Computation of dimensions and Ext functors

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Standard polynomials

- Let $\Lambda \subseteq R$ be a ring extension, $\mathcal{U}(\Lambda)$ the units of Λ,
- let $x_1, \ldots, x_p \in R$.
- The elements $X^\alpha = x_1^{\alpha_1} \cdots x_p^{\alpha_p}$ where $\alpha \in \mathbb{N}^p$ are called *(standard) monomials,*
- left Λ-linear combinations of them are called *(left) (standard) polynomials.*
- If $f = \sum_{\alpha \in \mathbb{N}^p} \lambda_\alpha X^\alpha$ is a polynomial, The *Newton diagram* of f is $\mathcal{N}(f) = \{\alpha \in \mathbb{N}^p \mid \lambda_\alpha \neq 0\}$.
- If $\preceq$ is an ordering on $\mathbb{N}^p$ then the exponent of f is defined as $\exp(f) = \max_\preceq \mathcal{N}(f)$.

Quantum relations

$$\begin{cases} x_j x_i = \tau_{ij} x_i x_j + p_{ij} \\ x_k \lambda = \lambda^{(k)} x_k + p_{k,\lambda} \end{cases}$$

$1 \leq i < j \leq p$, $1 \leq k \leq p$,
$\lambda, \lambda^{(k)} \in \Lambda$, $\tau_{ij} \in \mathcal{U}(\Lambda)$,
p_{ij} and $p_{k,\lambda}$ are standard polynomials

PBW rings

A left PBW Λ-ring is a $\Lambda \subseteq R$ extension such that there exists a set of elements $x_1, \ldots, x_p \in R$ satisfying the following properties

- R is a free Λ-module with basis $\{X^\alpha \mid \alpha \in \mathbb{N}^p\}$, the standard monomials
- the elements $x_1, \ldots, x_p$ satisfy the previous relations

$$\left\langle \begin{array}{l} x_j x_i = \tau_{ij} x_i x_j + p_{ij} \\ x_k \lambda = \lambda^{(k)} x_k + p_{k,\lambda} \end{array} \right\rangle$$

- There exists an admissible ordering $\preceq$ on $\mathbb{N}^p$ such that

$$\exp(p_{ij}) \prec \epsilon_i + \epsilon_j \quad \text{and} \quad \exp(p_{k,\lambda}) \prec \epsilon_k$$

Details in [1] and [2]

[1] J. L. Bueso, J. Gómez-Torrecillas and F. J. Lobillo, Homological Computations in PBW Modules, *Algebr. Represent. Theory*, **4**: 201–218, 2001

[2] J. L. Bueso, J. Gómez-Torrecillas and A. Verschoren, *Algorithmic methods in non-commutative algebra. Applications to quantum groups.*, Mathematical Modelling: Theory and Applications 17. Dordrecht: Kluwer Academic Publishers, 2003

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Example: Quantized affine space

Λ is a commutative ring, $\Lambda_q[x_1, \ldots, x_p]$ is a Λ-algebra and tails are zero. So

$$\begin{aligned} \lambda^{(k)} &= \lambda \\ p_{k,\lambda} &= 0 \\ p_{ij} &= 0 \end{aligned}$$

Hence relations are

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Example: Crossed product $R = \Lambda * \mathbb{N}^p$

Tails are zero. The map $\tau : \mathbb{N}^p \times \mathbb{N}^p \to \mathcal{U}(\Lambda)$ defined by $\tau(\epsilon_i, \epsilon_j) = \tau_{ij}$ is a 2-cocycle.
The relations are

$$\left\langle \begin{array}{c} x_j x_i = \tau_{ij} x_i x_j \\ x_k \lambda = \lambda^{(k)} x_k \end{array} \right\rangle$$

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The PBW ring extends the crossed product

$$\left\langle \begin{array}{l} x_j x_i = \tau_{ij} x_i x_j + p_{ij} \\ x_k \lambda = \lambda^{(k)} x_k + p_{k,\lambda} \end{array} \right\rangle$$

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The PBW ring extends the crossed product

$$\left\langle \begin{matrix} x_j x_i = \tau_{ij} x_i x_j + p_{ij} \\ x_k \lambda = \lambda^{(k)} x_k + p_{k,\lambda} \end{matrix} \right\rangle$$

Example: PBW algebras

They have been also known as solvable polynomial rings, and G-algebras in the literature. Λ is a commutative domain and R a Λ-algebra. The relations are then

$$\langle x_j x_i = \tau_{ij} x_i x_j + p_{ij} \rangle$$

Example: Ore algebras

Let $\Lambda = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k (if $n = 0$ then $\Lambda = k$). The skew polynomial ring $R = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called Ore algebra if the σ_i's and δ_j's commute for $1 \leq i, j \leq m$ and satisfy $\sigma_i(\partial_j) = \partial_j$, $\delta_i(\partial_j) = 0$ for $j < i$.

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Quantized matrix algebras

It's a special type of PBW algebra. The $n \times n$ quantized uniparametric matrix algebra $\mathcal{O}_q(M_n(\Lambda))$ is generated by x_{ij}, $1 \leq i, j \leq n$ with relations

$$\left\langle x_{ij}x_{kl} = \begin{cases} qx_{kl}x_{ij} & (k < i, j = l) \\ qx_{kl}x_{ij} & (k = i, j < l) \\ x_{kl}x_{ij} & (k < i, j > l) \\ x_{kl}x_{ij} + (q + q^{-1})x_{kj}x_{il} & (k < i, l < j) \end{cases} \right\rangle$$

The quantum determinant is

$$D_q = \sum_{\pi \in S_n} (-q)^{l(\pi)} x_{1\pi(1)} x_{2\pi(2)} \cdots x_{n\pi(n)}$$

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Example: Quantized enveloping algebras

The general presentation of the quantized enveloping algebra $U_q(C)$ associated to a Cartan matrix C can be seen in [3] and [4]

In [5] it is noted than those algebras are PBW Λ-ring where Λ is a McConnell-Pettit algebra.

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Example: Quantized enveloping algebras (Cont.)

When C is the Cartan matrix associated to $sl_3(k)$, we have ten generators $f_{12}, f_{13}, f_{23}, k_1, k_2, l_1, l_2, e_{12}, e_{13}, e_{23}$ and the following relations:

$$\begin{aligned} e_{13}e_{12} &= q^{-2}e_{12}e_{13} & f_{13}f_{12} &= q^{-2}f_{12}f_{13} \\ e_{23}e_{12} &= q^{2}e_{12}e_{23} - qe_{13} & f_{23}f_{12} &= q^{2}f_{12}f_{23} - qf_{13} \\ e_{23}e_{13} &= q^{-2}e_{13}e_{23} & f_{23}f_{13} &= q^{-2}f_{13}f_{23} \end{aligned}$$

$$\begin{aligned} e_{12}f_{12} &= f_{12}e_{12} + \frac{k_1^2 - l_1^2}{q^2 - q^{-2}} \\ e_{12}f_{13} &= f_{13}e_{12} + qf_{23}k_1^2 \\ e_{12}f_{23} &= f_{23}e_{12} \\ e_{13}f_{12} &= f_{12}e_{13} - q^{-1}l_1^2e_{23} \\ e_{13}f_{13} &= f_{13}e_{13} - \frac{k_1^2k_2^2 - l_1^2l_2^2}{q^2 - q^{-2}} \\ e_{13}f_{23} &= f_{23}e_{13} + qk_2^2e_{12} \\ e_{23}f_{12} &= f_{12}e_{23} \\ e_{23}f_{13} &= f_{13}e_{23} - q^{-1}f_{12}l_2^2 \\ e_{23}f_{23} &= f_{23}e_{23} + \frac{k_2^2 - l_2^2}{q^2 - q^{-2}} \end{aligned}$$

$$\begin{aligned} e_{12}k_1 &= q^{-2}k_1e_{12} & k_1f_{12} &= q^{-2}f_{12}k_1 \\ e_{12}k_2 &= qk_2e_{12} & k_2f_{12} &= qf_{12}k_2 \\ e_{13}k_1 &= q^{-1}k_1e_{13} & k_1f_{13} &= q^{-1}f_{13}k_1 \\ e_{13}k_2 &= q^{-1}k_2e_{13} & k_2f_{13} &= q^{-1}f_{13}k_2 \\ e_{23}k_1 &= qk_1e_{23} & k_1f_{23} &= qf_{23}k_1 \\ e_{23}k_2 &= q^{-2}k_2e_{23} & k_2f_{23} &= q^{-2}f_{23}k_2 \\ e_{12}l_1 &= q^{2}l_1e_{12} & l_1f_{12} &= q^{2}f_{12}l_1 \\ e_{12}l_2 &= q^{-1}l_2e_{12} & l_2f_{12} &= q^{-1}f_{12}l_2 \\ e_{13}l_1 &= ql_1e_{13} & l_1f_{13} &= qf_{13}l_1 \\ e_{13}l_2 &= ql_2e_{13} & l_2f_{13} &= qf_{13}l_2 \\ e_{23}l_1 &= q^{-1}l_1e_{23} & l_1f_{23} &= q^{-1}f_{23}l_1 \\ e_{23}l_2 &= q^{2}l_2e_{23} & l_2f_{23} &= q^{2}f_{23}l_2 \end{aligned}$$

$$\begin{aligned} l_1k_1 &= k_1l_1 & l_2k_1 &= k_1l_2 & k_2k_1 &= k_1k_2 \\ l_1k_2 &= k_2l_1 & l_2k_2 &= k_2l_2 & l_2l_1 &= l_1l_2 \end{aligned}$$

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Filtrations

A ($\mathbb{N}$-)filtration on a ring R is a family of subgroups

$$F = \{R_i \mid i \in \mathbb{N}\}$$

such that

- $R_i \subseteq R_{i+1}$ for all $i \in \mathbb{N}$
- $R_i R_j \subseteq R_{i+j}$
- $R = \bigcup_{i \in \mathbb{N}} R_i$

The associated graded ring is

$$\mathrm{gr}(R) = \bigoplus_{i \in \mathbb{N}} \frac{R_i}{R_{i-1}}$$

A $((\mathbb{N}^n, \preceq)$-)(multi)filtration on a ring R is a family of subgroups

$$F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$$

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- $F_\alpha(R) \subseteq F_\beta(R)$ for all $\alpha \preceq \beta \in \mathbb{N}^n$
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where $F_\alpha^-(R) = \bigcup_{\beta \prec \alpha} F_\beta(R)$

Filtrations

A ($\mathbb{N}$-)filtration on a ring R is a family of subgroups

$$F = \{R_i \mid i \in \mathbb{N}\}$$

such that

- $R_i \subseteq R_{i+1}$ for all $i \in \mathbb{N}$
- $R_i R_j \subseteq R_{i+j}$
- $R = \bigcup_{i \in \mathbb{N}} R_i$

The associated graded ring is

$$\mathrm{gr}(R) = \bigoplus_{i \in \mathbb{N}} \frac{R_i}{R_{i-1}}$$

A $((\mathbb{N}^n, \preceq))$-)(multi)filtration on a ring R is a family of subgroups

$$F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$$

such that

- $F_\alpha(R) \subseteq F_\beta(R)$ for all $\alpha \preceq \beta \in \mathbb{N}^n$
- $F_\alpha(R) F_\beta(R) \subseteq F_{\alpha+\beta}(R)$
- $R = \bigcup_{\alpha \in \mathbb{N}^n} F_\alpha(R)$

The associated graded ring is

$$G^F(R) = \bigoplus_{\alpha \in \mathbb{N}} \frac{F_\alpha(R)}{F_\alpha^-(R)}$$

where $F_\alpha^-(R) = \bigcup_{\beta \prec \alpha} F_\beta(R)$

Characterization

Theorem

Let Λ be a left noetherian ring and $\Lambda \subseteq R$ be a ring extension. Consider a fixed crossed structure $\Lambda * \mathbb{N}^p$. The following statements are equivalent

(I) *There is an admissible order $\preceq$ on some $\mathbb{N}^n$ and an $(\mathbb{N}^n, \preceq)$-filtration $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$ on R such that $F_0(R) = \Lambda$, every $F_\alpha(R)$ is finitely generated as a left Λ–module and $G^F(R) = \Lambda * \mathbb{N}^p$*

(II) *There is a filtration $\{R_n \mid n \in \mathbb{N}\}$ on R such that $R_0 = \Lambda$, every R_n is finitely generated as a left Λ-module and $\mathrm{gr}(R) = \Lambda * \mathbb{N}^p$ (the same structure).*

(III) *R is a PBW ring extending $\Lambda * \mathbb{N}^p$ where $\bigcup_{\lambda \in \Lambda} \mathcal{N}(p_{k,\lambda})$ is a finite subset of $\mathbb{N}^p$.*

See [6] and [7] for details.

As a consequence, if Λ is a domain, for each PBW Λ-ring R and all $f, g \in R$

$$\exp(fg) = \exp(f) + \exp(g)$$

[6] J. L. Bueso, J. Gómez-Torrecillas and F. J. Lobillo, Re-filtering and exactness of the Gelfand-Kirillov dimension. *Bull. Sci. Math.* 125, No.8, 689-715 (2001).

[7] J. Gómez-Torrecillas and F. J. Lobillo, Auslander-regular and Cohen-Macaulay quantum groups, *Algebr. Represent. Theory*, **7**(1): 35–42, 2004

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Modules and TOP order

let R^m be a free left R-module with free R-basis $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_m\}$. Any element $\mathfrak{f} \in$ can written in a unique way as

$$\mathfrak{f} = \sum_{\alpha, i} \lambda_{\alpha,i} X^{\alpha} \mathfrak{e}_i,$$

so R^m has a Λ-basis indexed in $\mathbb{N}^{p,(m)} = \mathbb{N}^p \times \{1, \ldots, m\}$. The order $\preceq$ is extended to $\mathbb{N}^{p,(m)}$ as follows:

$$(\alpha, i) \preceq (\beta, j) \iff \begin{cases} \alpha \prec \beta & \text{or} \\ \alpha = \beta, i \leq j \end{cases}$$

This order is known as TOP order in the literature. For every non-zero $\mathfrak{f} \in R^m$, we can define $\mathcal{N}(\mathfrak{f})$ and $\exp(\mathfrak{f}) \in \mathbb{N}^{n,(m)}$ using TOP order.

Gröbner basis

Corollary

Let $L \subseteq R^m$ be a left/right/twosided R-submodule. Then

$$\exp(M) + \mathbb{N}^p = \exp(M).$$

As in the commutative case:

- Gröbner bases are defined,
- normal forms can be computed,
- S-polynomials are defined,
- a Buchberger's like algorithm to compute them can be proved,

if Λ is a division ring. Again [8] and [9] for details.

[8] J. L. Bueso, J. Gómez-Torrecillas and F. J. Lobillo, Homological Computations in PBW Modules, *Algebr. Represent. Theory*, **4**: 201–218, 2001

[9] J. L. Bueso, J. Gómez-Torrecillas and A. Verschoren, *Algorithmic methods in non-commutative algebra. Applications to quantum groups.*, Mathematical Modelling: Theory and Applications 17. Dordrecht: Kluwer Academic Publishers, 2003

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Syzygies and kernels

Let L be a left submodule of R^m generated by $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_t\}$. Let φ be the morphism

$$\begin{aligned} \varphi : R^s &\longrightarrow R^m/L \\ \mathfrak{e}_i &\longmapsto \mathfrak{f}_i + L \end{aligned}$$

For some $\{\mathfrak{f}_1, \ldots, \mathfrak{f}_s\} \subseteq R^m$. Put

$$H = \{\mathfrak{f}_1, \ldots, \mathfrak{f}_s, \mathfrak{g}_1, \ldots, \mathfrak{g}_t\},$$

Proposition

If $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in R^{s+t}$ is a generating set for $\operatorname{syz}(H)$ and if $\mathfrak{h}_i \in R^s$ is the vector whose coordinates are the first s coordinates of $\mathfrak{p}_i$, with $1 \leq i \leq r$, then

$$\ker(\varphi) = R\mathfrak{h}_1 + \cdots + R\mathfrak{h}_r.$$

The computation of syzygies is completely analogous to commutative case.

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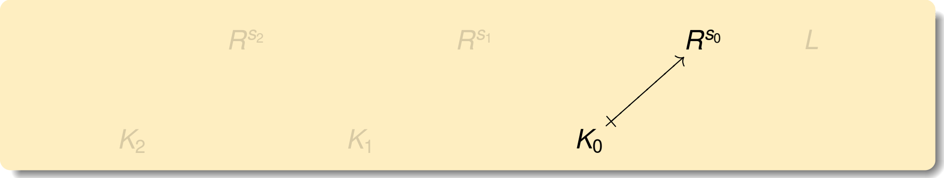
$$\begin{array}{ccccccc} & R^{s_2} & & R^{s_1} & & R^{s_0} & \twoheadrightarrow L \\ K_2 & & K_1 & & K_0 \rightarrowtail R^{s_0} & & \end{array}$$

iterate process

Computation of a free resolution up to some degree



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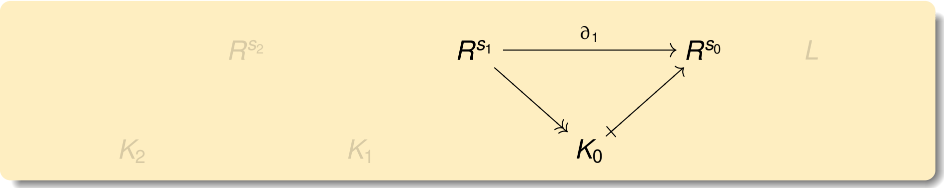

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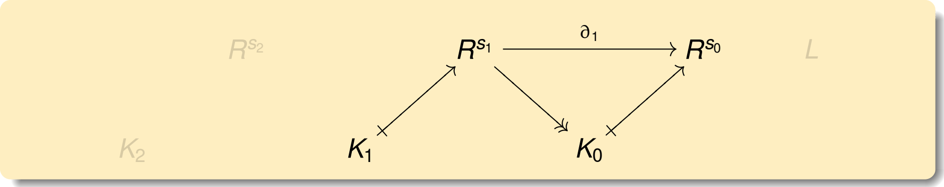

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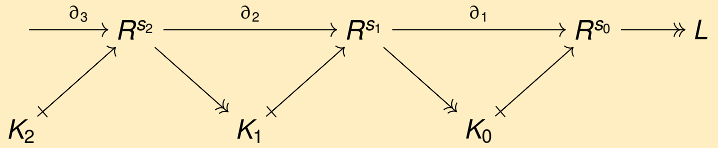

iterate process

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$\mathrm{Ext}^i(M, N)$

$$L \rightarrowtail R^m \twoheadrightarrow N$$

$$\cdots \longrightarrow R^{s_{i+1}} \xrightarrow{\partial_{i+1}} R^{s_i} \xrightarrow{\partial_i} R^{s_{i-1}} \longrightarrow \cdots \longrightarrow R^{s_1} \xrightarrow{\partial_1} R^{s_0} \xrightarrow{\partial_0} M$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(R^{s_{i-1}}, L) & \longrightarrow & \mathrm{Hom}(R^{s_{i-1}}, R^m) & \longrightarrow & \mathrm{Hom}(R^{s_{i-1}}, N) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \partial_i^* & & \\ 0 & \longrightarrow & \mathrm{Hom}(R^{s_i}, L) & \longrightarrow & \mathrm{Hom}(R^{s_i}, R^m) & \longrightarrow & \mathrm{Hom}(R^{s_i}, N) & \longrightarrow & 0 \end{array}$$

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$\operatorname{Ext}^i(M, N)$ (Cont.)

Since $\operatorname{Hom}(R^s, R^m) \cong (R^m)^s = R^{sm}$ and $\operatorname{Hom}(R^s, L) \cong L^s$, previous diagram is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^{s_{i-1}} & \longrightarrow & R^{s_{i-1}m} & \xrightarrow{\pi_{i-1}} & R^{s_{i-1}m}/L^{s_{i-1}} & \longrightarrow & 0 \\
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so

$$\operatorname{Ext}^i(M, N) = \ker \partial^*_{i+1} / \operatorname{im} \partial^*_i \cong \ker \pi_{i+1}\widetilde{\partial}_{i+1} / \operatorname{im} \widetilde{\partial}_i + L^{s_i}$$

The maps $\partial^*_i, \widetilde{\partial}_i, \overline{\partial_i}$ are R-module morphism if L is a subbimodule of R^m, i.e., N is a centralizing bimodule (a.k.a. bimodule in the sense of Artin)

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Degree number

The degree number of a left R-module M

$$j(M) = \inf\{i \mid \operatorname{Ext}^i(M, R) \neq 0\}$$

can be computed, although probably very slowly.

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Growth of a function

Let $f : \mathbb{N} \to \mathbb{R}^+$ be an eventually increasing function. The growth of f is defined as

$$\gamma(f) = \limsup_{n\to\infty} \log_n f(n) = \inf\{\rho \in \mathbb{R} \mid f(n) \leq n^\rho \text{ for all } n \gg 0\}$$

Let's point out just one property:

Lemma

Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two eventually increasing functions such that $f(n) \leq g(an+b)$ for all $n \gg 0$. Then $\gamma(f) \leq \gamma(g)$.

Gelfand–Kirillov dimension

- Λ is a field and A is a Λ–algebra.
- V is a finite dimensional generating vector space for A (we assume $1 \in V$).
- M is a left A–module.
- U is a finite dimensional vector space which generates U as A-module.

The chain

$$U \subseteq VU \subseteq V^2U \subseteq \cdots \subseteq V^nU \subseteq \cdots$$

is an exhaustive filtration on M. The Hilbert function of M associated to V and U is defined as

$$\mathrm{HF}^M_{V,U}(n) = \dim_\Lambda V^nU$$

The Gelfand–Kirillov dimension of M is defined as

$$\mathrm{GKdim}(M) = \gamma(\mathrm{HF}^M_{V,U})$$

This definition does not depend on V or U by previous Lemma.

Behavior

- It's a real number.
- For each $r \in \{0, 1\} \cup [2, \infty[$ there exists finitely generated Λ-algebra A_r such that $\mathrm{GKdim}(A_r) = r$.
- It was developed to prove a conjecture concerning Weyl algebras.
- More details in[10]

[10] G. R. Krause and T. H. Lenagan, "Growth of Algebras and Gelfand-Kirillov Dimension. Revised Edition"Graduated Studies in Mathematics Vol. 22 American Mathematical Society 2000

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Weighted admissible orders

Let $\omega \in (\mathbb{R}^+)^p$. The order $\preceq_\omega$ defined by

$$\alpha \preceq_\omega \beta \iff \begin{cases} \langle \alpha, \omega \rangle < \langle \beta, \omega \rangle & \text{or} \\ \langle \alpha, \omega \rangle = \langle \beta, \omega \rangle \text{ and } \alpha \leq_{\text{lex}} \beta & \end{cases}$$

is an admissible order.

Theorem

If R is a PBW algebra over a field Λ with respect to an admissible order $\preceq$ then there is an algorithm to compute $\omega \in (\mathbb{N}^*)^p$ such that R is a PBW algebra with respect to $\preceq_\omega$.

The proof follows from [11] and [12]

---

[11] J. L. Bueso, J. Gómez-Torrecillas, and F. J. Lobillo *Computing the Gelfand-Kirillov II* in Ring Theory and Algebraic Geometry", (Granja, Hermida, Verschoren eds.) Lecture Notes in Pure Appl. Math. 221, 33–57, Marcel-Dekker, 2001.

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Growth of Hilbert Functions again

Let $E \subseteq \mathbb{N}^{p,(m)}$ be a stable subset and let $\omega \in (\mathbb{R}^+)^p$. The Hilbert function of E with respect to ω is defined as

$$\mathrm{HF}_E^{\omega}(n) = \sharp\{(\alpha, i) \in \mathbb{N}^{p,(m)} \setminus E \mid \langle \alpha, \omega \rangle \leq n\}$$

Theorem

Let L be a left submodule of R^m where R is a PBW algebra over a field Λ with respect to $\preceq_\omega$. Then

$$\mathrm{GKdim}(R^m/L) = \gamma(\mathrm{HF}^{\omega}_{\exp(L)}).$$

The proof can be seen again in [13]

[13] J. L. Bueso, J. Gómez-Torrecillas, and F. J. Lobillo *Computing the Gelfand-Kirillov II* in Ring Theory and Algebraic Geometry", (Granja, Hermida, Verschoren eds.) Lecture Notes in Pure Appl. Math. 221, 33–57, Marcel-Dekker, 2001.

Effective computation

Let R be a PBW algebra over a field Λ generated by $\{x_1, \ldots, x_p\}$. Let M be a finitely presented left R-module provided as $M = R^m/L$ where a set of generators of L is known.

Let $S = \Lambda[x_1, \ldots, x_p]$. For each subset $E \subseteq \mathbb{N}^{p,(m)}$ let's denote $X^E = \{X^\alpha \mathfrak{e}_i \mid (\alpha, i) \in \mathbb{N}^{p,(m)}\}$. If E is stable SX^E is generated as S-module by elements corresponding to the generators of E.

1. Compute a Gröbner basis for L,
2. Compute the classical Krull dimension for the S-module $M_S = S^m/(SX^{\exp(L)})$.
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1. Compute a Gröbner basis for L,
2. Compute the classical Krull dimension for the S-module $M_S = S^m/(SX^{\exp(L)})$.
3. $\mathrm{GKdim}(M) = \gamma(\mathrm{HF}^{\omega}_{\exp(L)}) = \dim(M_S)$.

Index

Degree number again

A Λ-algebra R is called Cohen-Macaulay if for each finitely presented left R-module M

$$\text{GKdim}(R) = \text{GKdim}(M) + j(M)$$

PBW rings and Quantized enveloping algebras are Cohen-Macaulay[14][15], hence the degree number can be computed using Gelfand-Kirillov dimension.

[14] J. L. Bueso, J. Gómez-Torrecillas and F. J. Lobillo, Re-filtering and exactness of the Gelfand-Kirillov dimension. *Bull. Sci. Math*. 125, No.8, 689-715 (2001).

[15] J. Gómez-Torrecillas and F. J. Lobillo, Auslander-regular and Cohen-Macaulay quantum groups, *Algebr. Represent. Theory*, **7**(1): 35–42, 2004

Thank you for your attention!

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