14 February 2006, Josef Schicho, workshop on Gröbner bases, RICAM, Linz, Austria. Notes taken by Christiaan van de Woestijne.

1 Constructive resolution of singularities

For the main idea of constructive resolution algorithm by Villamayor, the speaker refers to the keynote talk by Herwig Hauser, a week before.

Experts on this: Herwig Hauser, Orlando Villamayor, Santiago Encinas.

Simultaneously with Villamayor, E. Bierstone and P. Milman invented a similar algorithm.

All of these rely on Hironaka's seminal paper on resolution of singularities in characteristic 0.

Implementation in Maple and Magma by G. Bodnár and the speaker; faster implementation jointly by A. Frühbis-Krüger and G. Pfister.

The speaker advertises a T-shirt produced by the company of Herwig Hauser, featuring nice pictures and the text "Überabzahlbar". Cost is €30.

2 Questions for today

Today: answer some questions that were left open in Hauser's talk.

1. Why do we bother about resolution of singularities?
2. What is blowing up?
   (a) What does the speaker's definition have to do with the definition in Hartshorne?
3. Prove the fact that the order does not increase.
4. How does one find a good hypersurface to do the induction on the dimension?
5. What is the coefficient ideal?
6. Prove that taking the coefficient ideal commutes with blowing up.

3 There we go.

Question 1. To compute birational invariants of a variety, you need a non-singular model.

An important application is the situation where a kind of numerical evaluation is done on a variety ("curve tracing", surface triangulation, etc). This usually relies on the smoothness of the variety; for example, Newton's method only works if the jacobian does not vanish.

However, resolution can be quite expensive, so in practice one might use only a partial version of resolution.

Another application is in Diophantine equations, viz., when the equation is reduced modulo a prime we want to lift it to characteristic 0 again using Hensel lifting. For this, the equation over the residue field must be nonsingular. Bruin's algorithm (implemented in Magma) for deciding local solvability automatically relies on blowing-up. Nils Bruin, "Some ternary Diophantine equations of signature $(n, n, 2)$", to appear in *Discovering Mathematics with Magma*, W. Bosma, J. Cannon (eds). (Springer).

But... the definition. Let X be a variety. A *resolution* of X is a map

$$\pi : \tilde{X} \to X$$

where $\tilde{X}$ is nonsingular, and π is a birational, regular, proper morphism of varieties.

Why proper? Definition: $f : X \to Y$ is *proper* iff f is not the restriction of $g : U \to Y$ to an open subset $X \subseteq U$. This ensures that we don't miss any interesting parts of X; for example, if we don't require properness, the inclusion of the nonsingular part of X in X would be a "resolution" as well!

Question 2. Given a ring R and an ideal $I \subseteq R$, how can we change R such that I becomes *principal*?

Say, $I = < g_1, \ldots, g_r >_R$; then replace R by

$$R_1 = R\left[\frac{g_2}{g_1}, \frac{g_3}{g_1}, \ldots, \frac{g_r}{g_1}\right].$$

This is clearly asymmetric; so, instead replace R by the *set* of rings R_i where g_i is a generator of IR_i, for $i = 1, \ldots, r$.

This seems to claim that the blowup of a variety cannot be represented by a single ring, but that we need several.

So what is Hartshorne's definition of blowup?

Let X be a variety, and $\mathcal{I}$ a sheaf of ideals on X. Define the sheaf of algebras

$$\mathcal{S} = \oplus_{i \geq 0} \mathcal{I}^i.$$

(We take $\mathcal{I}^0 = \mathcal{O}_X$.) Then the *blowing-up* of X along $\mathcal{I}$ is defined to be $\operatorname{Proj} \mathcal{S}$.

If $X = \operatorname{Spec} R$, then $\mathcal{I}$ is just the ideal sheaf belonging to some ideal $I \subseteq R$, and $\mathcal{S}$ comes from the graded algebra

$$S = \oplus_{i \geq 0} I^i.$$

If $I = < g_1, \ldots, g_r >$, then the g_i get degree 1 in S. The corresponding projective variety has one chart corresponding to each g_i.

The ring R_i defined above is the ring of all fractions of degree 0 in the localisation with respect to g_i.

We specialise to blowing up of $\mathbb{A}^n$ along a coordinate variety, i.e., its coordinate ring is generated by some of the coordinate functions.

Pictures: a double point blows up to a curve intersecting twice the exceptional divisor. A cusp blows up to a curve touching the exceptional divisor. After a second blowup, the total transform consists of three curves that meet in a point. After a third blowup, only simple transversal intersection points are left.

We need the *order* of an ideal at a point. For a single polynomial $f \in K[X]$, the order at a point P is the degree of the first monomial in the Taylor expansion that does not vanish. For an ideal I, we define $\mathrm{ord}_P(I) = \min\{\mathrm{ord}_P(f) \mid f \in I\}$.

Proposition. Assume that L is a coordinate variety, I is an ideal with $\mathrm{ord}_P(I) \geq m$ for all $P \in L$. Then the total transform of I with respect to L is a product $x^m \cdot I'$, where x is the equation of the exceptional divisor.

(The point was to make the generating ideal principal, at least locally. Now the equation of the exceptional divisor is the remaining generator. I' is called the *controlled transform of* I with respect to the control m. One may take m maximal, and then I' is called the *strict transform* of I.)

Proof. Let $f \in I$, and let $L = V(x_1, \ldots, x_r)$. If $f = \sum a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in I$, then we have $i_1 + \ldots + i_r \geq m$.

Now compute the total transform of f in the first chart, the other charts being similar. We have

$$\begin{aligned} f^{\mathrm{Tr}} &= \sum a_{i_1,\ldots,i_n} x_1^{i_1} (x_1 y_2)^{i_2} \ldots (x_1 y_r)^{i_r} x_{r+1}^{i_{r+1}} \cdots x_n^{i_n} \\ &= x_1^m \left(\sum a_{i_1,\ldots,i_n} x_1^{i_1+i_2+\ldots+i_n-m} \cdot \ (\text{other terms}) \right). \end{aligned}$$

This proves the Proposition.

Theorem. Let I be an ideal, and L a coordinate variety. Assume that $\mathrm{ord}_P(I) = m$ for all $P \in L$, and $\mathrm{ord}_P(I) \leq m$ for all $P \notin L$. Let I' be the controlled transform of I with respect to the control m. Then $\mathrm{ord}_{P'}(I') \leq m$ for all P' in the preimage of L.

(In other words, assume that L is contained in the *top locus* of the ideal I.)

Proof. We choose a coordinate system such that $L = (x_1, \ldots, x_r)$, and such that $P' = (x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n) = (0, \ldots, 0)$ in the first chart of the blowup.

Let $f \in I$ be such that in its Taylor expansion around the origin, all monomials with nonzero coefficients have degree at least m, and equality is achieved for at least one.

Then the total transform of f is

$$f^{\mathrm{TT}} = \sum a_{i_1,\ldots,i_n} x_1 i_1 + \ldots + i_r - m y_2^{i_2} \ldots y_r^{i_r} x_{r+1}^{i_{r+1}} \ldots x_n^{i_n},$$

and we have $i_2 + i_3 + \ldots + i_r \leq m$.