Discriminant Variety for Parametric Systems of Equations and Inequations

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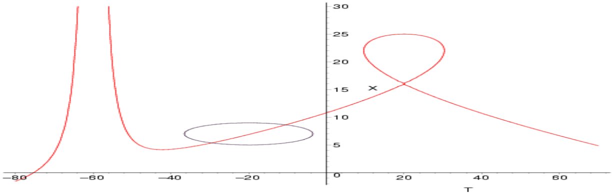
Parametric polynomial system



$$S : (E) \begin{cases} f_1 = 0 \\ \vdots \\ f_n = 0 \end{cases} \text{ and } (F) \begin{cases} g_1 \neq 0 \\ \vdots \\ g_r \neq 0 \end{cases} \in \mathbb{Q}[T_1, \cdots, T_s, X_1, \cdots, X_n]$$

- System of positive dimension
- Generically zero-dimensional and radical

$\Longrightarrow$ S is well behaved

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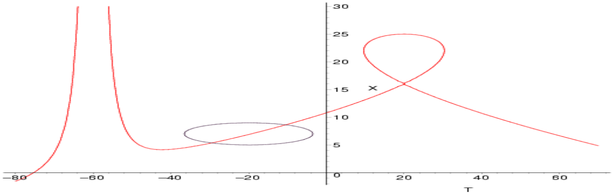

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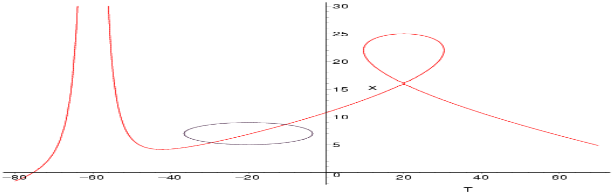

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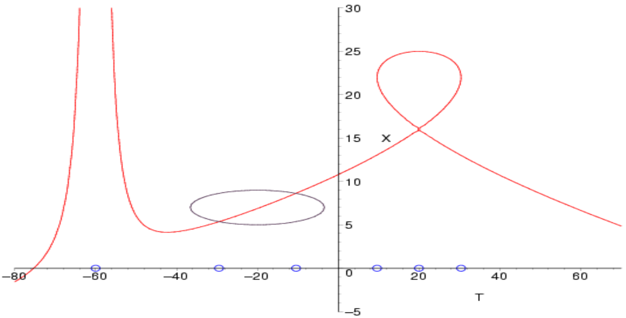
Disriminant variety: Definition



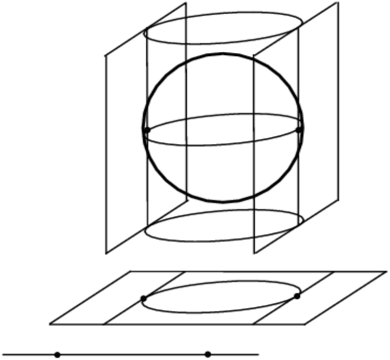
Definition [D. Lazard et F. Rouillier, 2004]

A variety $\mathcal{W} \subset \mathbb{C}^s$ is called discriminant variety of S w.r.t the parameters $[T_1, \cdots, T_s]$ and the unknowns $[X_1, \cdots, X_n]$ iff $\forall \mathcal{Q}$ being an open ball of $\mathbb{C}^s \setminus \mathcal{W}$, the canonical projection

$$\pi : \pi^{-1}(\mathcal{Q}) \cap \mathcal{C} \to \mathcal{Q} \text{ is an analytic covering of } \mathcal{Q}.$$

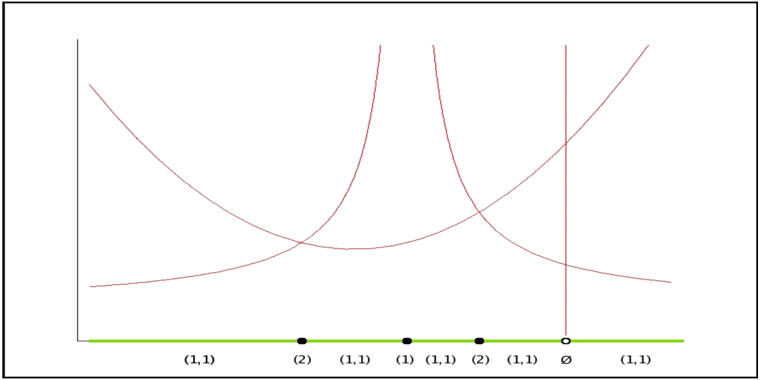
Examples of discriminant varieties

- C.A.D. (Collins)



Examples of discriminant varieties

- Map of vectors of multiplicities (Grigoriev)



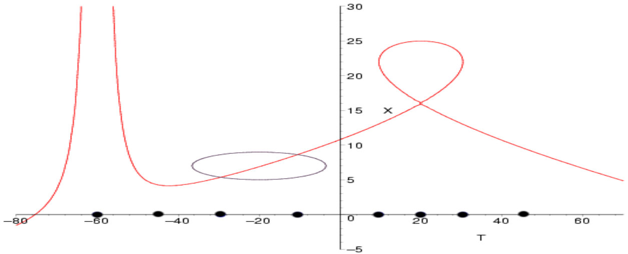
Minimal discriminant variety

Proposition

Let S be a well behaved system. The intersection of all the discriminant varieties is:

- *A discriminant variety*
- *Non trivial*

called the *minimal discriminant variety*



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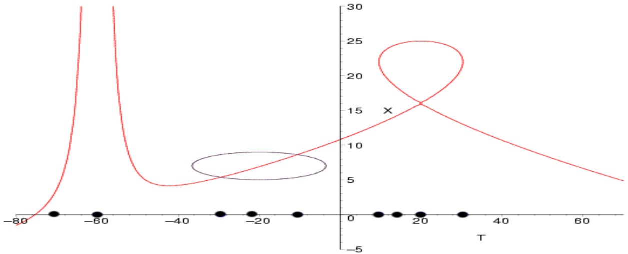

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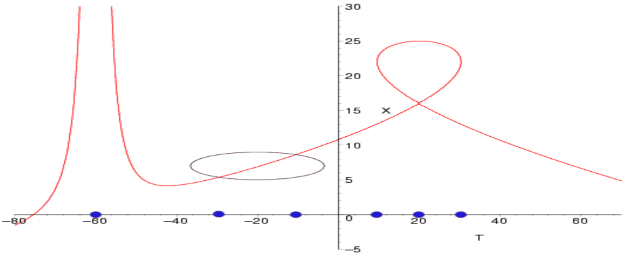

Minimal discriminant variety

In $\mathbb{C}$, the minimal discriminant variety of a well defined parametric system may be decomposed in:

- $\mathcal{O}_c$: critical values of the projection
- $\mathcal{O}_{ineq}$: projection of the intersections with the inequations
- $\mathcal{O}_\infty$: projection of the points at the infinity
- The union $\mathcal{O}_\infty \cup \mathcal{O}_{ineq} \cup \mathcal{O}_c$ is a Zariski closed subset: the minimal discriminant variety.

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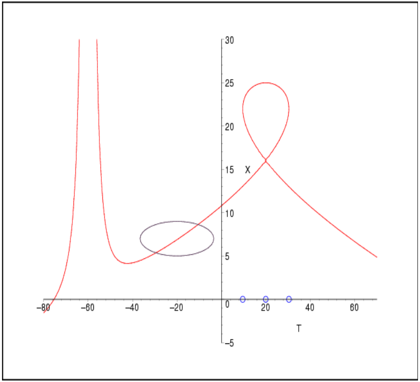

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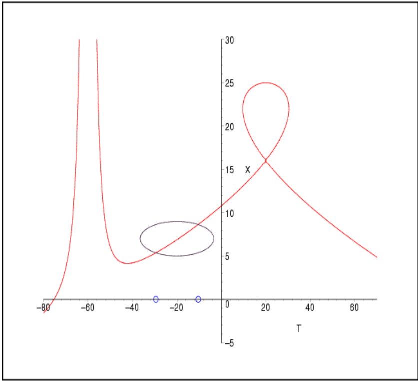

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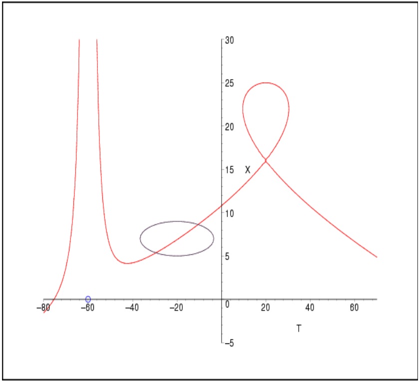

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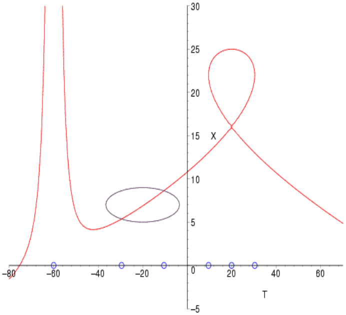
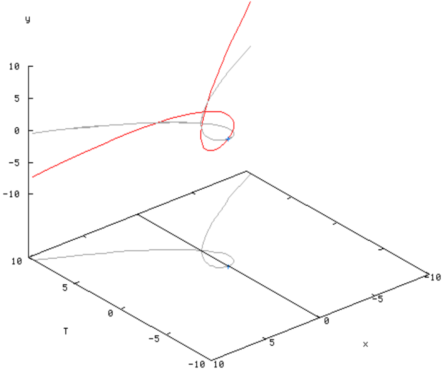


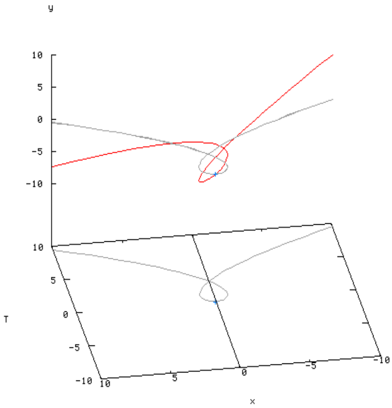
Example with 1 parameter and 2 unknowns



$$(E)\begin{cases} T-(X+Y)^2 & = & 0 \\ (X+Y)^3-5X-3Y & = & 0 \end{cases}$$

Parameters : T
Unknowns : X,Y

Example with 1 parameter and 2 unknowns



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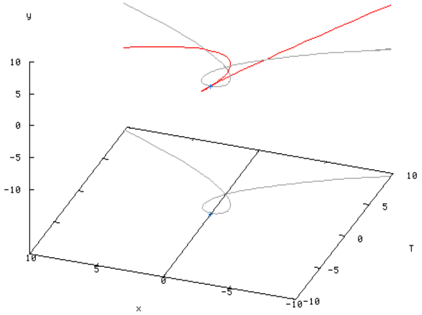

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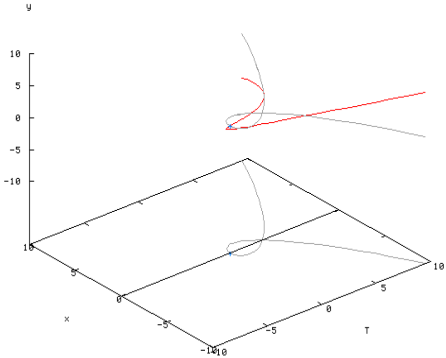

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Parameters : T
Unknowns : X,Y

One unknown: almost like the partial C.A.D

For 1 unknown, we have the following parametric system:

$$S : (E) \quad f = 0 \quad \text{and} \quad (F) \quad g \neq 0 \qquad f, g \in \mathbb{Q}[T_1, \cdots, T_s, X]$$

DV	Partial C.A.D
$V_{crit} = \mathbf{V}\left(\langle f, \frac{\partial f}{\partial X}\rangle \cap \mathbb{Q}[T_1, \ldots, T_s]\right)$	$\mathbf{V}\left(resultant\left(f, \frac{\partial f}{\partial X}\right)\right)$ $\mathbf{V}\left(resultant\left(g, \frac{\partial g}{\partial X}\right)\right)$
$V_{ineq} = \mathbf{V}\left(\langle f, g\rangle \cap \mathbb{Q}[T_1, \ldots, T_s]\right)$	$\mathbf{V}\left(resultant\left(f, g\right)\right)$
$V_{ineq} = \mathbf{V}\left(ldcf\left(f\right)\right)$	$\mathbf{V}\left(ldcf\left(f\right)\right)$ $\mathbf{V}\left(ldcf\left(g\right)\right)$

Critical points

Mathematical characterization:

- Partial jacobian:

$$J_X = \begin{array}{c} \\ f_1 \\ \vdots \\ f_n \end{array} \begin{array}{c} \begin{array}{ccc} \frac{\partial}{\partial X_1} & \cdots & \frac{\partial}{\partial X_n} \end{array} \\ \left| \begin{array}{ccc} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{array} \right| \end{array}$$

- Elimination of $X_1, \cdots, X_n$ in

$$I_c = < f_1, \cdots, f_n, J_X >$$

Implementation:

- Gröbner basis of I_c for a block DRL ordering

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Inequations

Mathematical characterization:

- Saturation of $\langle f_1, \cdots, f_n \rangle$ by $g_1 \cdots g_r$:

$$I_{sat} = \langle f_1, \cdots, f_n \rangle : \left(\prod_{i=1}^{r} g_i\right)^\infty$$

- Elimination of $X_1, \cdots, X_n$ in

$$I_{g_i} = I_{sat} + \langle g_i \rangle$$

  for $1 \leq i \leq s$

Implantation:

- Gröbner basis of $\langle f_1, \cdots, f_n, Zg_1 \cdots g_r - 1 \rangle$ for an elimination order
- Gröbner basis of I_{g_i} for a block DRL ordering

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Points at the infinity

Mathematical characterization:

$\overline{\mathcal{V}}^p$: Partial projective closure in $\mathbb{C}^s \times \mathcal{P}_n$:

$$I^h =< f_1, \cdots, f_n >^h \subset \mathbb{Q}[T_1, \cdots, T_s][X_1, \cdots, X_n, H]$$

$\mathcal{H}_\infty$: Hyperplane at the infinity: $\mathbf{V}(< H >)$

$\pi(\overline{\mathcal{V}}^p \cap \mathcal{H}_\infty)$: Projective elimination of $X_1, \cdots, X_n$ in

$$I^h + < H >$$

Implementation:

- Computation of G the Gröbner basis of $< f_1, \cdots, f_n >$ for a block DRL ordering
- $E_{tete} = \{$head monomials of G for a simple DRL ordering w.r.t $[X_1, \cdots, X_n]\}$
- $I_i =< P \in \mathbb{Q}[T_1, \cdots, T_s]$ such that $PX_i^k \in E_{tete} >$

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Points at the infinity: why it does work

The Gröbner basis of $\langle f_1, \dots, f_n \rangle$:

$$
\begin{array}{c}
\hline
c_{n,i_n}(\mathbf{T})X_n^{d_{n,i_n}} + h_{n,i_n}^{d_{n,i_n}}(\mathbf{T}, X_1, \dots, X_n) + p_{n,i_n}^{<d_{n,i_n}}(\mathbf{T}, \mathbf{X}) \\
\vdots \\
c_{n,1}(\mathbf{T})X_n^{d_{n,1}} + h_{n,1}^{d_{n,1}}(\mathbf{T}, X_1, \dots, X_n) + p_{n,1}^{<d_{n,1}}(\mathbf{T}, \mathbf{X}) \\
\hline
\vdots \\
\vdots \\
\hline
c_{2,i_2}(\mathbf{T})X_2^{d_{2,i_2}} + h_{2,i_2}^{d_{2,i_2}}(\mathbf{T}, X_1, X_2) + p_{2,i_2}^{<d_{2,i_2}}(\mathbf{T}, \mathbf{X}) \\
\vdots \\
c_{2,1}(\mathbf{T})X_2^{d_{2,1}} + h_{2,1}^{d_{2,1}}(\mathbf{T}, X_1, X_2) + p_{2,1}^{<d_{2,1}}(\mathbf{T}, \mathbf{X}) \\
\hline
c_{1,i_1}(\mathbf{T})X_1^{d_{1,i_1}} + p_{1,i_1}^{<d_{1,i_1}}(\mathbf{T}, \mathbf{X}) \\
\vdots \\
c_{1,1}(\mathbf{T})X_1^{d_{1,1}} + p_{1,1}^{<d_{1,1}}(\mathbf{T}, \mathbf{X}) \\
\hline
c_{0,i_0}(\mathbf{T}) \\
\vdots \\
c_{0,1}(\mathbf{T}) \\
\hline
\end{array}
$$

Head term not a pure power:

$$
\begin{array}{c}
q_k(\mathbf{T}, \mathbf{X}) \\
\vdots \\
q_1(\mathbf{T}, \mathbf{X})
\end{array}
$$

Points at the infinity: why it does work

A Gröbner basis of $I_h + \langle H \rangle$:

$$c_{n,i_n}(\mathbf{T})X_n^{d_{n,i_n}} + h_{n,i_n}^{d_{n,i_n}}(\mathbf{T}, X_1, \ldots, X_n)$$
$$\vdots$$
$$c_{n,1}(\mathbf{T})X_n^{d_{n,1}} + h_{n,1}^{d_{n,1}}(\mathbf{T}, X_1, \ldots, X_n)$$
$$\vdots$$
$$\vdots$$
$$c_{2,i_2}(\mathbf{T})X_2^{d_{2,i_2}} + h_{2,i_2}^{d_{2,i_2}}(\mathbf{T}, X_1, X_2)$$
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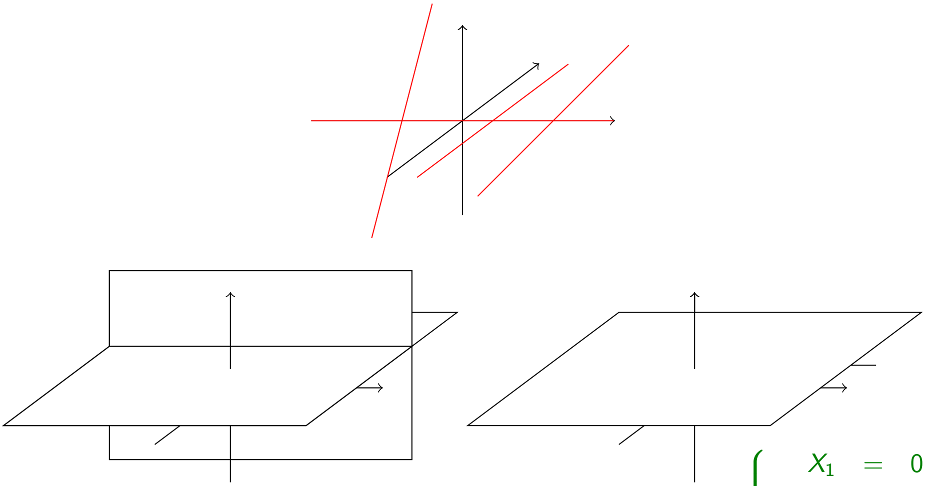
$$\vdots$$
$$c_{1,1}(\mathbf{T})X_1^{d_{1,1}}$$
$$c_{0,i_0}(\mathbf{T})$$
$$\vdots$$
$$c_{0,1}(\mathbf{T})$$

Head term not a pure power:

$$r_k(\mathbf{T}, \mathbf{X})$$
$$\vdots$$
$$r_1(\mathbf{T}, \mathbf{X})$$

Points at the infinity: why it does work



$S_1 : X_1 = 1, \ldots, S_n : X_n = 1$

$$S_1 : X_1 = 1, \ldots, S_n : \begin{cases} X_1 & = & 0 \\ & \vdots & \\ X_{n-1} & = & 0 \\ X_n & = & 1 \end{cases}$$

Points at the infinity: why it does work

A Gröbner basis of $I_h + \langle H \rangle$:

$$c_{n,i_n}(\mathbf{T})X_n^{d_{n,i_n}} + h_{n,i_n}^{d_{n,i_n}}(\mathbf{T}, X_1, X_2, \dots, X_n)$$
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$$\vdots$$
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Head term not a pure power:

$$r_k(\mathbf{T}, \mathbf{X})$$
$$\vdots$$
$$r_1(\mathbf{T}, \mathbf{X})$$

$$S_1 : X_1 = 1$$

Points at the infinity: why it does work

A Gröbner basis of $I_h + \langle H \rangle$:

$$c_{n,i_n}(\mathbf{T})X_n^{d_{n,i_n}} + h_{n,i_n}^{d_{n,i_n}}(\mathbf{T}, 0, 1, \ldots, X_n)$$
$$\vdots$$
$$c_{n,1}(\mathbf{T})X_n^{d_{n,1}} + h_{n,1}^{d_{n,1}}(\mathbf{T}, 0, 1, \ldots, X_n)$$
$$\vdots$$
$$\vdots$$
$$c_{2,i_2}(\mathbf{T})$$
$$\vdots$$
$$c_{2,1}(\mathbf{T})$$
$$0$$
$$\vdots$$
$$0$$
$$c_{0,i_0}(\mathbf{T})$$
$$\vdots$$
$$c_{0,1}(\mathbf{T})$$

Head term not a pure power:

$$r_k(\mathbf{T}, \mathbf{X})$$
$$\vdots$$
$$r_1(\mathbf{T}, \mathbf{X})$$

$$S_2 : X_2 = 1, X_1 = 0$$

Points at the infinity: why it does work

A Gröbner basis of $I_h + \langle H \rangle$:

$c_{n,i_n}(\mathbf{T})$
$\vdots$
$c_{n,1}(\mathbf{T})$

0
$\vdots$
$\vdots$
$\vdots$
$\vdots$
$\vdots$
$\vdots$
0
$c_{0,i_0}(\mathbf{T})$
$\vdots$
$c_{0,1}(\mathbf{T})$

Head term not a pure power:

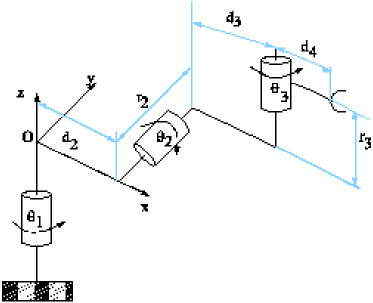
$r_k(\mathbf{T}, \mathbf{X})$
$\vdots$
$r_1(\mathbf{T}, \mathbf{X})$

$S_n : X_n = 1, X_{n-1} = 0, \ldots, X_1 = 0$

High level algorithm

- Saturation of $\langle f_1, \cdots, f_n \rangle$ by $g_1 \cdots g_r$:
$$I_{sat} = \langle f_1, \cdots, f_n \rangle : (\prod_{i=1}^{r} g_i)^\infty$$
- For $1 \leq i \leq s$:
Elimination of $X_1, \cdots, X_n$ in
$$I_{g_i} = I_{sat} + \langle g_i \rangle$$
- Elimination of $X_1, \cdots, X_n$ in
$$I_c = I_{sat} + \langle J_X \rangle$$
- Projective elimination of $X_1, \cdots, X_n$ in
$$I_c^h + < H >$$

A practical problem



Goal: characterize the values of the parameters where the robot can change its position without meeting any singularities.

Polynomial modelization

$$P = at^4 + bt^3 + ct^2 + dt + e$$

$$\begin{cases} P & = & 0 \\ \frac{\partial P}{\partial t} & = & 0 \\ \frac{\partial^2 P}{\partial t^2} & = & 0 \end{cases}$$

- Paramètres: d_3, r_2, d_4
- Inconnues: t, r, z

Polynomial modelization

$$P = at^4 + bt^3 + ct^2 + dt + e$$

$$\begin{aligned}
P &= at^4 + bt^3 + ct^2 + dt + e \\
a &= m5 - m2 + m0 \\
b &= -2m3 + 2m1 \\
c &= -2m5 + 4m4 + 2m0 \\
d &= 2m3 + 2m1 \\
e &= m5 + m2 + m0
\end{aligned}$$

$$\begin{aligned}
m0 &= Z - R + r2^2 + \frac{(R+1-L)^2}{4} \\
m1 &= 2r2d4 + (L - R - 1)d4r2 \\
m2 &= (L - R - 1)d4d3 \\
m3 &= 2r2d3d4^2 \\
m4 &= d4^2(r2^2 + 1) \\
m5 &= d4^2 d3^2 \\
L &= d4^2 + d3^2 + r2^2 \\
Z &= z^2 \\
R &= r^2 + z^2
\end{aligned}$$

Polynomial modelization

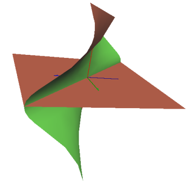
$$
(E)\begin{cases}
(d_4^2d_3^2-(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4d_3-r^2+r_2^2+1/4(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2)t^4+ \\
(-4r_2d_3d_4^2+4r_2d_4+2(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4r_2)t^3+(-2d_4^2d_3^2+4d_4^2(r_2^2+1)-2r^2+2r_2^2+ \\
1/2(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2)t^2+(4r_2d_3d_4^2+4r_2d_4+2(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4r_2)t+ \quad = \quad 0 \\
d_4^2d_3^2+(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4d_3-r^2+r_2^2+1/4(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2 \\
\\
4(d_4^2d_3^2-(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4d_3-r^2+r_2^2+1/4(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2)t^3+ \\
3(-4r_2d_3d_4^2+4r_2d_4+2(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4r_2)t^2+2(-2d_4^2d_3^2+4d_4^2(r_2^2+1)-2r^2+2r_2^2+ \quad = \quad 0 \\
1/2(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2)t+4r_2d_3d_4^2+4r_2d_4+2(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4r_2 \\
\\
12(d_4^2d_3^2-(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4d_3-r^2+r_2^2+1/4(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2)t^2+ \\
6(-4r_2d_3d_4^2+4r_2d_4+2(d_4^2+d_3^2+r_2^2-r^2-z^2-1)d_4r_2)t-4d_4^2d_3^2+ \quad = \quad 0 \\
8d_4^2(r_2^2+1)-4r^2+4r_2^2+(r^2+z^2+1-d_4^2-d_3^2-r_2^2)^2
\end{cases}
$$

$$
(F)\begin{cases} r2 \neq 0 \\ d3 \neq 0 \\ d4 \neq 0 \end{cases}
$$

Parameters : r_2, d_3, d_4
Unknowns : r, t, z

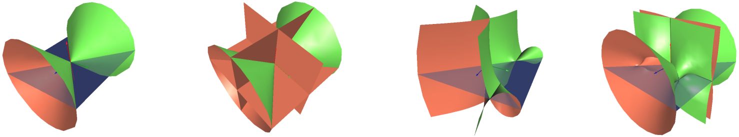
Parametric system for a robotics problem

Discriminante variety



$\mathcal{O}_\infty$

$\mathcal{O}_{ineq}$

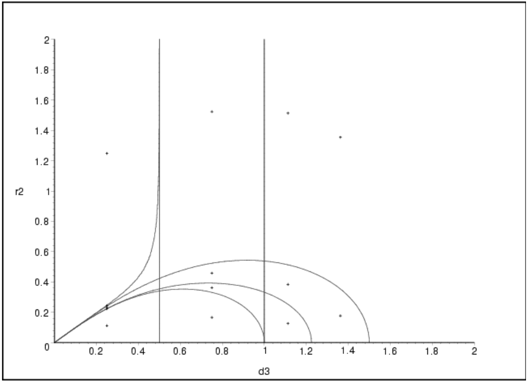


$\mathcal{O}_c$

The different componants of its minimal discriminant variety

Description of the discriminant variety

- The partial C.A.D of the discriminant variety $[r_2, d_3]$



Description of the discriminant variety

- The number of solutions above each cell

$(d_3, r_2) \setminus d_4$	1	2	3	4	5	6	7
(1,1)	0	0	4	4	2	0	0
(1,2)	0	4	4	4	2	0	0
(1,3)	0	4	4	4	2	0	0
(1,4)	0	4	4	2	2	0	0
(1,5)	0	4	4	2	0	0	0
(2,1)	0	0	4	4	2	2	0
(2,2)	0	4	4	4	2	2	0
(2,3)	0	4	4	4	2	2	0
(2,4)	0	4	4	2	2	2	0
(3,1)	0	4	4	4	2	2	4
(3,2)	0	4	4	4	2	2	4
(3,3)	0	4	4	2	2	2	4
(4,1)	0	4	4	4	2	2	4
(4,2)	0	4	4	2	2	2	4
(5,1)	0	4	4	2	2	2	4

A frist upper bound

Theorem [D. Grigoriev et N. Vorobjov, 2000]

Let

$$(E)\begin{cases} f_1 &= 0 \\ & \vdots \\ f_s &= 0 \end{cases} \qquad f_i \in \mathbb{Z}[T_1, \cdots, T_s][X_1, \cdots, X_n]$$

be a generically zero-dimensional system.
Then, the map of vectors of multiplicities of (E) *may be computed in* $\sigma^{\mathcal{O}(1)} d^{\mathcal{O}(n^2 s)}$ *steps on a Turing machine.*

Corollary

Let S be a well behaved parametric system with s parameters and n unknowns.
Then, a discriminant variety of S may be computed in $\sigma^{\mathcal{O}(1)} d^{\mathcal{O}(n^2 s)}$ *steps on a Turing machine.*

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Improved bound

Theorem

With the notations of the first part, the union of:

- *the critical values of the projection* ($\mathcal{O}_{crit}$)
- *the variety induced by the inequations* ($\mathcal{O}_{ineq}$)
- *the variety induced by the points at the infinity* ($\mathcal{O}_{\infty}$)

may be computed in $\sigma^{\mathcal{O}(1)} d^{\mathcal{O}(\max(n^2, ns))}$ *steps on a Turing machine.*

Lemma

Let $I = < f_1, \cdots, f_n, f >$ *such that* $dim(\pi(V(I)) = s - 1$*, then one may find a polynomial* $P \in \mathbb{Q}[T_1, \cdots, T_s]$ *such that* $\pi(V(I)) = V(P)$ *in*

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High level algorithm

- Saturation of $\langle f_1, \cdots, f_n \rangle$ by $g_1 \cdots g_r$:
$$I_{sat} = \langle f_1, \cdots, f_n \rangle : (\prod_{i=1}^{r} g_i)^{\infty}$$
- For $1 \leq i \leq s$:
Elimination of $X_1, \cdots, X_n$ in
$$I_{g_i} = I_{sat} + \langle g_i \rangle$$
- Elimination of $X_1, \cdots, X_n$ in
$$I_c = I_{sat} + \langle J_X \rangle$$
- Projective elimination of $X_1, \cdots, X_n$ in
$$I_c^h + < H >$$

Conclusion

Initial conditions:

- May be verified during the computation
- Often verified in practice

Complexity:

- $\sigma^{\mathcal{O}(1)} d^{\mathcal{O}(\max(n^2, ns))}$ steps on a Turing machine

Real point of view:

- Not minimal
- Extraction of test points:

Bad news: Real dimension $\neq$ Complex dimension

Good news: Some simplifications may appear

$\Rightarrow$ Critical point method (Safey El Din)

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