000*df(c1,x)*x^10 - 27121336584773760000000*df(c1,x)*x^9 + 1200119132954592000000
*df(c1,x)*x^8 - 339400508861196000000*df(c1,x)*x^7 + 47353641490332000000*df(c1,x)
*x^6 - 6820303914720000000*df(c1,x)*x^5 + 653201176320000000*df(c1,x)*x^4 - 388526803200000000
*df(c1,x)*x^3 + 160*df(c2,x,3)*x^18 - 2848*df(c2,x,3)*x^17 + 23776*df(
c2,x,3)*x^16 - 123344*df(c2,x,3)*x^15 + 444258*df(c2,x,3)*x^14 -
1176034*df(c2,x,3)*x^13 + 2361347*df(c2,x,3)*x^12 - 3652430*df(c2,x,3)*x
^11 + 4361637*df(c2,x,3)*x^10 - 3974500*df(c2,x,3)*x^9 + 2683410*df(c2,
```

Example 4, solution 2 continued

```
x,3)*x^8 - 1267704*df(c2,x,3)*x^7 + 374112*df(c2,x,3)*x^6 - 51840*df(c2
,x,3)*x^5 + 5280*df(c2,x,2)*x^17 - 85920*df(c2,x,2)*x^16 + 650016*df(c2
,x,2)*x^15 - 3023520*df(c2,x,2)*x^14 + 9636186*df(c2,x,2)*x^13 - 22194618
*df(c2,x,2)*x^12 + 37922301*df(c2,x,2)*x^11 - 48411105*df(c2,x,2)*x^10
+ 45630078*df(c2,x,2)*x^9 - 30556566*df(c2,x,2)*x^8 + 13264164*df(c2,x,2
)*x^7 - 2836296*df(c2,x,2)*x^6 - 155520*df(c2,x,2)*x^5 + 155520*df(c2,x
,2)*x^4 + 52800*df(c2,x)*x^16 - 778560*df(c2,x)*x^15 + 5280768*df(c2,x)
*x^14 - 21734016*df(c2,x)*x^13 + 60271260*df(c2,x)*x^12 - 118155132*df(
c2,x)*x^11 + 166701780*df(c2,x)*x^10 - 168143040*df(c2,x)*x^9 +
116761536*df(c2,x)*x^8 - 50685264*df(c2,x)*x^7 + 10427868*df(c2,x)*x^6
+ 311040*df(c2,x)*x^4 - 311040*df(c2,x)*x^3 + 884382105600*c1*x^19 - 16940830556160
*c1*x^18 + 152003484057600*c1*x^17 - 845227264573440*c1*x^16 + 3248316804096000
*c1*x^15 - 9110526149836800*c1*x^14 + 19191055602216960*c1*x^13 - 30763796669337600
*c1*x^12 + 37575744238218240*c1*x^11 - 34604949519820800*c1*x^10 + 23424252115622400
*c1*x^9 - 11090058063920640*c1*x^8 + 3328568452620000*c1*x^7 - 509433032525760
*c1*x^6 + 34091611009200*c1*x^5 - 21221280853200*c1*x^4 + 7236121478400
*c1*x^3 + 158400*c2*x^15 - 2093760*c2*x^14 + 12563712*c2*x^13 -
44986368*c2*x^12 + 106178436*c2*x^11 - 171904260*c2*x^10 + 191722740*
c2*x^9 - 142641660*c2*x^8 + 64421892*c2*x^7 - 13419132*c2*x^6 -
311040*c2*x^3 + 311040*c2*x^2)/(8*x - 8)
```

Example 4, solution 2 continued

```
f=( - 4096*df(c1,x,7)*x^23 + 118784*df(c1,x,7)*x^22 - 1656832*df(c1,x,7)
*x^21 + 14772224*df(c1,x,7)*x^20 - 94392064*df(c1,x,7)*x^19 + 459420416*
df(c1,x,7)*x^18 - 1767313664*df(c1,x,7)*x^17 + 5503475968*df(c1,x,7)*x^
16 - 14095779056*df(c1,x,7)*x^15 + 30005571376*df(c1,x,7)*x^14 - 53423104520
*df(c1,x,7)*x^13 + 79792675240*df(c1,x,7)*x^12 - 99970601089*df(c1,x,7)*
x^11 + 104736819293*df(c1,x,7)*x^10 - 91162715698*df(c1,x,7)*x^9 + 65223853062
*df(c1,x,7)*x^8 - 37744293168*df(c1,x,7)*x^7 + 17243538336*df(c1,x,7)*x
^6 - 5991293952*df(c1,x,7)*x^5 + 1488886272*df(c1,x,7)*x^4 - 235892736*df
(c1,x,7)*x^3 + 17915904*df(c1,x,7)*x^2 - 557056*df(c1,x,6)*x^22 +
15425536*df(c1,x,6)*x^21 - 205000704*df(c1,x,6)*x^20 + 1737175040*df(c1,
x,6)*x^19 - 10520671232*df(c1,x,6)*x^18 + 48379915776*df(c1,x,6)*x^17 - 175217517056
*df(c1,x,6)*x^16 + 511645788416*df(c1,x,6)*x^15 - 1223231788800*df(c1,x,
6)*x^14 + 2417895408928*df(c1,x,6)*x^13 - 3973308654160*df(c1,x,6)*x^12 +
 5438736840720*df(c1,x,6)*x^11 - 6192568827028*df(c1,x,6)*x^10 + 5836634991370
*df(c1,x,6)*x^9 - 4513714358664*df(c1,x,6)*x^8 + 2824579916067*df(c1,x,6
)*x^7 - 1400714241912*df(c1,x,6)*x^6 + 533370221712*df(c1,x,6)*x^5 - 148430050560
*df(c1,x,6)*x^4 + 27760133376*df(c1,x,6)*x^3 - 2960603136*df(c1,x,6)*x^
2 + 116453376*df(c1,x,6)*x - 30826496*df(c1,x,5)*x^21 + 813293568*df(c1,x
,5)*x^20 - 10273136640*df(c1,x,5)*x^19 + 82518036480*df(c1,x,5)*x^18 - 472244295168
*df(c1,x,5)*x^17 + 2044985980416*df(c1,x,5)*x^16 - 6946814947328*df(c1,x
,5)*x^15 + 18941013131520*df(c1,x,5)*x^14 - 42065410701600*df(c1,x,5)*x^
13 + 76779376097952*df(c1,x,5)*x^12 - 115697422679520*df(c1,x,5)*x^11 + 144030615526320
*df(c1,x,5)*x^10 - 147677724419890*df(c1,x,5)*x^9 + 123832577985090*df(
```

Example 4, solution 2 continued

```
c1,x,5)*x^8 - 83916885803016*df(c1,x,5)*x^7 + 45125703307487*df(c1,x,5)*
x^6 - 18733460395896*df(c1,x,5)*x^5 + 5756061448080*df(c1,x,5)*x^4 - 1222835339520
*df(c1,x,5)*x^3 + 159125762304*df(c1,x,5)*x^2 - 9828366336*df(c1,x,5)*x
+ 140341248*df(c1,x,5) - 897269760*df(c1,x,4)*x^20 + 22499328000*df(c1,x,
4)*x^19 - 269402112000*df(c1,x,4)*x^18 + 2045091840000*df(c1,x,4)*x^17 - 11023172520960
*df(c1,x,4)*x^16 + 44782791413760*df(c1,x,4)*x^15 - 142086979276800*df(
c1,x,4)*x^14 + 360000750950400*df(c1,x,4)*x^13 - 738580711838400*df(c1,x
,4)*x^12 + 1236820182771840*df(c1,x,4)*x^11 - 1696112907705600*df(c1,x,4)*
x^10 + 1903008578140800*df(c1,x,4)*x^9 - 1737880244058900*df(c1,x,4)*x^8
+ 1278971829458400*df(c1,x,4)*x^7 - 746459504035920*df(c1,x,4)*x^6 + 337173366369120
*df(c1,x,4)*x^5 - 113564815281000*df(c1,x,4)*x^4 + 26893073174400*df(c1,
x,4)*x^3 - 4050783302400*df(c1,x,4)*x^2 + 319323617280*df(c1,x,4)*x - 8069621760
*df(c1,x,4) - 14789836800*df(c1,x,3)*x^19 + 351530188800*df(c1,x,3)*x^18
 - 3978097459200*df(c1,x,3)*x^17 + 28445330227200*df(c1,x,3)*x^16 - 143867220264960
*df(c1,x,3)*x^15 + 546030342604800*df(c1,x,3)*x^14 - 1610354270054400*df(
c1,x,3)*x^13 + 3770544695424000*df(c1,x,3)*x^12 - 7100474074329600*df(c1
,x,3)*x^11 + 10827194062064640*df(c1,x,3)*x^10 - 13391872446979200*df(c1,x
,3)*x^9 + 13395888736804800*df(c1,x,3)*x^8 - 10751038665018600*df(c1,x,3)*
x^7 + 6827114738236200*df(c1,x,3)*x^6 - 3356215067718720*df(c1,x,3)*x^5 +
```

Example 4, solution 2 continued

```
 1235193972936900*df(c1,x,3)*x^4 - 322883216942400*df(c1,x,3)*x^3 + 54898847452800
*df(c1,x,3)*x^2 - 5141530598400*df(c1,x,3)*x + 177122350080*df(c1,x,3) - 137570549760
*df(c1,x,2)*x^18 + 3090123325440*df(c1,x,2)*x^17 - 32939977605120*df(c1,
x,2)*x^16 + 221035466588160*df(c1,x,2)*x^15 - 1044576392970240*df(c1,x,2)*
x^14 + 3686066463559680*df(c1,x,2)*x^13 - 10049372382597120*df(c1,x,2)*x^
12 + 21606491817400320*df(c1,x,2)*x^11 - 37068737315281920*df(c1,x,2)*x^10
 + 51014853290135040*df(c1,x,2)*x^9 - 56303841920636160*df(c1,x,2)*x^8 + 49552920072426240
*df(c1,x,2)*x^7 - 34370823853394880*df(c1,x,2)*x^6 + 18426571003846800*df(
c1,x,2)*x^5 - 7406164862343360*df(c1,x,2)*x^4 + 2126972234926680*df(c1,x
,2)*x^3 - 402906966549120*df(c1,x,2)*x^2 + 43348977878400*df(c1,x,2)*x - 1848196776960
*df(c1,x,2) - 665993871360*df(c1,x)*x^17 + 140901521817760*df(c1,x)*x^16
- 14095175122944O*df(c1,x)*x^15 + 883849583001600*df(c1,x)*x^14 - 3884188999864320
*df(c1,x)*x^13 + 12673696420270080*df(c1,x)*x^12 - 31738523342561280*df(
c1,x)*x^11 + 62195784461015040*df(c1,x)*x^10 - 96358605890803200*df(c1,x
)*x^9 + 118418624307832320*df(c1,x)*x^8 - 115106586538152960*df(c1,x)*x^7
 + 87675481310780160*df(c1,x)*x^6 - 51443011987065360*df(c1,x)*x^5 + 22614142144885200
*df(c1,x)*x^4 - 7124372023625280*df(c1,x)*x^3 + 1493450545767480*df(c1,x
)*x^2 - 181342852764480*df(c1,x)*x + 9130379875200*df(c1,x) - 160*df(c2,x
,3)*x^15 + 2848*df(c2,x,3)*x^14 - 23776*df(c2,x,3)*x^13 + 123344*df(c2,
x,3)*x^12 - 444258*df(c2,x,3)*x^11 + 1176034*df(c2,x,3)*x^10 - 2361347*df
(c2,x,3)*x^9 + 3652430*df(c2,x,3)*x^8 - 4361637*df(c2,x,3)*x^7 +
3974500*df(c2,x,3)*x^6 - 2683410*df(c2,x,3)*x^5 + 1267704*df(c2,x,3)*x^
```

Example 4, solution 2 continued

```
4 - 374112*df(c2,x,3)*x^3 + 51840*df(c2,x,3)*x^2 - 6240*df(c2,x,2)*x^14
 + 103488*df(c2,x,2)*x^13 - 800736*df(c2,x,2)*x^12 + 3826848*df(c2,x,2)*
x^11 - 12608502*df(c2,x,2)*x^10 + 30276828*df(c2,x,2)*x^9 - 54592479*df(
c2,x,2)*x^8 + 74907630*df(c2,x,2)*x^7 - 78175245*df(c2,x,2)*x^6 +
61113132*df(c2,x,2)*x^5 - 34578558*df(c2,x,2)*x^4 + 13278816*df(c2,x,2)*
x^3 - 3055968*df(c2,x,2)*x^2 + 311040*df(c2,x,2)*x - 73920*df(c2,x)*x^
13 + 1135680*df(c2,x)*x^12 - 8090496*df(c2,x)*x^11 + 35346432*df(c2,x)*x
^10 - 105564900*df(c2,x)*x^9 + 227453724*df(c2,x)*x^8 - 363428712*df(c2
,x)*x^7 + 435098580*df(c2,x)*x^6 - 388536678*df(c2,x)*x^5 + 253426320*df(
c2,x)*x^4 - 115623864*df(c2,x)*x^3 + 34035552*df(c2,x)*x^2 - 5489856*df
(c2,x)*x + 311040*df(c2,x) - 1288671068160*c1*x^16 + 25582507130880*c1*
x^15 - 239144848588800*c1*x^14 + 1394590069555200*c1*x^13 - 5667865261793280
*c1*x^12 + 16991779060039680*c1*x^11 - 38797432144035840*c1*x^10 + 68689303899955200
*c1*x^9 - 95090700527078400*c1*x^8 + 103011973067796480*c1*x^7 - 86765147241615360
*c1*x^6 + 56002136812231680*c1*x^5 - 27011544281762400*c1*x^4 + 9342648436046400
*c1*x^3 - 2161254222126720*c1*x^2 + 293368428038880*c1*x - 17004885474240
*c1 - 262080*c2*x^12 + 3704064*c2*x^11 - 24097920*c2*x^10 + 95325120*
c2*x^9 - 255134340*c2*x^8 + 486442656*c2*x^7 - 676941048*c2*x^6 +
691687500*c2*x^5 - 513369810*c2*x^4 + 268382640*c2*x^3 - 92933496*c2*
x^2 + 18823104*c2*x - 1622592*c2)/8
```

Why such a difference in complexity?

Euclid: multiplication with $\frac{c_{in_i}}{c_{jn_j}}$

new: multiplication with $\frac{c_{i0}}{c_{j0}}$

each followed by partial integrations
$\longrightarrow$ both algorithms complement each other

Why such a difference in complexity?

Euclid: multiplication with $\frac{c_{in_i}}{c_{jn_j}}$

new: multiplication with $\frac{c_{i0}}{c_{j0}}$

each followed by partial integrations
$\longrightarrow$ both algorithms complement each other

Both methods work on same data structure
$\longrightarrow$ Euclid steps and new steps can both be applied in one computation
$\longrightarrow$ hybrid method: use as next step whichever looks cheapest

Outline

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Outline

References

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