New Ideas In The Enumeration Of Coherent Subalgebras Of A Prescribed Coherent Algebra

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Outline

1. The Problem
2. The Traditional Algorithm
3. New Ideas

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Coherent Algebras

A *coherent algebra* W is a matrix algebra over $\mathbb{C}$ such that

- W has a basis $\mathcal{A} = \{A_1, \ldots, A_r\}$ of $(0,1)$-matrices (standard basis);
- W contains I, the identity matrix;
- $\sum_{i=1}^{r} A_i = J$, the all-one matrix;
- W is closed under transposition.

Sources of coherent algebras

- Centralizer rings of permutation groups
- Strongly regular graphs
- Distance regular graphs
- Partial Geometries
- Maximal arcs in projective planes
- ...

Coherent subalgebras

- A merging (or fusion) of W is a coherent subalgebra of W.
- Thus, its standard basis consists of sums of distinct basis elements of W.
- A merging is uniquely determined by a partition of the basis $\mathcal{A}$.

The problem

Problem

Given a coherent algebra W, determine all mergings

The solution helps, e.g., to find all distance regular graphs invariant under a given permutation group

A running example

Example

Consider the Schur ring W over the cyclic group C_8

We have elements $\underline{0}, \underline{1}, \ldots, \underline{7}$ in the group ring

Multiplying two elements is straighforward, e.g.,

$$\underline{2} \cdot \underline{3} = \underline{5}$$

More complicated:

$$\begin{aligned} \underline{2,4} \cdot \underline{4,6} &= \underline{6} + \underline{0} + \underline{0} + \underline{2} \\ &= 2 \cdot \underline{0} + \underline{2,6} \end{aligned}$$

Outline

COCO

- Computer package COCO
- Available for DOS and Unix
- Very efficient for certain tasks

History of COCO

1979-80 (I. Faradžev, M. Klin, V.A. Zaichenko)

1981-84 (Faradžev, A.A.Ivanov, Klin.)
Primitive representations of simple groups of order up to 10^6

1984-89 (A.V.Ivanov + above) Sporadic computations

1990-92 (Faradžev, Klin, D. Pasechnik) Implementation for PC

1992 (Brouwer) Unix port

1996-98 (Faradžev, F. Fiedler, Ch. Pech, Klin)
Slight improvements

Outline Of The Algorithm

Mergings (partitions) are constructed as follows:

- Find sets that can form classes in the mergings ("good sets").
- Find partitions of the basis which consist of good sets.
- Check each to see whether it gives a merging.

Finding good sets

- Subsets of the basis are considered, of size up to half the rank.
- Only consider symmetrical or antisymmetrical sets
- Check that the set isn't subdivided by squaring

Example

In W, take the set $\{1, 2\}$. We compute the square:

$$\underline{1, 2} \cdot \underline{1, 2} = \underline{2, 4} + 2 \cdot \underline{3}$$

2 occurs once, 1 doesn't occur $\longrightarrow$ not good! Other set:

$$\underline{2, 4, 6} \cdot \underline{2, 4, 6} = 3 \cdot \underline{0} + 2 \cdot \underline{2, 4, 6}$$

Here, each of $2, 4, 6$ occurs exactly twice $\longrightarrow$ good!

Constructing mergings from good sets

- Construct partitions from good sets
- Check which are closed under multiplication and transposition

Example

Let $S_1 = \underline{0}$, $S_2 = \underline{1, 3, 5, 7}$, $S_3 = \underline{2, 4, 6}$. Then, e.g.,

$$\begin{aligned} S_2 \cdot S_2 &= 4 \cdot S_1 + 4 \cdot S_3 \\ S_2 \cdot S_3 &= 3 \cdot S_2 \end{aligned}$$

In fact, this is a merging

Limitations of COCO

- Only constructs homogeneous mergings
- Does not exploit symmetries
- Restriction to 100 good subsets

Example

- We have $(\underline{3,5})^2 = 2 \cdot \underline{0} + \underline{2 + 6}$
- Moreover, $(\underline{1})^2 = \underline{2}$, so both sets are good sets.
- So, we would consider all partitions containing both sets...

Outline

Bannai-Muzychuk method

- Instead of tensor, consider character table
- Work with quadratic instead of cubic information
- Partition of rows induces partition of colors; sizes have to coincide
- Only applicable in the commutative case!
- How do we find the eigenvalues?

Finding eigenvalues

- In the commutative case the algebra is diagonalizable
- Irreducible representations are one-dimensional
- Recall that
$$A_i \cdot A_j = \sum p_{ij}^k A_k$$
- Get quadratic equation for eigenvalues: For all $i \leq j$,
$$x_i x_j - \sum p_{ij}^k x_k = 0$$
- Solve using Gröbner bases

Example

Take the two sets $\underline{1}$ and $\underline{3,5}$. We have

$$\begin{aligned} \underline{1}^2 &= \underline{2} \\ \underline{1} \cdot \underline{2} &= \underline{3} \end{aligned}$$

Thus, $\underline{3,5}$ is split $\longrightarrow$ no merging containing both sets!
In fact, $\underline{1}$ generates all of W

Stabilization

- Problem: Find the coarsest merging containing a given partition
- Goes back to Weisfeiler-Leman
- Idea: Multiply any two classes; refine as necessary
- If no further refinement occurs, we are done
- Complexity?

Example

Taking the set $\underline{3,5}$, we get the partition

$$\underline{0};\underline{3,5};\underline{1,2,4,6,7}$$

Stabilization gives

$$\underline{0};\underline{3,5};\underline{1,7};\underline{2,6};\underline{4}$$

Using groups

- Interested in mergings up to isomorphism
- Sufficient to determine up to $CAut(W)$
- In fact, mergings depend only on tensor, hence use $AAut(W)$
- Compute groups using partition backtracking (Leon)

Example

In W, $(1,7)(2,6)(3,5)$ is an algebraic automorphism

Set orbit enumeration

- For good sets, want to check only one representative from each orbit
- Reduced to a canonicity check
- Can check (rather) efficiently if a set is smallest in its orbit
- Enumerate orbits depth-first

Independent sets

- Idea: Remove redundancy in the classical partition enumeration
- A generating set is independent if no element can be omitted
- Lemma: Any generating set contains an independent set
- Enumerate independent sets instead of partitions

An algorithm

- Find “real good” sets up to isomorphism under AAut(W)
- Incrementatlly construct partial partitions
- Stabilize, check independence
- Check that there is no “smaller” generating set
- In each step, get a new merging

Example (E_{32})

- Consider regular action of elementary abelian group E_{32}
- Take its centralizer ring (“Schur ring”)
- $AAut = CAut = SL(5, 2)$, size 10^7
- 614,482 good sets in 31 orbits
- Found 402 mergings (up to alg. isomorphisms) in three hours

Summary

- We are considering the enumeration of mergings
- One important ingredient: Stabilization
- Other ingredient: Symmetry

- Implementation in COCO-II,
  a GAP package due in Spring ’07