Syzygy Computations in Analysis of Elliptic PDE systems

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Boundary Symbol
Ellipticity
Overdetermined Operators
Quasicomplexes
Elliptic Quasicomplexes
Euler Characteristic
And finally...

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The canonical projections: $\pi_X : T^*X \to X$ and $\pi_Y : T^*Y \to Y$.

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Consider a differential operator

$$A : \; C^\infty(X, V) \to C^\infty(X, \tilde{V})$$

in the space of C^∞ sections of $V, \tilde{V} \in \mathrm{Vect}(X)$.

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A can be written in a coordinate system in the following general form

$$Au = \sum_{|\mu| \leq m} a_\mu(x) D^\mu u = f.$$

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Definition. The principal symbol of A is

$$\sigma_\psi(A)(x, \xi) = \sum_{|\mu| = m} a_\mu(x) \xi^\mu.$$

$\sigma_\psi(A)(x, \xi) : \mathbb{C}^k \to \mathbb{C}^l$ for each $(x, \xi) \in T^*X$.

$\sigma_\psi(A) : \pi_X^* V \to \pi_X^* \tilde{V}$

Classical Boundary Problems

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A boundary value problem for A is classically regarded as an operator

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : \ C^\infty(X,V) \to \begin{matrix} C^\infty(X,\tilde{V}) \\ \oplus \\ C^\infty(Y,W) \end{matrix}$$

where $W \in \mathrm{Vect}(Y)$, $T : C^\infty(X,V) \to C^\infty(Y,W)$ is a trace operator that defines boundary conditions in the problem $Au = f$, $Tu = g$.

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The principal interior symbol of $\mathcal{A}$: $\sigma_\psi(\mathcal{A}) = \sigma_\psi(A)$.

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The principal interior symbol of $\mathcal{A}$: $\sigma_\psi(\mathcal{A}) = \sigma_\psi(A)$.

Locally near Y: ($x = (x', x_n) \in \Omega \times \overline{\mathbb{R}}_+$)

the homogeneous principal boundary symbol of A:

$$\sigma_\partial(A)(x',\xi') = \sigma_\partial(A)(x',0,\xi',D_{x_n}) : \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^k \to \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \mathbb{C}^l$$

$\mathcal{S}(\overline{\mathbb{R}}_+)$ is the Schwartz space.

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Globally on Y:

$$\sigma_\partial(A) : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V' \to \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \tilde{V}',$$

$V' = V|_Y$, $\tilde{V}' = \tilde{V}|_Y$.

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T is a column of operators $T_j = r\tilde{T}_j$, $j = 1, \ldots, \nu$, where $ru = u|_Y$ is the restriction operator.

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$$\sigma_\partial(\tilde{T}_j) : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V' \to \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes W_j$$

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$$\sigma_\partial(T) = \left(r\sigma_\partial(\tilde{T}_j)\right)_{j=1,\dots,\nu} : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V' \to \pi_Y^* W.$$

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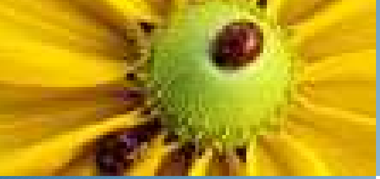
The principal boundary symbol of $\mathcal{A}$:

$$\sigma_\partial(\mathcal{A}) = \begin{pmatrix} \sigma_\partial(A) \\ \sigma_\partial(T) \end{pmatrix} : \pi_Y^* \mathcal{S}(\overline{\mathbb{R}}_+) \otimes V' \to \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes \tilde{V}' \\ \oplus \\ W \end{pmatrix}$$



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Definition. The boundary problem $\mathcal{A}$ is elliptic if

1. A is elliptic, i.e. $\sigma_\psi(A) : \pi^* V \to \pi^* \tilde{V}$ is an isomorphism away from the zero section of T^*X;

2. the boundary conditions are elliptic, i.e. the boundary symbol $\sigma_\partial(\mathcal{A})$ is an isomorphism away from the zero section of T^*Y.

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(2) is called the Shapiro-Lopatinskij condition.

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$\mathcal{A}$ induces a continuous map

$$\mathcal{A} : H^s(X, V) \to \begin{array}{c} H^{s-m}(X, \tilde{V}) \\ \oplus \\ H^{s-\mu-1/2}(Y, W) \end{array}$$

where $H^{s-\mu-1/2}(Y, W) = \bigoplus_{j=1}^{\nu} H^{s-\mu_j-1/2}(Y, W_j)$, $\mu_j = \text{ord}\, \tilde{T}_j$.

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Theorem. $\mathcal{A}$ is elliptic if and only if it is Fredholm (for sufficiently large s).

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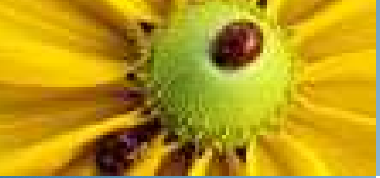
$\Longrightarrow$ we need to consider a compatibility complex for A.



Overdetermined Operators

Let A be an overdetermined operator on X.

$\Longrightarrow$ we need to consider a compatibility complex for A.

Example.

$$\text{gradient}: \quad \nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}\right)$$

is overdetermined, since $\nabla \times (\nabla y) = 0$.

$$\text{curl}: \ \nabla \times f = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$

is overdetermined, since $\nabla \cdot (\nabla \times f) = 0$.

$$\text{divergence}: \quad \nabla \cdot h = \left(\frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_2} + \frac{\partial h}{\partial x_3}\right)$$

Compatibility complex for ∇:

$$C^\infty(\Omega, \mathbb{R}) \xrightarrow{\nabla} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\Omega, \mathbb{R}) \longrightarrow 0$$

Construction of Compatibility Complex

$A_0 \;:\; \mathsf{S}(X, \mathbb{R}^{k_0}) \to \mathsf{S}(X, \mathbb{R}^{k_1})$ with constant coefficients.

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$A_0 \,:\, \mathsf{S}(X, \mathbb{R}^{k_0}) \to \mathsf{S}(X, \mathbb{R}^{k_1})$ with constant coefficients.

The full symbol of A_0:

$$\tilde{A}_0 = \sum_{|\mu| \le m} a_\mu \xi^\mu$$

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$a^1, \ldots, a^{k_1}$: the rows of $\tilde{A}_0$.

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$$\tilde{\mathsf{A}}_0 = \sum_{|\mu| \le m} a_\mu \xi^\mu$$

$a^1, \dots, a^{k_1}$: the rows of $\tilde{\mathsf{A}}_0$.

Construct a free resolution of the module $M_0 = \langle a^1, \dots, a^{k_1} \rangle$:

$$0 \longrightarrow \mathbb{A}^{k_r} \dots \xrightarrow{\tilde{\mathsf{A}}_2^T} \mathbb{A}^{k_2} \xrightarrow{\tilde{\mathsf{A}}_1^T} \mathbb{A}^{k_1} \xrightarrow{\tilde{\mathsf{A}}_0^T} \mathbb{A}^{k_0} \longrightarrow \mathbb{A}^{k_0}/M_0 \longrightarrow 0$$

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A_i: a d.o. corresponding to the syzygy matrix $\tilde{A}_i$.

Theorem. A complex of differential operators with constant coefficients is a compatibility complex for A_0 if and only if the operators A_i are associated to the syzygy matrices of the free resolution of $\mathbb{A}$-module M_0.

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Example. $Ay = (\nabla \times y, \nabla \cdot y)$.
Let $b^1, \dots, b^4$ be rows of the principal symbol of A

$$\sigma_\psi(A) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}, \quad M_0 = \langle b^1, \dots, b^4 \rangle.$$

Computing the syzygy of module M_0, we get

$$S = (\xi_1, \xi_2, \xi_3, 0), \quad M_1 = \langle S \rangle \quad - \text{ the 1-st syzygy module.}$$

Computing the syzygy of M_1, we get $M_2 = 0$. Hence we have the following free resolution for M_0:

$$0 \longrightarrow \mathbb{A}^1 \xrightarrow{S^T} \mathbb{A}^4 \xrightarrow{A^T} \mathbb{A}^3 \longrightarrow \mathbb{A}^3/M_0 \longrightarrow 0 \,.$$

The compatibility complex is

$$0 \longrightarrow C^\infty(X, \mathbb{R}^3) \xrightarrow{A} C^\infty(X, \mathbb{R}^4) \xrightarrow{(\nabla \cdot, 0)} C^\infty(X, \mathbb{R}) \longrightarrow 0 \,.$$

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Let A be an overdetermined operator on X. Consider a boundary problem operator $\mathcal{A} = (A, T)$.

$\Longrightarrow$ we need to consider a compatibility complex for $\mathcal{A}$ and study whether the cohomology of the complex is finite dimensional (Fredholm complex).

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Let A be an overdetermined operator on X. Consider a boundary problem operator $\mathcal{A} = (A, T)$.

$\Longrightarrow$ we need to consider a compatibility complex for $\mathcal{A}$ and study whether the cohomology of the complex is finite dimensional (Fredholm complex).

Fix sufficiently large s_0. Then the compatibility complex for $\mathcal{A}$ is given by

$$0 \longrightarrow H^{s_0}(X, V^0) \xrightarrow{\mathcal{A}^0} \begin{matrix} H^{s_1}(X, V^1) \\ \oplus \\ H^{t_1}(Y, W^1) \end{matrix} \ldots \xrightarrow{\mathcal{A}^N} \begin{matrix} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{matrix} \longrightarrow 0$$

$$\mathcal{A}^i = \begin{pmatrix} A^i & 0 \\ T^i & d^i \end{pmatrix}, \quad i = 1, \ldots, N,$$

where A^i and d^i are differential operators on X and on Y, respectively, T^i are trace operators.

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1. Construct a compatibility complex for a normalised BVP-operator.
2. Using it to construct a compatibility complex for the original BVP- operator.

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Definition. A differential operator $A : \mathsf{S}(X, V) \to \mathsf{S}(X, \tilde{V})$ is called *normalised* if

1. A is a first order operator;
2. A is involutive;
3. there are no (explicit or implicit) algebraic (i.e., non-differential) relations between dependent variables in the system.

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Definition. A BVP-operator $\mathcal{A} = (A, T)$ is normalised if A is normalised and T contains only differentiation in directions tangent to the boundary.

Compatibility Operator for Normalised $\mathcal{A}$

1. Compute a compatibility operator A_1 for A.

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1. Compute a compatibility operator A_1 for A.

2. Compute the tangent part A^τ of A. This is a Gröbner bases computations with a suitable module ordering.

Example. $A = \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $X = \{x \in \mathbb{R}^3 : x_3 \geq 0\}$.
Then $A^\tau y = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$.

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3. Compute a compatibility operator A_1^τ for A^τ. This is syzygy computations, since A^τ is defined on Y.

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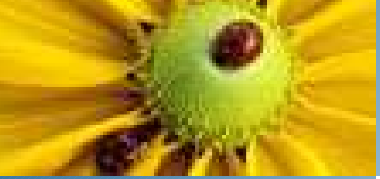

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Construction of Compatibility Operator

To construct a compatibility operator for a BVP-operator $\mathcal{A} = (A, T)$ we should

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To construct a compatibility operator for a BVP-operator $\mathcal{A} = (A, T)$ we should

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To construct a compatibility operator for a BVP-operator $\mathcal{A} = (A, T)$ we should

1. Construct the involutive form of A.

2. Prolong the system (if necessary) until the order of the system is higher than the the order of normal derivatives in the boundary operator.

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3. Construct an equivalent first order system.

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5. Construct the compatibility operator for the normalised BVP.

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5. Construct the compatibility operator for the normalised BVP.

6. Construct the compatibility operator for the original BVP-operator.

Elliptic Complexes

Consider a compatibility complex for $\mathcal{A} = (A, T)$.

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Consider a compatibility complex for $\mathcal{A} = (A, T)$.

The interior symbol sequence $\sigma_\psi(\mathfrak{A})$:

$$0 \longrightarrow \pi_X^* V^0 \xrightarrow{\sigma_\psi(\mathcal{A}^0)} \pi_X^* V^1 \xrightarrow{\sigma_\psi(\mathcal{A}^1)} \cdots \xrightarrow{\sigma_\psi(\mathcal{A}^{N-1})} \pi_X^* V^N \longrightarrow 0$$

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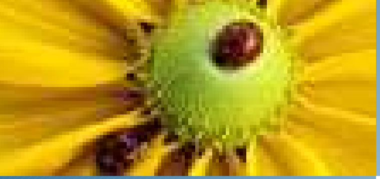

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$$0 \longrightarrow \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}_+}) \otimes V^{0\prime} \\ \oplus \\ W^0 \end{pmatrix} \xrightarrow{\sigma_\partial(\mathcal{A}^0)} \ldots \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}_+}) \otimes V^{N\prime} \\ \oplus \\ W^N \end{pmatrix} \longrightarrow 0$$

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Definition. A compatibility complex for $\mathcal{A}$ is called elliptic if symbol sequences $\sigma_\psi(\mathfrak{A})$ and $\sigma_\partial(\mathfrak{A})$ are exact.

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Theorem. (K.K, N.Tarkhanov, J.Tuomela, 2006) An elliptic compatibility complex for an overdetermined boundary problem is Fredholm for any sufficiently large s_0.

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$\mathcal{P}$ is a parametrix for $\mathcal{A}$ if

$$\mathcal{A}\mathcal{P} - \mathcal{I} \quad \text{and} \quad \mathcal{P}\mathcal{A} - \mathcal{I}$$

are compact

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For boundary problem operators, parametrices are Boutet de Monvel operators.

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$$\mathcal{A} = \begin{pmatrix} P+G & K \\ \\ T & S \end{pmatrix}$$

P is a pseudo-differential operator on X satisfying the transmission property with respect to Y and T, K, G are *trace*, *Poisson* and *singular Green* operators, respectively, and S is a pseudo-differential operator on Y.

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$\Longrightarrow$ under which condition a complex of Boutet de Monvel operators is Fredholm?

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And finally...

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And finally...

Let us consider a sequence

$$\mathcal{E}: \quad 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^{N-1} \xrightarrow{d^{N-1}} E^N \longrightarrow 0$$

E^i are Hilbert spaces and d^i are continuous linear maps.

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Instead of complexes it is natural to consider a sequence $\mathcal{E}$ with the property that the composition $d^i d^{i-1}$ is “small” in some sense. By “small” operators one usually means compact operators.

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Let us denote by $\mathcal{L}(E, \tilde{E})$ a space of continuous linear maps and by $\mathcal{K}(E, \tilde{E})$ the subspace of $\mathcal{L}(E, \tilde{E})$ consisting of compact operators.

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Definition. $(\mathcal{E}, d)$ is a quasicomplex if $d^i d^{i-1} \in \mathcal{K}(E^{i-1}, E^{i+1})$.

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An operator $d \in \mathcal{L}(E, \tilde{E})$ is Fredholm if and only if its image in the Calkin algebra $\mathcal{L}(E, \tilde{E})/\mathcal{K}(E, \tilde{E})$ is invertible.

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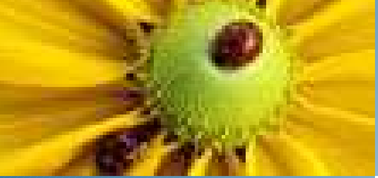
Any compact perturbation of a Fredholm quasicomplex is a Fredholm quasicomplex.



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Definition. The sequence

$$0 \longleftarrow E^0 \xleftarrow{p^1} E^1 \xleftarrow{p^2} \cdots \longleftarrow E^{N-1} \xleftarrow{p^N} E^N \longleftarrow 0$$

with $p^i \in \mathcal{L}(E^i, E^{i-1})$ is called a parametrix of the quasicomplex $(\mathcal{E}, d)$, if

$$d^{i-1}p^i + p^{i+1}d^i = I_{E^i} - C^i, \quad i = 0, \ldots, N,$$

where $C^i \in \mathcal{K}(E^i, E^i)$.

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Theorem. The quasicomplex $(\mathcal{E}, d)$ is Fredholm if and only if it possesses a parametrix.

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Choose $s_0 \in \mathbb{N}$ sufficiently large.

$$0 \longrightarrow \begin{matrix} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{matrix} \xrightarrow{\mathcal{A}^0} \begin{matrix} H^{s_1}(X, V^1) \\ \oplus \\ H^{t_1}(Y, W^1) \end{matrix} \ldots \xrightarrow{\mathcal{A}^{N-1}} \begin{matrix} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{matrix} \longrightarrow 0$$

$$\mathcal{A}^i \mathcal{A}^{i-1} \in \mathcal{K} \begin{pmatrix} H^{s_{i-1}}(X, V^{i-1}) & H^{s_{i+1}}(X, V^{i+1}) \\ \oplus \quad , & \oplus \\ H^{t_{i-1}}(Y, W^{i-1}) & H^{t_{i+1}}(Y, W^{i+1}) \end{pmatrix}$$

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The boundary symbol sequence $\sigma_\partial(\mathfrak{A})$:

$$0 \longrightarrow \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}_+}) \otimes V^{0\prime} \\ \bigoplus \\ W^0 \end{pmatrix} \xrightarrow{\sigma_\partial(\mathcal{A}^0)} \ldots \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}_+}) \otimes V^{N\prime} \\ \bigoplus \\ W^N \end{pmatrix} \longrightarrow 0$$

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Definition. A quasicomplex $(\mathfrak{A}, \mathcal{A})$ is called elliptic if symbol sequences $\sigma_\psi(\mathfrak{A})$ and $\sigma_\partial(\mathfrak{A})$ are exact.

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Theorem. (K.K, N.Tarkhanov, J.Tuomela, 2006) Let $(\mathfrak{A}, \mathcal{A})$ be an elliptic quasicomplex with Boutet de Monvel operators. Then it is Fredholm for a sufficiently large s_0.

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Theorem. (K.K, N.Tarkhanov, J.Tuomela, 2006) Given any two exact sequences of symbols $\{\sigma_\psi^i\}_{i=0}^{N-1}$ and $\{\sigma_\partial^i\}_{i=0}^{N-1}$ there is a complex of operators $\mathcal{D}^i$ such that the principal interior symbols $\sigma_\psi(\mathcal{D}^i) = \sigma_\psi^i$ and boundary symbols $\sigma_\partial(\mathcal{D}^i) = \sigma_\partial^i$ for all $i = 0, \ldots, N-1$.

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Let us now consider an elliptic quasicomplex:

$$0 \longrightarrow \begin{array}{c} H^{s_0}(X, V^0) \\ \oplus \\ H^{t_0}(Y, W^0) \end{array} \xrightarrow{\mathcal{A}^0} \begin{array}{c} H^{s_1}(X, V^1) \\ \oplus \\ H^{t_1}(Y, W^1) \end{array} \xrightarrow{\mathcal{A}^1} \ldots \begin{array}{c} H^{s_N}(X, V^N) \\ \oplus \\ H^{t_N}(Y, W^N) \end{array} \longrightarrow 0$$

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Thus, the complex of operators $\mathcal{D}^i$ is elliptic and hence Fredholm.

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For the Fredholm complex of $\mathcal{D}^i$, the Euler characteristic is defined by

$$\chi(\mathcal{D}) = \sum_{i=0}^{N} (-1)^i \dim H^i.$$

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Definition. The Euler characteristic of the elliptic quasicomplex of $\mathcal{A}^i$ is called the Euler characteristic of the Fredholm complex of $\mathcal{D}^i$, i.e. $\chi(\mathcal{A}) = \chi(\mathcal{D})$.

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Theorem. (K.K, N.Tarkhanov, J.Tuomela, 2006) The Euler characteristic is well-defined, i.e. independent of the particular choice of complex.

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Thank you for your attention!