14 February 2006, Anne Frühbis-Krüger, workshop on Gröbner bases, RICAM, Linz, Austria. Notes taken by Christiaan van de Woestijne, Universiteit Leiden. Please send comments to `cvdwoest@math.LeidenUniv.nl`.

Resolution of singularities, part II.

Main structure of the proof is an induction on the dimension of the ambient space: it proceeds by *order reduction* for $I \subseteq \mathcal{O}_W$, where W is the (smooth) ambient space, having dimension n.

Reduce this to order reduction for ideals in $I \subseteq \mathcal{O}_Z$, where $\dim Z = n-1$.

But there is an intermediate step: order reduction for *marked ideals* where $\dim Z = n-1$.

We want an algorithm that tells us how to choose the centres for the blow-ups.

In the course of the algorithm for the choice of the centres, we associate to each point of the variety $V(I)$ a list of invariants

$$(\mathrm{ord}, N_E; \mathrm{ord}, N_E; \ldots)$$

which are the order of I at this point and a count of certain exceptional divisors from the various blowups. The ";" marks the descent in dimension. More on this below. The set of points where this invariant has maximal value is then used as the centre for the upcoming blow-up.

The important point in reduction of the ambient dimension is the choice of a hypersurface of maximal contact, and the construction of the coefficient ideal. (Already touched on by Hauser in his keynote talk.)

Contents:

1. How to compute the order of an ideal
2. How to choose the special hypersurface
3. How to construct the coefficient ideal
4. A simple example: resolving $V(< z^2 - x^2y^2 >) \subseteq \mathbb{A}^3$
5. Exercises

References:

- János Kollár, *Resolution of singularities — Seattle lecture*, arXiv:math. AG/0508332. Nice introduction to the field.
- Anne Frühbis-Krüger and Gerhard Pfister, *Auflösung von Singularitäten*, DMV-Mitteilungen 13-2/2005, 98–105. For a general audience. Note: figure 12 left is the wrong picture! It should show the three hyperplanes meeting along a line.

1 Order of an ideal at a point, and the locus of maximal order

Situation: W an ambient space, equidimensional and nonsingular, I an ideal, b is a "marker" for marked ideals, E is a chronologically ordered list of exceptional divisors. If b is not defined otherwise, it is set to the maximal order of I, as defined below.

Definition. The order of an ideal at w is defined as

$$\mathrm{ord}_w(I) = \max\{m \in \mathbb{N} \mid I \subseteq \mathfrak{m}_{W,w}^m\}.$$

The "higher singular loci" for integers m are defined as

$$\mathrm{Sing}_m(I) = \{w \in W \mid \mathrm{ord}_w(I) \geq m\}.$$

We may assume, without loss of generality, that W is embedded in $\mathbb{A}^\ell$ for some ℓ, as W is covered by affine charts. In practice, this is especially true, as we consider one chart at a time, encoding all data by means of ideals in a polynomial ring.

Now given a closed point $w \in W$, and a regular system of parameters $x_1, \ldots, x_d$ at w, and a set of generators $g_1, \ldots, g_s$ for I at w, then

$$\Delta(I_w) =_{\mathrm{def}} < g_1, \ldots, g_s, \frac{\partial g_i}{\partial x_j} \mid 1 \leq i \leq s, 1 \leq j \leq d > .$$

(This ideal is implicit in the proof of the Proposition in Schicho's talk.)

Notation: $\Delta^i(I) =_{\mathrm{def}} \Delta(\Delta^{i-1}(I))$, for $i \geq 1$.

Now: $\mathrm{ord}_w(I) = \max\{m \in \mathbb{N} \mid 1 \notin \Delta^{m-1}(I_w)\}$.

Problem. Use of a (local) regular system of parameters (i.e., algebra generators of the coordinate ring) at each point. These do not exist generally for the whole chart, and it is of course infeasible to compute them for all points locally. Therefore, we look for an open covering $U_1, \ldots, U_r$ of W such that on each U_i, there exists a (global) system $x_1, \ldots, x_d$ giving rise to a local regular system of parameters for all $w \in U_i$.

Actually, we already assume $W = V(f_1, \ldots, f_r) \subseteq \mathbb{A}^\ell$. Write $I =< g_1, \ldots, g_s >$.

Case 1: $W = \mathbb{A}^\ell$: done, as the coordinates $x_1, \ldots, x_n$ give local parameters everywhere.

Case 2: $W \subsetneq \mathbb{A}^\ell$: we assume that W is nonsingular, so at each point $w \in W$ at least one $(\ell - \dim W) \times (\ell - \dim W)$-minor M_k of the Jacobian matrix of the f_i does not vanish. On the open set $M_k \neq 0$, we can use

$$\{x_i \mid \text{ the column index } i \text{ does } \textit{not} \text{ appear in } M_k\}$$

as a regular system of parameters.

Determine square matrix A such that $A \cdot M_k = \det M_k \cdot E_{n-\dim W}$.

Now define

$$\tilde{\Delta}(I, M_k) = I + < \det M_k \cdot \frac{\partial g_i}{\partial x_j} - \sum_{\substack{l \text{ row of } M \\ m \text{ column of } M}} \frac{\partial f_m}{\partial x_j} A_{ml} \frac{\partial g_i}{\partial x_l} \mid {}^{1 \leq i \leq s}_{j \text{ not a row of } M_k} > .$$

We want to drop all contributions inside $V(\det M_k)$ which might have appeared. These cannot be trusted, as we computed on $W \backslash V(\det M_k)$. Thus, we set

$$\Delta(I, M_k) = \left(\tilde{\Delta}(I, M_k) : (\det M_k)^{\infty} \right),$$

and we will have

$$\Delta(I) = \cap_k \Delta(I, M_k),$$

where we form the union of all contributions on the different U_i.

Remark by F.-O. Schreyer: this could be explained much easier by using power series. This exposition is chosen because it shows computability.

2 Descent in dimension, maximal contact and the coefficient ideal

Having obtained the locus of maximal order, we now want to pass to an ambient space Z that has one less dimension, but that retains all information on the locus of maximal order. We construct the centre of blowup on this smaller ambient space, using the induction hypothesis to the effect that we know how to find a suitable centre on such a smaller space.

How do we choose Z? Conditions on the lower-dimensional ambient space Z:

- $\mathrm{Sing}_{\text{max ord of } I}(I) \subseteq Z$, and this should still hold after a finite sequence of blow-ups at centres *inside* the maximal order locus
- Z is normal crossing with exceptional divisors from a subset[1] $E_{(1)} \subseteq E$
- $\{Z \cap E_i \mid E_i \notin E_{(1)}\}$ is normal crossing

The latter two conditions are easy to verify once an explicit hypersurface has been chosen.

By definition, we know that $\Delta^{\text{max ord of } I}$ contains 1 at each point $w \in W$. So locally at a point $w \in \mathrm{Sing}_{\text{max ord}}(I)$, there exists $f \in \Delta^{\text{max ord } -1}(I)$ such that $\mathrm{ord}_w(f) = 1$.

[1] $E_{(1)}$ is the subset of the set E of exceptional divisors consisting of all elements being present before the order of I attained the current value. These are being counted as N_E in the invariant tuple right at the beginning.

Same problem as before. So to construct f, we pass to an open covering $U_1, \ldots, U_r$ such that on each U_i, we can use the same Z for all points in $\mathrm{Sing}_{\text{max ord}}(I) \cap U_i$.

Suppose that Δ^m contains 1, with m minimal. Then because Δ^{m-1} has an element of order 1, we know that the singular loci of its generators do not have common points. So, choose generators $h_1, \ldots, h_t$ for $\Delta^{\text{max ord of } I-1}(I)$; we will have $\cap_{i=1}^{t} \mathrm{Sing}(h_i) = \emptyset$.

Then, choose U_i to be in the complement[2] of $\mathrm{Sing}(h_i)$ and put $Z = V(h_i)$.

Problem: how to recombine the data from the different U_i. It turns out that we cannot directly recombine, because the spaces Z on the different U_i are truly different. We can only combine the contributions to the centres right before the upcoming blow-up, discarding those charts where the invariants do not have maximal value.

And: we cannot simply take the intersection of our variety $X = V(I)$ with Z; see a counterexample later on. Instead, we define the *coefficient ideal*

$$I' =_{\text{def}} \mathrm{Coeff}_Z(I) = \sum_{i=0}^{\text{max ord }(I)-1} \left(\Delta^i(I) \cdot \mathcal{O}_Z\right)^{\frac{\text{max ord}(I)!}{\text{max ord }(I)-i}},$$

where we mark the ideal I' by

$$b' = \text{max ord }(I)!,$$

obtaining a "marked ideal", and obtain a new set of exceptional divisors

$$E' = \{E_j|_Z \mid E_j \notin E_{(1)}\}.$$

Example: if I is generated by

$$z^k + z^{k-2}a_2(x,y) + \ldots + a_k(x,y)$$

of order k, this implies in particular that the order of $a_i(x,y)$ is at least i. Now choosing $Z = V(z)$, we only achieve order reduction in the (x,y)-chart if the order of one of $a_i(x,y)$ drops under i after a blowup. Here a_k does not play a special role, we must consider all a_i. This shows that we cannot only use the term with Δ^0 in the above formula for I'.

The new ideal is marked b', which is the "order reduction goal" for this ideal. If b' is less than the actual maximal order of I', we resort to the reduction procedure for marked ideals outlined below.

3 Order reduction for marked ideals, and reduction to the monomial case

I' is transformed under blow-ups by using the controlled transform with respect to the marking b'. We split the transform J in a monomial and a non-monomial

[2]In practice, this may be a complement of a suitably chosen hypersurface containing $\mathrm{Sing}(h_i)$.

part:

$$J = M(J) \cdot N(J)$$

where $M(J)$ factors as

$$\prod_{i=1}^{t} I(E_i)^{\alpha_i}$$

and no $I(E_i)$ is a factor of $N(J)$.

Case 1: max ord $(N(J)) \geq b'$ along $\mathrm{Sing}_{b'}(J)$:
Use induction hypothesis to obtain order reduction for the (now non-marked) $N(J)$, iterating this process until we arrive in Case 2 or 3.

Case 2: $1 \leq$ max ord $(N(J)) \leq b' - 1$ along $\mathrm{Sing}_{b'}(J)$:
The idea is to separate $V(N(J))$ from $\mathrm{Sing}_{b'}(J)$ by using induction hypothesis for

$$M(J)^{\text{max ord }(N(J))} + N(J)^{b'}$$

and push order down until max ord $< b' \cdot$ max ord $(N(J))$, iterating this process until we arrive in Case 3.

Case 3: max ord $(N(J)) = 0$ along $\mathrm{Sing}_{b'}(J)$.
So, there is only a contribution to the order of J from the monomial part.

Simple example illustrating the special problem here:

$$W = \mathbb{A}^2,$$
$$I = < x^3 y^3 >,$$
$$b = 2,$$
$$E = \{V(x), V(y)\}.$$

So locus of maximal order of I is $V(x, y)$. Blow up with centre $V(x, y)$.
Chart 1: $E_3 = V(x)$, $I_{\text{ctrl}} = x^4 \cdot y^3$; chart 2: $E_3 = V(y)$, $I_{\text{ctrl}} = x^3 \cdot y^4$. So we made the situation worse!

Use different invariant $(-c, \rho, J)$. We put

$$c =_{\text{def}} \min\{k \mid \text{there exists } J \subseteq \{1, \ldots, t\} \text{ with } |J| = k \text{ and } \sum_{j \in J} \alpha_j \geq b\},$$

$$\rho =_{\text{def}} \max\{\sum_{j \in J} \alpha_j \mid J \subseteq \{1, \ldots, t\}, |J| = c\};$$

now let $J \in \{1, \ldots, t\}$ have c elements and attain the maximum ρ. Then the ideal sum of ideals $I(E_i)$ corresponding to J gives the next centre. If there are more possibilities for J, decide lexicographically.

Back to example: $c = 1$, $\rho = 3$ and both $J = (0, 1)$ and $(1, 0)$ attain the maximum. Now, blow up at $E_1 = V(x)$; get

$$I_{\text{ctrl}} = x \cdot y^3.$$

4 Example

We resolve the following surface in affine 3-space:

$$W = \mathbb{A}^3, \qquad I =< z^2 - x^2y^2 > .$$

We start afresh, so there are no exceptional divisors arising from earlier blowups that we must take care of: $E = \emptyset$. Pictures for this example are taken from the speaker's home page,
`http://www.mathematik.uni-kl.de/~anne/Aufl-Bilder/DMV.html`.
Higher resolution ones can be downloaded from the workshop web page,
`http://www.ricam.oeaw.ac.at/srs/groeb/schedule_A.html`.

To find the locus of maximal order, we compute

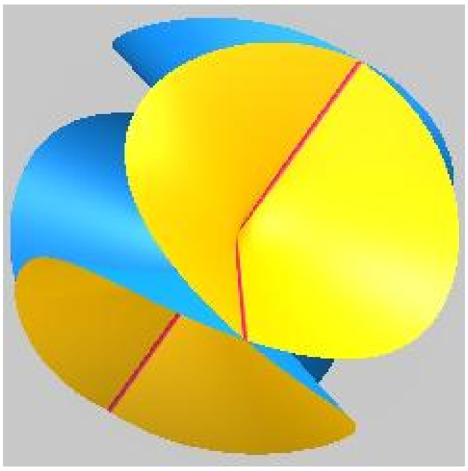
$$\Delta(I) =< z^2 - x^2y^2, z, xy^2, x^2y >=< z, xy^2, x^2y > .$$

From this, it is obvious that $\Delta^2(I)$ will contain 1, and hence the maximal order of I is 2.

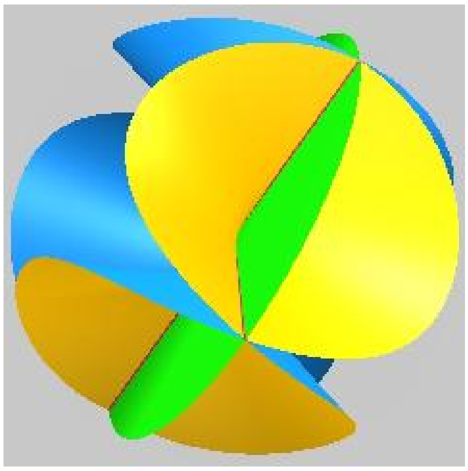
The locus is given by the radical of $\Delta(I)$. It is

$$V(< z, xy^2, x^2y >) = V(< z, xy >)$$

(the red lines in the picture on the left below).



The initial singularity



With hyperplane of maximal contact

If, as here, there are no exceptional divisors to take care of, any generator of order 1 of $\Delta(I)$ will generate a suitable hypersurface. So here, our hypersurface of maximal contact will be

$$H = V(z), \text{ and } \mathcal{O}_H = K[x, y, z]/I(H) = K[x, y].$$

The coefficient ideal I' is now

$$I' = (I \cdot \mathcal{O}_H)^{\frac{2!}{2-0}} + (\Delta(I) \cdot \mathcal{O}_H)^{\frac{2!}{2-1}} .$$

It evaluates to just $< x^2y^2 >$ as all other generators turn out to be zero modulo the first. The ideal I' is marked with $b' = 2! = 2$. The splitting into a monomial and a non-monomial part is trivial as E is trivial. Thus, we have

$$I' = N(I')$$

of order 4, whereas the order reduction goal b' is 2. We are thus in Case 1 of the order reduction for marked ideals.

Compute the maximal order of $N(I')$: we see that

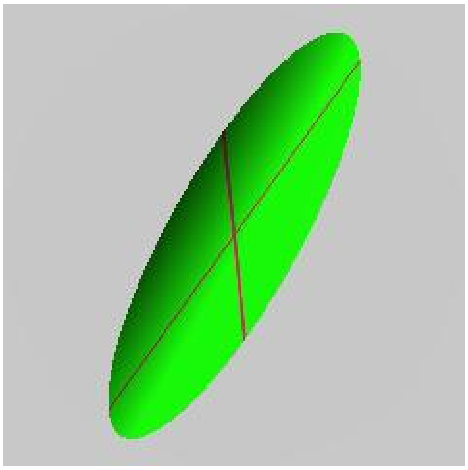
$$\Delta^3(N(I')) =< x, y > .$$

It follows at once that the maximal order is 4. Also, the ideal just computed describes just one point, so we don't need to descend any further. We can now decide that the centre of blowup must be $V(< x, y, z >)$, i.e., the origin.

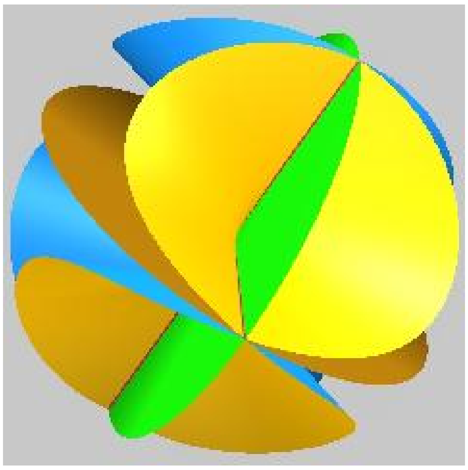
Let's do the blow-up. We get a new ambient space

$$W_1 \subseteq \mathbb{A}^3 \times \mathbb{P}^2$$

defined by equations $uz-wx$, $uy-vx$, $vz-wy$, where (x, y, z) are the coordinates of $\mathbb{A}^3$ and $(u : v : w)$ those of $\mathbb{P}^2$.



The coefficient ideal I'



After blowup: the first chart, with the exceptional divisor (brown)

First chart: $u \neq 0$**.** Here we have

$$z = \frac{w}{u}x, \qquad y = \frac{v}{u}x,$$

and we introduce new variables $z_1 = w/u$ and $y_1 = v/u$. Now we are back in $\mathbb{A}^3$ with coordinates x, y_1, z_1. The total transform of the ideal I is

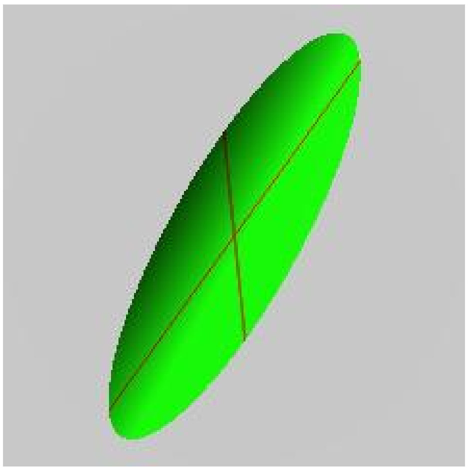
$$I_{\text{total}} =< z_1^2x^2 - y_1^2x^4 >,$$

and factoring this, we find the strict transform

$$I_{\rm str} =< z_1^2 - y_1^2 x^2 >$$

and the exceptional divisor $E = V(x)$.

The strict transform still defines a singular variety; in fact, the ideal has not changed at all! So we continue the process. The only difference with the initial situation is that now we have an exceptional divisor that we carry with us. Of the two lines that were on the hypersurface H, one of them is now contained in the exceptional divisor, while the other is not: this fact allows us to distinguish between the two[3].



Here is the new coefficient ideal: notice the different colours of the lines! (I promise you: the "horizontal" line is now brown, because it is in the exceptional divisor. OK, it's not really *visible*...)

The strict transform of H is

$$H_{\rm str} = V(z_1),$$

and by some theorem, this is at the same time our new hypersurface of maximal contact. The coefficient ideal with respect to the new H is

$$I'_{\rm ctrl} =< y_1^2 x^2 >,$$

which splits in a monomial and a non-monomial part as

$$M \cdot N, \qquad M(I'_{\rm str}) = x^2, \qquad N(I'_{\rm str}) = y_1^2.$$

We are thus in Case 1, with the marked ideal

$$(N(I'_{\rm str}), b') = (< y_1^2 >, 2).$$

The order has dropped to 2, which is equal to the marking b'!

We now take the centre $V(y_1, z_1)$ on this chart.

[3]Here our calculation does not strictly follow the original algorithm of O. Villamayor, but uses a computational improvement on the use of the exceptional divisors due to G. Bodnár.

Second chart: $v \neq 0$**.** This is analogous to $u \neq 0$.

Third chart: $w \neq 0$**.** Here we have

$$x = \frac{u}{w}z, \qquad y = \frac{v}{w}z.$$

The strict transform of I is

$$I_{\text{str}} = < 1 - x_1^2 y_1^2 z^2 >,$$

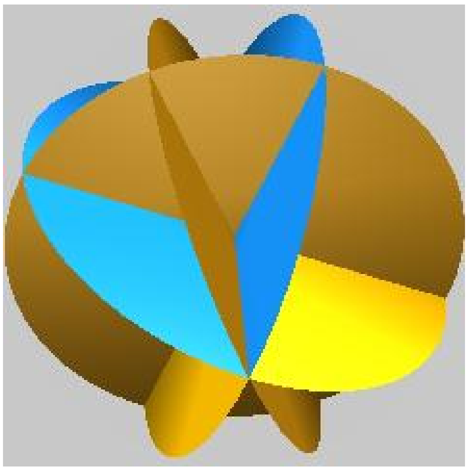
which is nonsingular! Also, it does not meet the exceptional divisor $E_1 = V(z)$, so in particular all crossings with it are normal.

Second blow-up. We continue with the chart $u \neq 0$ above, taking it as ambient $\mathbb{A}^3$. This time, we blow up in a one-dimensional centre, so we get a new ambient

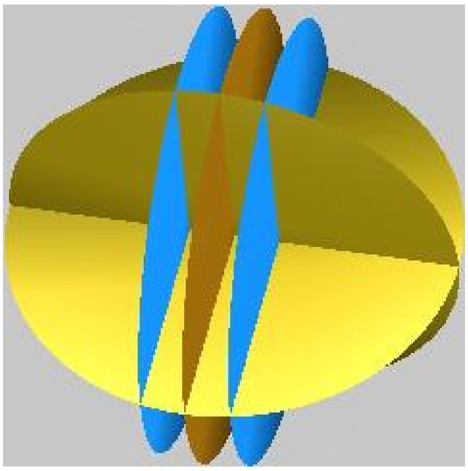
$$W_2 \subseteq \mathbb{A}^3 \times \mathbb{P}^1$$

defined by $uz_1 - vy_1$.

First chart. For $v \neq 0$, we find a similar behaviour to $w \neq 0$ in the previous step.



After the second blowup: the second chart



After the third blowup

Second chart. For $u \neq 0$, we get

$$z_1 = \frac{v}{u}y_1,$$

and we introduce the new variable $z_2 = v/u$. The new exceptional divisor is

$$E_2 = V(y_1)$$

whereas the strict transform of the old one is

$$E_{1,\mathrm{str}} = V(x).$$

Now abusing notation (we leave one "str" out), we write

$$I_{\mathrm{str}} =< z_2^2 - x^2 > .$$

It follows that $\Delta(I_{\mathrm{str}}) =< z_2, x >$. Here z_2 is a generator of order 1, so still $< z_2 >$, which is equal to the strict transform of H, is a hypersurface of maximal contact.

We have

$$I'_{\mathrm{ctrl}} =< x^2 >,$$

so

$$N(I'_{\mathrm{ctrl}}) =< 1 >, \qquad M(I'_{\mathrm{ctrl}}) =< x^2 > .$$

Now we have three hyperplanes meeting along $< x, z_2 >$; this is reduced to a normal crossing situation by finally blowing up once more along $< x, z_2 >$.

5 Exercises

Exercise 1. Choice of hypersurface of maximal contact.

Consider $I =< z^2 - y^3 - x^6 >\subseteq K[x, y, z]$. Decide whether the following hypersurfaces can be chosen as Z:

- $V(z)$
- $V(z - x^2)$

If not, why not?

Exercise 2. The coefficient ideal.

Given $I =< xy - z^5 >$, compute $\Delta^i(I)$, choose a hypersurface of maximal contact Z and compute $\mathrm{Coeff}_Z(I)$.

Exercise 3.

Compute a resolution of $V(z^2 + y^2 - x^4) \subseteq \mathbb{A}^3$ by Villamayor's algorithm. How many different exceptional divisors do you see?