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From Surfaces to Interpolation

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Let X be a projective smooth surface, let L be an ample line bundle on X.

Take the blowup $\Pi : \widetilde{X} \to X$ of X in r points in general position with exceptional divisors $E_1, \ldots, E_r$. Denote by $\widetilde{L}$ the strict transform of L.

Given $d, m_1, \ldots, m_r \in \mathbb{N}$ compute

$$\dim |d\widetilde{L} - \sum_{i=1}^{r} m_i E_i|$$

For $d, m_1, \ldots, m_r \in \mathbb{N}$ and $p_1, \ldots, p_r \in \mathbb{C}^2$ in general position define

$$\mathcal{L}_d(m_1, \ldots, m_r) = \{C \subset \mathbb{C}^2 \mid C \text{ is a curve, } \deg C \leq d, \operatorname{mult}_{p_i} C \geq m_i,\ i = 1, \ldots, r\}.$$

Problem: Compute $\dim_{\mathbb{C}} \mathcal{L}_d(m_1, \ldots, m_r)$.

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$$\dim_{\mathbb{C}} \mathcal{L}_d(m_1, \ldots, m_r) = \max\left(\binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2}, 0\right),$$

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when conditions are independent!

The DEGLEX ordering puts $\mathbb{N}^2$ into a sequence $(\alpha_1, \alpha_2, \dots)$.

For $m_1, \dots, m_r \in \mathbb{N}$ and $p_1, \dots, p_r \in \mathbb{C}^2$ in general position define

$$I(m_1, \dots, m_r) = \{f \in \mathbb{C}[x_1, x_2] \mid \frac{\partial^{|\beta|} f}{\partial x^\beta}(p_i) = 0,\ |\beta| < m_i,\ i = 1, \dots, r\}.$$

Let $k = \sum_{i=1}^r \binom{m_i+1}{2}$.

Main Problem: When $\{x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_k}\}$ is a basis for $\mathbb{C}[x_1, x_2]/I(m_1, \dots, m_r)$?

Problems and questions:

- find fast algorithm (or a formula) to check if a given system $\mathcal{L}_d(m_1, \ldots, m_r)$ has expected dimension;
- prove (or disprove...) the Hirschowitz–Harbourne Conjecture;
- find fast algorithm to compute the basis for $\mathbb{C}[x_1, x_2]/I(m_1, \ldots, m_r)$ with respect to DEGLEX;
- do it in higher dimensions.