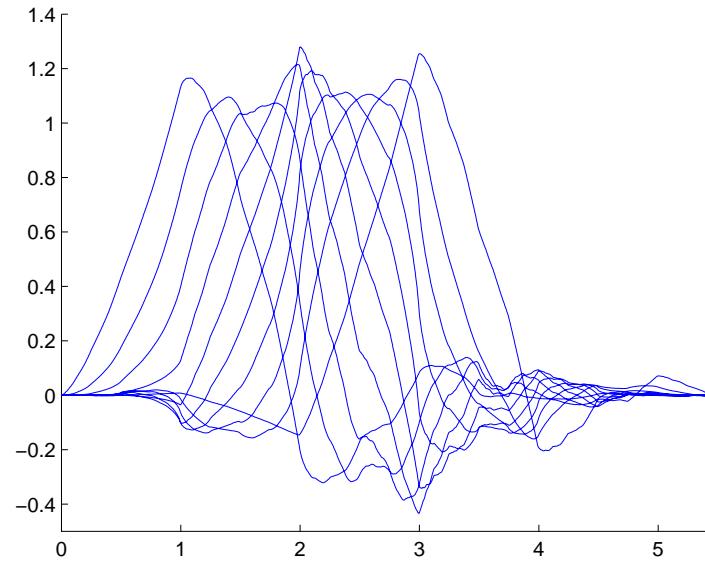
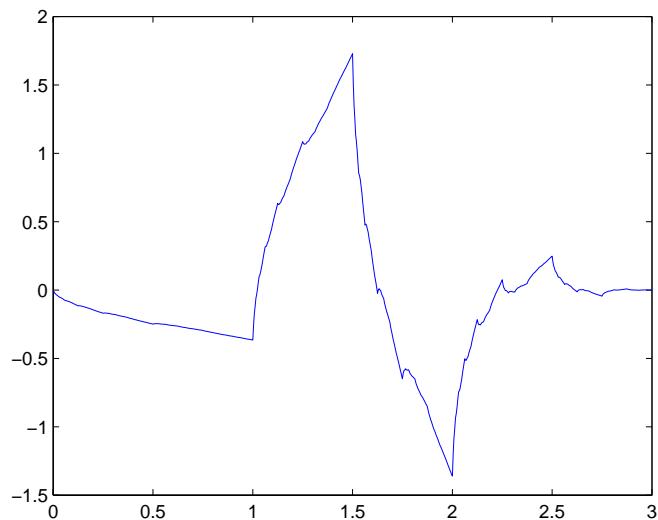


Parametrizing orthonormal wavelets by moments



$$h_0 = 1/4(1 + \sqrt{3}) = 0.6830127$$

$$h_0 = -3/32 - 1/8 a + 1/32 a^2 + 1/32 w$$
$$w = \sqrt{-a^4 + 14 a^2 + 15}$$

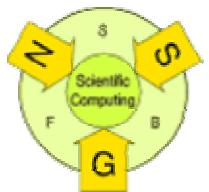
Georg Regensburger

Johann Radon Institute for Computational and Applied Mathematics (RICAM)

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georg.regensburger@oeaw.ac.at

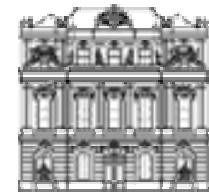
SFB F1322



FWF

Der Wissenschaftsfonds.

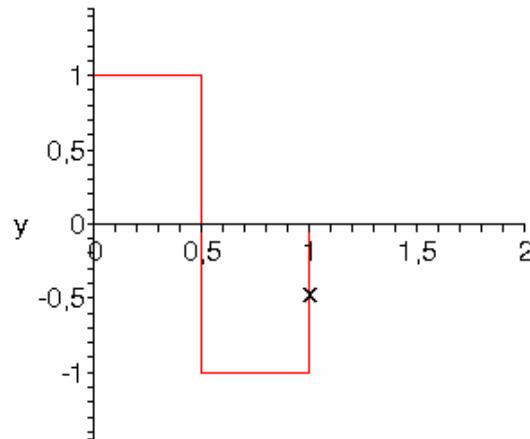
*Special Semester on Gröbner Bases 2006
Gröbner Bases in Control Theory and Signal Processing
Linz, Austria, 19th May, 2006*



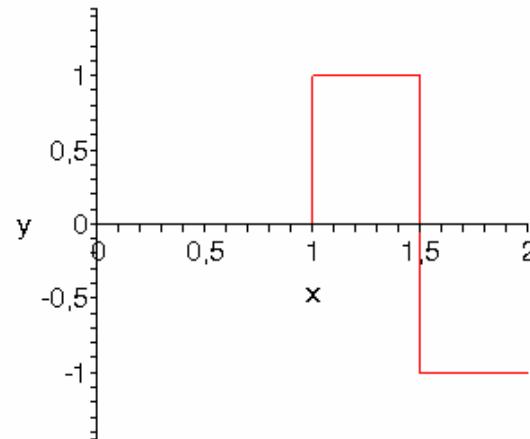
RICAM

Wavelets – translation and dilation

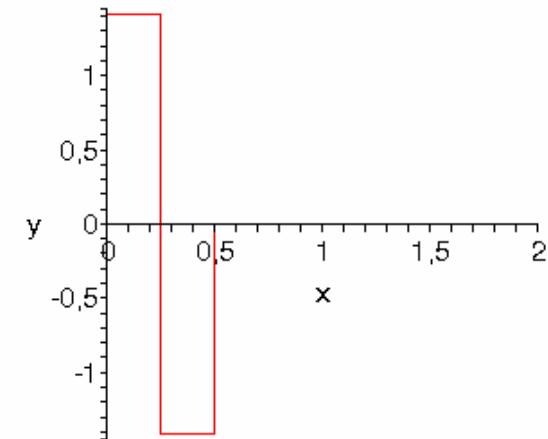
Haar [Haa10] Daubechies [Dau88]



$$\psi(x)$$



$$\psi(x - 1)$$



$$\sqrt{2}\psi(2x)$$

$\psi \in L^2(\mathbb{R})$ is an (*orthogonal*) *wavelet* if

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad \text{for } j, k \in \mathbb{Z}$$

is an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$

Scaling functions – dilation equation, normalization

Construction of wavelets based on *scaling functions*

$$\phi(x) = \sum_{k=0}^N h_k \phi(2x - k) \quad \text{dilation equation}$$

(real) filter coefficients / \ \\\\
scaled and translated

Nonzero Integral

$$\int \phi \neq 0$$

$$\int \phi = \sum_{k=0}^N h_k \int \phi(2x - k) dx = \frac{1}{2} \sum_{k=0}^N h_k \int \phi(x) dx$$

$$\int \phi \neq 0 \Rightarrow \sum_{k=0}^N h_k = 2$$

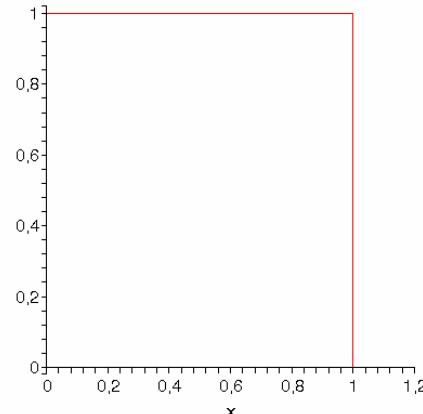
dilation equation

Conditions on $\phi(x)$

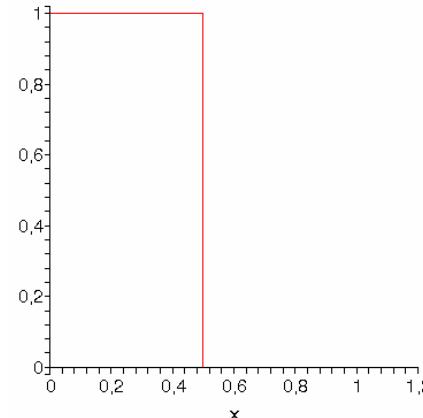
$$\Rightarrow$$

Conditions on h_k

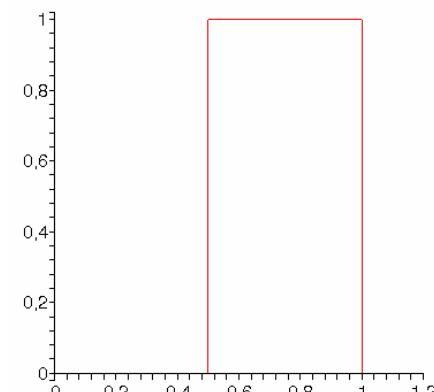
Scaling functions – examples, Haar scaling and hat function



=

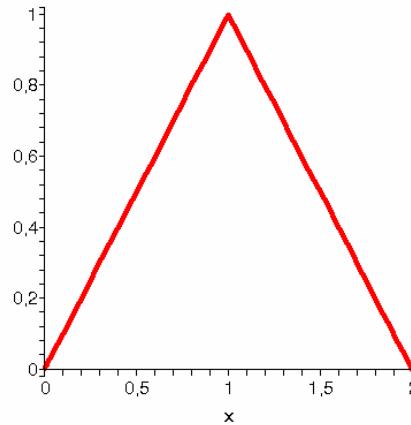


+

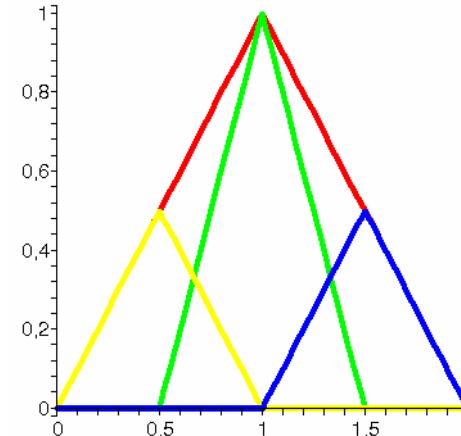


$$\phi(x) = \phi(2x) + \phi(2x - 1)$$

$$h_0 = h_1 = 1$$



=



$$\phi(x) = 1/2\phi(2x) + \phi(2x - 1) + 1/2\phi(2x - 2)$$

$$h_0 = h_2 = 1/2, h_1 = 1$$

Scaling functions – construction, cascade algorithm

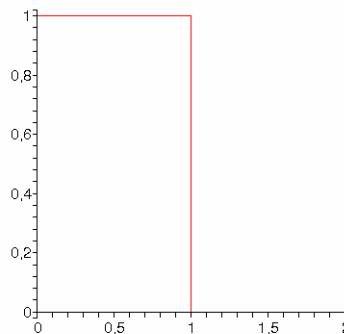
Given filter coefficients h_0, \dots, h_N

Start with $\phi^0(x) = \text{box function on } [0, 1]$

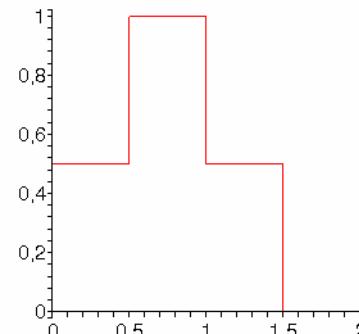
Iterate

$$\phi^{i+1}(x) = \sum_{k=0}^N h_k \phi^i(2x - k)$$

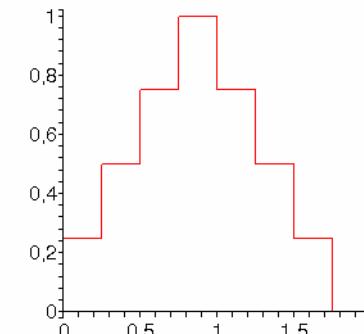
Example $h_0 = h_2 = 1/2, h_1 = 1$



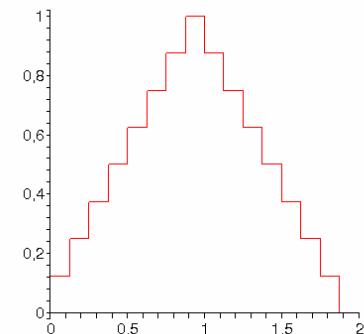
$\phi^0(x)$



$\phi^1(x)$



$\phi^2(x)$



$\phi^3(x)$

Scaling functions – support

Dilation Equation

$$\phi(x) = \sum_{k=0}^N h_k \phi(2x - k)$$

Suppose ϕ has compact support $[a, b]$

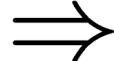
Then RHS has support $[a/2, (b + N)/2]$

$$\Rightarrow [a, b] = [0, N]$$

Scaling functions – orthonormality

$$\{\phi(x - l)\}_{l \in \mathbb{Z}} \text{ orthonormal, } \int \phi(x) \phi(x - l) dx = \delta_{0,l}$$

dilation equation



$$\sum_{k=0}^N h_k h_{k-2l} = 2\delta_{0,l} \quad N \text{ odd}$$

N=3:

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 2$$

„double-shift orthogonality“

$$h_0 h_2 + h_1 h_3 = 0$$

quadratic equations

necessary conditions for orthonormality

Scaling functions – existence

Orthonormality $\sum_{k=0}^N h_k h_{k-2l} = 2\delta_{0,l}$ (1) $l = 0, \dots, (N-1)/2$

Normalization $\sum_{k=0}^N h_k = 2$ (2)

Then there exists a unique solution of the dilation equation

$$\phi \in L^2(\mathbb{R}) \quad \text{supp } \phi = [0, N] \quad \int \phi = 1 \quad \text{Lawton [Law90]}$$

For almost all h_k with (1) and (2) $\{\phi(x - l)\}$ is orthonormal

Counterexample, $N=3$,

$$h_0 = h_3 = 1, \quad h_1 = h_2 = 0$$

$$\phi(x) = \begin{cases} 1/3, & \text{for } 0 \leq x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Necessary and sufficient conditions for orthormality
(convergence of the cascade algorithm)

Cohen [Coh90], [Law91]

Scaling functions and wavelets – wavelet equation

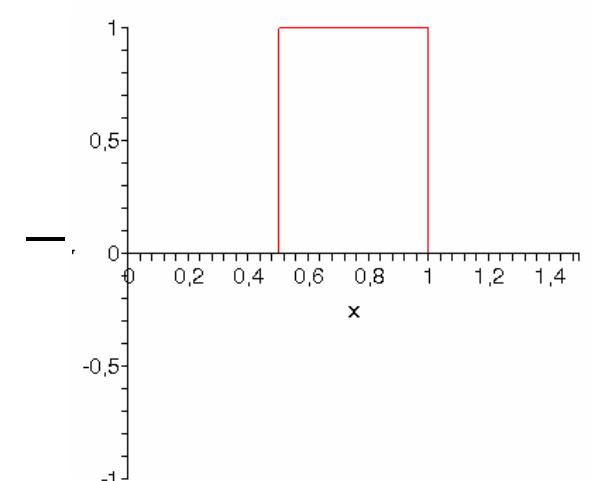
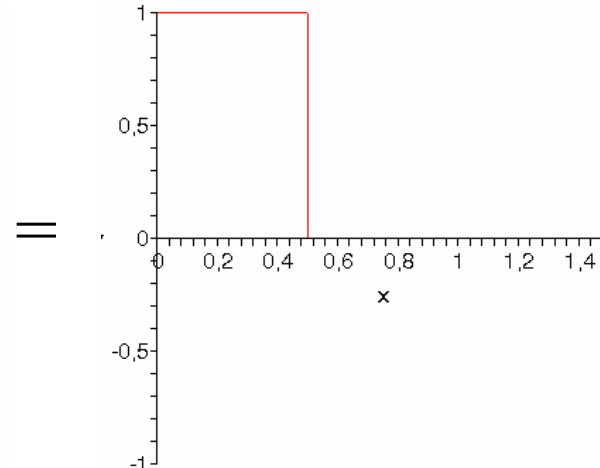
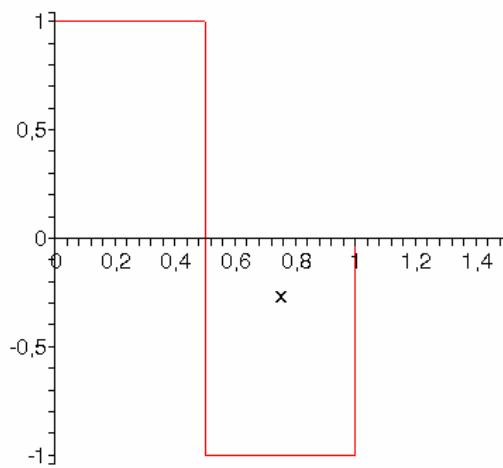
$\phi(x)$ scaling function, $\{\phi(x - k)\}_{k \in \mathbb{Z}}$ orthonormal

Then

$$\psi(x) = \sum_{k=0}^N (-1)^k h_{N-k} \phi(2x - k)$$

wavelet equation

is a wavelet



$$\psi(x) = \phi(2x) - \phi(2x - 1)$$

Scaling functions and wavelets – multiresolution analysis

$\phi(x)$ scaling function, $\{\phi(x - k)\}_{k \in \mathbb{Z}}$ orthonormal

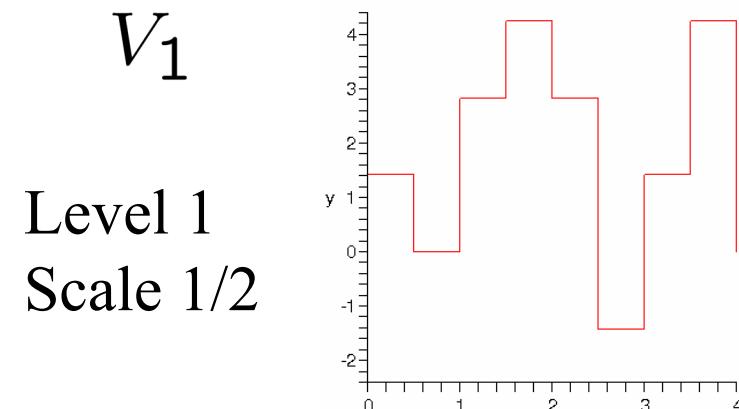
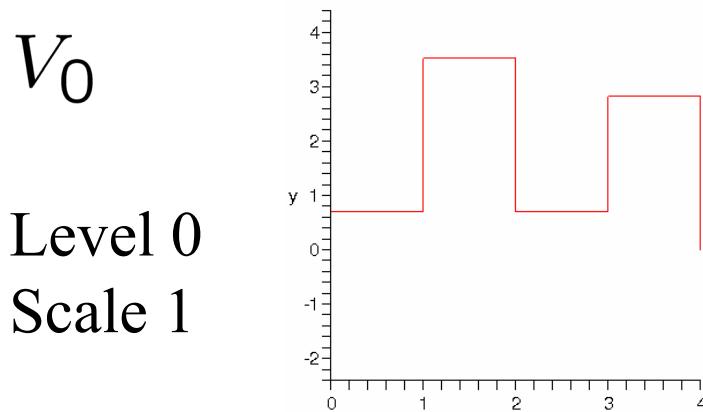
$$\phi_{jk}(x) = 2^{j/2} \phi(2^j x - k), \quad \text{for } j, k \in \mathbb{Z}$$

$$V_j = \overline{\text{span}}\{\phi_{jk}(x)\} \subset L^2(\mathbb{R}) \quad \{\phi_{jk}\}_{k \in \mathbb{Z}} \text{ ONB } V_j$$

$$V_0 \subset V_1 = \overline{\text{span}}\{\phi(2x - k)\} \quad (\text{dilation eq.})$$

$$V_j \subset V_{j+1} \quad \bigcap V_j = \{0\} \text{ and } \overline{\bigcup V_j} = L_2(\mathbb{R})$$

V_j with $\phi(x)$ are a *multiresolution analysis* Meyer 86, Mallat [Mal88]



Scaling functions and wavelets –wavelet subspace

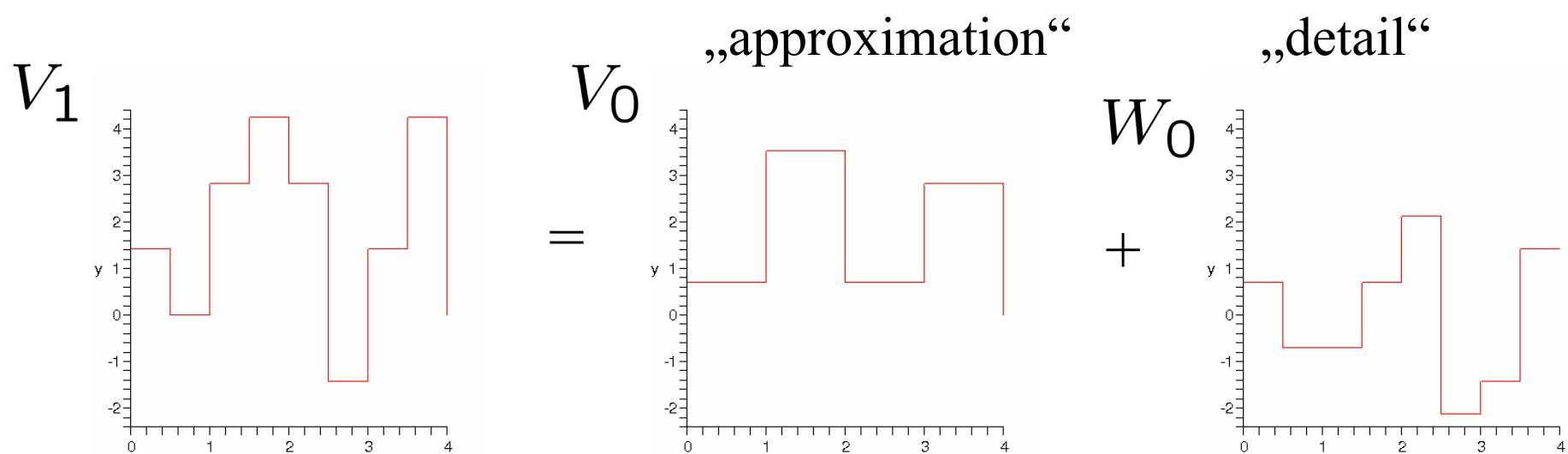
$V_j \subset V_{j+1}$ orthogonal complement W_j wavelet subspace

$V_{j+1} = V_j \oplus W_j$ and $W_j \perp V_j$

W_j difference between V_{j+1} and V_j

Wavelet equation:

$$W_j = \overline{\text{span}}\{\psi_{jk}(x)\} \subset L^2(\mathbb{R}) \quad \{\psi_{jk}\}_{k \in \mathbb{Z}} \text{ ONB } W_j$$



Fast wavelet transform (FWT) I

$$\begin{array}{ccccc} & V_j = V_{j-1} \oplus W_{j-1} & & & \\ & \nearrow & \uparrow & \uparrow & \\ \text{ONB} & \{\phi_{jk}\} & \{\phi_{j-1,k}\} & \{\psi_{j-1,k}\} & \end{array}$$

$$\begin{array}{ll} f \in L^2(\mathbb{R}) & a_{jk} = \langle f, \phi_{jk} \rangle \text{ and } d_{jk} = \langle f, \psi_{jk} \rangle \\ & \text{approximation} \qquad \qquad \text{detail coefficients} \end{array}$$

$$f_j(x) = \sum_k a_{jk} \phi_{jk}(x) \in V_j$$

$$f_j(x) = \sum_k a_{j-1,k} \phi_{j-1,k}(x) + \sum_k d_{j-1,k} \psi_{j-1,k}(x)$$

How to compute $a_{j-1,k}$ and $d_{j-1,k}$ from a_{jk} ?

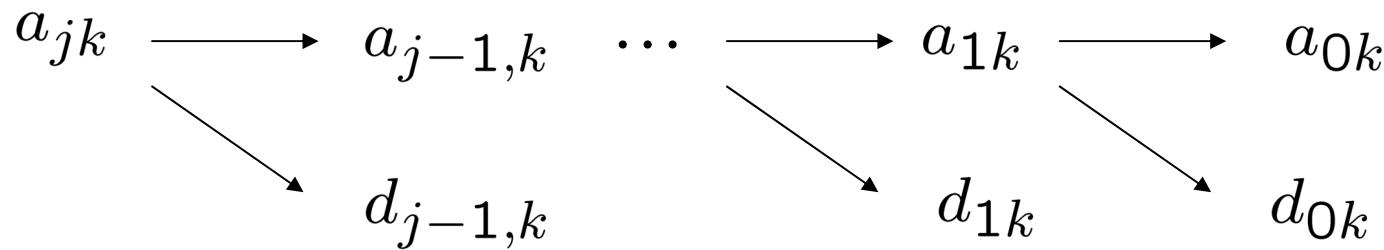
Fast wavelet transform II – Haar scaling function and wavelet

$$a_{jk} = \langle f, \phi_{jk} \rangle \text{ and } d_{jk} = \langle f, \psi_{jk} \rangle$$

$$\phi(x) = \phi(2x) + \phi(2x - 1) \Rightarrow \phi_{j-1,k}(x) = \frac{1}{\sqrt{2}}(\phi_{j,2k}(x) + \phi_{j,2k+1}(x))$$

$$\psi(x) = \phi(2x) - \phi(2x - 1) \Rightarrow \psi_{j-1,k}(x) = \frac{1}{\sqrt{2}}(\phi_{j,2k}(x) - \phi_{j,2k+1}(x))$$

$$a_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} + a_{j,2k+1}) \quad d_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} - a_{j,2k+1})$$



$$V_j = V_{j-1} \oplus W_{j-1}$$

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

Mallat's pyramid algorithm

Mallat [Mal89], Strang and Nguyen [SN96]

Fast wavelet transform III – example, compression

$$a_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} + a_{j,2k+1}) \quad d_{j-1,k} = \frac{1}{\sqrt{2}}(a_{j,2k} - a_{j,2k+1})$$

$$\begin{array}{ccc}
 a_2 = [1.3, 1, -1, -1.3] & \xrightarrow{\hspace{3cm}} & a_1 = \frac{1}{\sqrt{2}}[2.3, -2.3] \\
 & \searrow & \xrightarrow{\hspace{3cm}} \\
 & & d_1 = \frac{1}{\sqrt{2}}[0.3, -0.3] \\
 & & = [.2121, .2121] \\
 & & \swarrow \\
 & & a_0 = [0] \\
 & & \xrightarrow{\hspace{3cm}} \\
 & & d_0 = [2.3]
 \end{array}$$

$$[1.3, 1, -1, -1.3] \xrightarrow{\text{FWT}} [a_0, d_0, d_1] = [0, 2.3, 0.2121, 0.2121]$$

Compression = „Set small wavelet coefficients to zero“

$$\begin{matrix} & \text{Inverse FWT} \\ [0, 2.3, 0, 0] & \xrightarrow{\hspace{1cm}} & [1.15, 1.15, -1.15, -1.15] \end{matrix}$$

Wavelets – vanishing moments

$$\int x^l \psi(x) dx = 0, \text{ for } l = 0, \dots, p-1 \quad (1)$$

,,ψ(x) has p vanishing moments“

$1, x, \dots, x^{p-1}$ are „linear combinations“ of $\{\phi(x - k)\}$

smooth function can be approximated by $\{\phi_{jk}(x)\}_{k \in \mathbb{Z}}$

wavelet (detail) coefficients decrease rapidly for smooth function

Related to smoothness of the scaling function [SN96], Strang [Str89],
Unser and Blu [UB03]

$$(1) \Leftrightarrow \sum_{k=0}^N (-1)^k k^l h_k = 0, \text{ for } l = 0, \dots, p-1$$

wavelet equation

linear equations

Daubechies Wavelets have max. number of vanishing moments, $(N-1)/2$

finitely many solutions for the filter coefficients

Daubechies [Dau88], [Dau92]

Parametrizing all filter coefficients

Parametrize all filter coefficients that correspond to compactly supported orthonormal wavelets

$$\sum_{k=0}^N h_k h_{k-2l} = 2\delta_{0,l} \quad \sum_{k=0}^N h_k = 2$$

Lai and Roach [LR02], Lina and Mayrand [LM93],
Pollen [Pol90], Schneid and Pittner [SP93],
Sherlock and Monro [SM98],
Wells [Wel93], Zou and Tewfik [ZT93]

Express filter coefficients in terms of trigonometric functions

No natural interpretation of the angular parameters for the resulting scaling function and wavelet

Solve transcendental constraints for wavelets with more than one vanishing moment

Parametrizing filter coefficients with several vanishing moments

Omit vanishing moment condition(s) for $\psi(x)$

= linear constraint(s) on the filter coefficients h_k

Introduce the first discrete moments as parameter

Solve the resulting (parametrized) algebraic equations
for h_k using symbolic computation,

in particular Gröbner bases

Buchberger [Buc65], [Buc89]

Wavelet and filter design using Gröbner bases:

Charoenlarpnopparut and Bose [CB99],

Chyzak et al. [CPSSZ01],

Faugère, Moreau de Saint-Martin, and Rouillier [FMR98],

Lebrun and Selesnick [LS04]

Park [Par04], [Par02], Park, Kalker, and Vetterli [PKV97],

with Scherzer [RS05], [Reg05]

Parametrized wavelets – moments of the scaling function

Normalization $M_0 = \int \phi(x) dx = 1 \Rightarrow m_0 = \sum_{k=0}^N h_k = 2$

moments of $\phi(x)$	(discrete) moments of h_k
----------------------	-----------------------------

$$M_1 = \int x\phi(x) dx \quad m_1 = \sum_{k=0}^N h_k k = 2M_1$$

$$M_n = \int x^n \phi(x) dx \quad m_n = \sum_{k=0}^N h_k k^n$$

$$M_n = \frac{1}{2^{n+1} - 2} \sum_{i=1}^n \binom{n}{i} m_i M_{n-i}$$

$$m_n = (2^{n+1} - 2) M_n - \sum_{i=1}^{n-1} \binom{n}{i} m_i M_{n-i}$$

with Scherzer [RS05]

Parametrized wavelets – relation between moments

Th. If $\psi(x)$ has two vanishing moments Sweldens and Piessens [SP94]

$$\text{then } M_2 = M_1^2 \Rightarrow m_2 = \frac{1}{2}m_1^2$$

Th. “The even moments are determined by the odd up to the number of vanishing moments”. with Scherzer [RS05]

Introduce parameter(s)

$$m_1 = \sum_{k=0}^N h_k k \quad m_3 = \sum_{k=0}^N h_k k^3$$

Use the relation(s)

$$\sum_{k=0}^N h_k k^2 = m_2 = \frac{1}{2}m_1^2 \quad m_4 = -\frac{3}{8}m_1^4 + 2m_1 m_3$$

Parametrized wavelets – Gröbner bases

Solve linear equations

vanishing moments

discrete moments (parameters)

Obtain parametrized quadratic equations (orthonormality)

Maple

Groebner package

MATLAB

Wavelet Toolbox

Discuss 4, 6, 8 and 10 filter coefficients

Maple, MATLAB

Parametrized wavelets – applications, rational filter coefficients

4 filter coefficients: Rational parametrization

$$h_0 = \frac{t(t-1)}{t^2+1}, h_1 = \frac{1-t}{t^2+1}, h_2 = \frac{t+1}{t^2+1}, h_3 = \frac{t(t+1)}{t^2+1} \quad [\text{Dau88}]$$

$[3/5, 6/5, 2/5, -1/5]$ smoothest (Hölder regularity at least 0.586)
[Dau92]

6: Rational filter coefficients with at least two vanishing moments?

Find rational points on a variety

Parametrization, find rational points on $w = \sqrt{-a^4 + 14a^2 + 15}$

$$f(x, y) = x^2 + (y^2 - 15)(1 + y^2) \quad \text{genus } f = 1$$

There are no rational points on f with Josef Schicho

8: Would have to find rational points on a curve with genus 3

Faltings' theorem: There are only finitely many rational points
[Fal83]

Parametrized wavelets – design, compression

The Haar wavelet is the only symmetric orthonormal wavelet

[Dau92]

Find the most symmetric wavelet Maple

Parametrized wavelets and compression MATLAB

Future work

- optimization for „wavelets compression“
- „symbolic“ wavelet transform
- more filter coefficients
- biorthogonal wavelets,
- multiwavelets,
- multidimensional FIR ...

Thank you!

Further details, technical report [Reg05]

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