Abstracts

Workshop on Gröbner Bases in Control Theory and Signal Processing
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Sum of roots with positive real part in system and control theory

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In this talk, we explain the relationship between the sum of roots with positive real part (SORPRP) of an even polynomial and several important control problems such as polynomial spectral factorization and stability analysis. We obtain SORPRP without computing each root of the polynomial by using Groebner bases theory. Therefore, this approach provides a promising direction for realizing guaranteed accuracy computation and also parametric approach in control system design. The developed result is demonstrated on some optimal control problems.

Two Decades (1985-2005) of Gröbner Bases in Multidimensional Systems

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From about 1975 - 1985, there was considerable use of resultants-subresultants (Sylvester, inners, bigradients) and Bezoutiants to implement old and newly developed tests for multidimensional system stability, multivariate polynomial positivity and tests for relative primeness and greatest common divisor extraction of multivariate polynomials. The software available to implement the algorithms requiring symbolic as well as numerical manipulations included REDUCE, SAC, MACSYMA, etc.

With the introduction of Gröbner - Buchberger bases and algorithmic algebra, probably the first use of Gröbner bases in system theory was by J. P. Guiver and N. K. Bose in 1985 for stabilization of 2-D control systems, characterized by bivariate rational matrices, with compensators, also similarly characterized, either stable (strong stabilizability) or unstable. With the development of software like SINGULAR, COCOA, MACUALAY 2, QEPCAD, the implementation of algorithms based on theory of Gröbner bases was considerably expedited and diversified into many domains of application of multidimensional systems theory. This included tests for multivariate matrix polynomial left/right comprimeness, and, where possible, multivariate polynomial matrix left/right greatest common divisor extraction in multidimensional filter bank design. The topic has recently expanded to Linear Matrix Inequality (LMI) and Semidefinite Programming (SDP) problems where positive polynomials and sum of squares (SOS) representation (SOSTOOLS, SeDuMi) can be used to formulate and solve a host of problems in robust control and signal processing, nonlinear control and convex optimization.

Following the descriptions of the post 1985 developments in the deployment of Gröbner bases in multidimensional systems and signal processing problems, this talk focuses on bihermitian forms, linear maps and "sum of squares" representation. This provides a unified approach to the relationships existing between positive maps, completely positive maps and their finite sum of congruences representation.

Normal form methods in statistical signal processing

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We will present in this talk an algebraic approach based on normal forms to some problems arising in statistical signal processing and communications that can be described as systems of multivariate quadratic polynomial equations. This approach achieves a full description of the solution space and thus avoids the local minima issue of adaptive algorithms. Furthermore, the computational cost is kept low by a split of the problem into two stages. First, a symbolic pre-computation is done off line once for all, to get a more convenient parametric trace-matrix representation of the problem using normal forms. The solutions of the problem are then easily obtained from this representation by solving a single univariate polynomial equation. This approach is quite general and can be applied to a wide variety of problems: SISO channel identification of PSK modulations, filter design and also MIMO blind source separation by deflation.

Genericity of Parameters in Control Theory

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In systems, containing parameters, it often happens, that some structural properties (like the controllability) hold only for the generic case (i.e. for almost all values of parameters).

It means, that there might exist some parameter constellations, such that a generically controllable system, specialized at these constellations becomes non–controllable. We provide an algorithmic way to detect such and similar phenomena, which we call "genericity violation", not restricting ourselves to the generically controllable systems.

Let $\mathbb{K}$ be field. By introducing parameters $p_1, \ldots, p_n$, we define its extension, $\mathbb{K}(p_1, \ldots, p_n)$. This field extension is transcendental, if and only if $\{p_i\}$ are algebraically independent; otherwise the extension is algebraic.

Let R be a ring, which is Noetherian integral domain over $\mathbb{K}(p_1, \ldots, p_n)$ and M be a finitely presented (left) R–module. The fact that a Gröbner basis of M as well as many other objects, derived from Gröbner basis (like e.g. syzygies), is generic, means the following. We assume during the computations, that $\{p_i\}$ are algebraically independent, and hence, we allowe any operations with the elements of $\mathbb{K}(p_1, \ldots, p_n)$, in particular we allow divisions by any polynomial involving p_i.

There exist several variations of Comprehensive Gröbner bases (going back to V. Weispfennig). The result of a typical C. G. B is a tree, which consists of different Gröbner bases together with the constraints on parameters, which imply the particular form of the attached Gröbner basis. Algorithmically, this approach is complicated, complex, and aimed at parametric Gröbner bases in their full generality. In addition, a reasonable implementation of this method is hardly available.

We propose an approach, which works in the situation, where we expect that there are several parameter constellations, leading to the bases, different from the generic Gröbner basis. On the other side, it seems reasonable to require, that a sequence of substitutions of parameters (chosen randomly) leads us to results with the same leading submodule.

We propose to assume the algebraic independence of parameters, and compute the Gröbner basis, say G of a left module, generated by F. It is known, that essentially with the same Gröbner basis algorithm we can obtain the so–called Gröbner trinity, namely the Gröbner basis G, the first syzygy module and the transformation matrix T between the original set of generators F and G. The last algorithm is often referred as LIFT. Requiring from G to have all leading coefficients 1, we compute the matrix T, extract all the denominators of its non–zero entries, factorize them and remove redundant polynomials from the total list of factors.

We end up with a set, say $\{f_1, \ldots, f_m\} \subset \mathbb{K}[p_1, \ldots, p_n]$. Then, we have to solve $2^m - 1$ systems of equations and inequalities of the form

$$\{f_1 = 0, \ldots, f_k = 0; f_{k+1} \neq 0, \ldots, f_m \neq 0\}.$$

In the case of generically controllable system, each system, having nontrivial solutions, leads to the special case, where an annihilator of the torsion submodule can be computed. We present the implementation of the ideas above as the function `genericity` of the library CONTROL.LIB, written for the Computer Algebra System SINGULAR ([2]).

In [3], Quadrat et. al. showed, how to detect parameter constellations in the flat case, i.e. when a system module admits a generalized inverse. We show the approach, which is not limited to the flat situation and, moreover, can be applied not only to Gröbner bases, but to almost all applications, involving Gröbner bases, e.g. for syzygy modules.

We also demonstrate the impact of genericity on some interesting examples, e. g. on the "two pendula" example from the book of Poldermann–Willems, and comment on the other examples.

We discuss the general algorithm and investigate its separate algebraic components from the computational point of view.

References

[1] Chyzak, F., Quadrat, A. and Robertz, D. Linear control systems over Ore algebras.

Effective algorithms for the computation of parametrizations. In *Proc. of Workshop on Time-Delay Systems (TDS03)*. INRIA, 2003.

[2] Greuel, G.-M., Pfister G., and Schönemann H. SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, 2005. Available from `http://www.singular.uni-kl.de`.

Canonical State Representations and Hilbert Functions of Multidimensional Systems

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A basic and substantial theorem of one-dimensional systems theory, due to R. Kalman, says that an arbitrary input/output behavior with proper transfer matrix admits an observable state representation which, in particular, is a realization of the transfer matrix. The state equations have the characteristic property that any local, better temporal, state at time zero and any input give rise to a unique global state or trajectory of the system or, in other terms, that the global state is the unique solution of a suitable Cauchy problem. With an adaption of this state property to the multidimensional situation or rather its algebraic counter-part we prove that any behavior governed by a linear system of partial differential or difference equations with constant coefficients is isomorphic to a canonical state behavior which is constructed by means of Gröbner bases. In contrast to the one-dimensional situation and to J.C. Willems' multidimensional state space models the canonical state behavior is not necessarily a first order system. Further Kalman representations and first order models are due to J.-F- Pommaret and E. Zerz. As a by-product of the state space construction we derive a new algorithm for the computation of the Hilbert function of any finitely generated polynomial module or behavior. J. Wood, P. Rocha et al. recognized the systems theoretic significance of this Hilbert function in context with complexity and structure indices. The theorems are constructive and have been implemented in MAPLE in the two-dimensional case and demonstrated in a simple, but instructive example. For the standard one-dimensional systems the present algorithms compare well with those from the literature.

References

[1] J. Apel, *The Theory of Involutive Divisions and an Application to Hilbert Function Computations*, J.Symb.Computation 25(1998), 683-704

[2] J. Apel, *On a conjecture of R.P. Stanley II: Quotients modulo monomoial ideals*, J.Algebr.Comb. 17(2003), 57-74

[3] V.D. Blondel, A. Megretski (Editors), *Unsolved Problems in Mathematical Systems and Control Theory*, Princeton University Press, 2004

[4] Y.V.Egorov, M.A. Shubin (Editors), *Partial Differential Equations I*, Springer-Verlag, Berlin, 1992

[5] Y.V.Egorov, M.A. Shubin (Editors), *Partial Differential Equations III*, Springer-Verlag, Berlin, 1991

[6] I.M. Gelfand, G.E. Shilov, *Generalized Functions III. Theory of Differential Equations* , Academic Press, New York, 1967

[7] H. Gluesing-Luerssen, *Linear Delay-Differential Systems with Commensurate Delays: An Algebraic Approach*, Lecture Notes in Mathematics 1770, Springer-Verlag, Berlin, 2002

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983

[9] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer-Verlag, Berlin, 1983

[10] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980

[11] R.E. Kalman,*Mathematical description of linear dynamical systems*, SIAM Journal on Control 1(1963), 152-192

[12] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, Berlin, 2000

[13] P. Lu, M. Liu, U. Oberst, *Linear Recurring Arrays, Linear Systems and Multidimensional Cyclic Codes over Quasi-Frobenius Rings*, Acta Appl. Math. 80(2004), 175-198

[14] F.S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, Cambridge, 1916

[15] U. Oberst, *Multidimensional Constant Linear Systems*, Acta Appl. Math. 20(1990), 1 - 175

[16] U. Oberst, *Variations on the Fundamental Principle for Linear Systems of Partial Differential and Difference Equations with Constant Coefficients*, AAECC 6(1995), 211-243

[17] U. Oberst, F. Pauer, *The Constructive Solution of Linear Systems of Partial Difference and Differential Equations with Constant Coefficients*, Multidimensional Systems and Signal Processing 12(2001), 253-308

[18] V.P. Palamodov, *Linear Differential Operators*, Springer-Verlag, Berlin, 1970

[19] J.W. Polderman, J.C. Willems, *Introduction to Mathematical Systems Theory*, Springer-Verlag, New York, 1998

[20] J.-F. Pommaret, *Localization and Transfer Matrix Computation for Linear Multidimensional Control Systems*, Proceedings MTNS 2004

[21] P. Rapisarda, J.C. Willems, *State Maps for Linear Systems*, SIAM J. Control Optim. 35(1997), 1053-1091

[22] C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Gauthiers - Villars, Paris, 1910

[23] R.P. Stanley, *Combinatorics and Commutative Algebra*, 2.Edition, Birkhäuser, Boston, 1996

[24] B. Sturmfels, *Solving Systems of Polynomial Equations*, American Math.Soc., 2002

[25] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, New York, 1975

[26] A.I.G. Vardulakis, *Linear Multivariable Control*, John Wiley & Sons, Chichester, 1991

[27] W.V. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry*, Springer-Verlag, Berlin, 1998

[28] J.C. Willems, *State and First Order Representations*, [1, p.54-57]

[29] W.A. Wolovich, *Linear Multivariable Systems*, Springer-Verlag, New York, 1974

[30] J. Wood, P. Rocha, E. Rogers, and D.H. Owens, *Structure Indices for Multidimensional Systems*, IMA J.Math. Control Inform. 17(2000), 227-256

[31] E. Zerz, *Topics in Multidimensional Linear Systems Theory*, Lecture Notes in Control and Information Sciences 256, Springer-Verlag, London, 2000

Gröbner bases and algebraic analysis : new perspectives in control theory

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We start recalling and illustrating the difference existing between Grobner bases and involutive bases. In fact, while Grobner bases highly depend on the coordinate system, the huge advantage of involutive bases, especially for applications, is that they are intrinsic, that is to say do not depend on the coordinate system.

Accordingly, they fit much better with the formal theory of systems of partial differential equations and "Algebraic analysis", that is the intrinsic module theoretic study of systems of partial differential equations independently of their presentation.

We then provide a few techniques from algebraic analysis and illustrate them on specific control problems, in particulat controllability and observability, in order to show out their usefulness.

This elementary talk can be considered as a summary of the mini-course (12 hours) given the week before.

References

[1] J-F Pommaret, *Advanced topics in Control Systems Theory*, Lecture Notes in Control and Information Sciences, LNCIS 311, Springer, 2005, Chapter 5, "Algebraic Analysis of Control Systems Defined by Partial Differential Equations", p 155-223.

On the Stafford and the Quillen-Suslin theorems and flat multidimensional linear systems

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A control system is called *flat* if there exists an injective parametrization of its solutions [7]. Such a class of control systems plays an important role in the motion planning and the tracking problems [7]. This concept was first developed for non-linear systems of ordinary differential equations [7] but it has been extended to multidimensional linear systems (e.g., ordinary differential time-delay systems [13], under-determined systems of partial differential equations [16], under-detemined functional systems [3]). It was proved that a linear mutidimensional control system is flat if and only if the cokernel module defined by the system matrix is free [3, 7, 13, 16]. Therefore, the problem to recognize whether or not a multidimensional linear system is flat is equivalent to check whether or not a certain finitely presented module over a (non-commutative) ring of functional operators is free. Moreover, the computation of the bases of this module is equivalent to compute the so-called *flat outputs* of the system, from which we can solve the motion planning problem [2, 7, 13] or do some dynamic placement or optimal control [19].

Based on *algebraic analysis*, constructive algorithms were obtained in [3, 16, 18] in order to classify finitely presented modules over some Ore algebras in terms of their structural properties (e.g., torsion, with torsion, torsion-free, reflexive, projective, stably free modules). Moreover, in [3, 17], some heuristics were given in order to check whether or not a finitely presented module over some Ore algebras is free. The main purpose of this talk is to explain the recent progress in this direction obtained in [5, 6, 22, 23] and, specially, to give a constructive algorithm for the computation of bases of free modules over the Weyl algebras $D = A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0 (i.e., rings of differential operators with polynomial or rational coefficients). A well-known result due to Stafford asserts that a stably free left module M over the Weyl algebras $D = A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, with $\text{rank}_D(M) \geq 2$ is free [25]. We shall present a new constructive proof of this result as well as an effective algorithm for the computation of the bases of M. This algorithm, based on the new constructive proofs [8, 9] of Stafford's result on the number of generators of left ideals over D [25], performs Gaussian eliminations on the columns of the formal adjoint of a "minimal presentation matrix" of M [22, 23]. In order to compute such a minimal presentation matrix (full row rank matrix), we give an algorithm which computes the left projective dimension of a general left module defined by means of a finite free resolution. In particular, it allows us to check whether or not a left D-module is stably free [23]. We illustrate the previous results on explicit examples computed by means of the new package Stafford of the library OreModules [2].

We then give some applications of flat multidimensional systems. We first recall that the computation of the flat outputs of a flat shift-invariant multidimensional linear system can be done by applying a constructive version of the Quillen-Suslin theorem [11, 24, 26] over a commutative polynomial ring with coefficients in a field k. An implementation of the Quillen-Suslin theorem has recently been done by A. Fabiańska (Aachen University) which allows us to handle these difficult computations. We explain the general idea of the Quillen-Suslin theorem by showing that a flat shift-invariant multidimensional linear system is algebraically equivalent to a controllable 1-D linear system obtained by setting all but one functional operators to a given value in the system matrix [5]. We illustrate this result on examples of flat ordinary differential time-delay systems which are then proved to be equivalent to controllable ordinary differential systems obtained by setting all the delay amplitudes to 0, i.e., they are equivalent to the corresponding systems without delays. This result is shown to be sometimes useful for the computation of stabilizing controllers. Finally, we explain how the extension of these results can be done for linear systems over Laurent polynomial rings using the Park transformation developed in [14] and we explain how to use it for the computation of flat outputs of π-free shift-invariant multidimensional linear systems [3, 5, 13].

Based on Pommaret's proof [15] of Lin-Bose's conjecture [10], which generalizes Serre's conjecture, we give a general algorithm which constructively solves this problem. Moreover, using results obtained in [20], we give a constructive algorithm which computes (weakly) doubly coprime factorizations of multivariate rational transfer matrices over the commutative polynomial ring. These algorithms have been implemented in Maple. For more details, see [6].

Moreover, we show how to use the previous results on the Stafford theorem in order to give a constructive answer to Datta's question [4] on the possibility to generalize the results of [12] for multi-input multi-output linear systems. Indeed, we prove that every controllable ordinary differential linear system with polynomial coefficients and at least two inputs is flat [21]. The extension of this result to the case of coefficients belonging to the ring of locally convergent series seems to be likely [1]. But, we personally do not know if the corresponding ring D of differential operators is very simple, namely, if for every a_1, a_2, $a_3 \in D$, there exist λ and $\mu \in D$ such that $D\,a_1 + D\,a_2 + D\,a_3 = D\,(a_1 + \lambda\,a_3) + D\,(a_2 + \mu\,a_3)$. If so, our algorithm directly applies to this case.

Finally, we conclude our talk by giving some open questions concerning the extension of the previous results to flat multidimensional nonlinear systems.

This work has been done in separated collaborations with Anna Fabiańska and Daniel Robertz from Aachen University (Germany).

References

[1] J. Briançon, P. Maisonobe, "Idéaux de germes d'opérateurs différentiels à une variable", *L'enseignement Mathématique*, 30 (1984), 7-38.

[2] F. Chyzak, A. Quadrat, D. Robertz, OREMODULES project, `http://wwwb.math.rwth-aachen.de/OreModules`.

[3] F. Chyzak, A. Quadrat, D. Robertz, "Effective algorithms for parametrizing linear control systems over Ore algebras", *Applicable Algebra in Engineering, Communications and Computing*, 16 (2005), 319-376.

[4] K. B. Datta, MathSciNet review of [12].

[5] A. Fabiańska, A. Quadrat, "Flat shift-invariant multidimensional linear systems are algebraically equivalent to controllable 1-D linear systems", *Proceedings of the* 17^{th} *International Symposium on Mathematical Theory of Networks and Systems (MTNS2006)*, Kyoto (Japon) (24-28/07/06).

[6] A. Fabiańska, A. Quadrat, "Applications of the Quillen-Suslin's theorem in multidimensional systems theory", in preparation.

[7] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "Flatness and defect of nonlinear systems: introductory theory and examples", *Internatial Journal of Control*, 61 (1995), 1327-1361.

[8] A. Hillebrand, W. Schmale, "Towards an effective version of a theorem of Stafford", *Journal of Symbolic Computation*, 32 (2001), 699-716.

[9] A. Leykin, "Algorithmic proofs of two theorems of Stafford", *Journal of Symbolic Computation*, 38 (2004), 1535-1550.

[10] Z. Lin, N. K. Bose, "A generalization of Serre's conjecture and related issues", *Linear Algebra and its Applications*, 338 (2001), 125-138.

[11] A. Logar, B. Sturmfels, "Algorithms for the Quillen-Suslin Theorem", *Journal of Algebra*, 145 (1992), 231-239.

[12] F. Malrait, P. Martin, P. Rouchon, "Dynamic feedback transformations of controllable linear time-varying systems", in *Lecture Notes in Control and Information Sciences*, 259, Springer, 2001, 55–62.

[13] H. Mounier, *Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques*, Thèse de doctorat, Université de Paris XI, 1995.

[14] H. Park, "Symbolic computation and signal processing", *Journal of Symbolic Computation*, 37 (2004), 209-226.

[15] J.-F. Pommaret, "Solving Bose conjecture on linear multidimensional systems", *Proceedings of the European Control Conference (ECC)*, Porto (Portugal), 04-07/09/01.

[16] J.-F. Pommaret, A. Quadrat, "Generalized Bezout Identity", *Applicable Algebra in Engineering, Communications and Computing*, 9 (1998), 91–116.

[17] J.-F. Pommaret, A. Quadrat, "Localization and parametrization of linear multidimensional control systems", *Systems and Control Letters*, 37 (1999), 247–260.

[18] J.-F. Pommaret, A. Quadrat, "Algebraic analysis of linear multidimensional control systems", *IMA Journal of Matematical Control and Information*, 16 (1999), 275–297.

[19] J.-F. Pommaret, A. Quadrat, "A differential operator approach to multidimensional optimal control", *International Journal of Control*, 77 (2004), 821-836.

[20] A. Quadrat, "The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. Part I: (Weakly) doubly coprime factorizations", *SIAM Journal of Control and Optimization*, 42 (2003), 266-299.

[21] A. Quadrat, D. Robertz, "On the blowing-up of stably free behaviours", *Proceedings of the Conference on Decision and Control (CDC) and European Control Conference (ECC)*, Seville (Espagne), 12-15/12/05.

[22] A. Quadrat, D. Robertz, "Constructive computation of flat outputs of a class of multidimensional linear systems with variable coefficients", *Proceedings of the* 17^{th} *International Symposium on Mathematical Theory of Networks and Systems (MTNS2006)*, Kyoto (Japon) (24-28/07/06).

[23] A. Quadrat, D. Robertz, "Constructive computation of bases of free modules over the Weyl algebras", *INRIA report RR-5786*, `http://www.inria.fr/rrrt/rr-5786.html`, 2005, submitted for publication.

[24] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.

[25] J. T. Stafford, "Module structure of Weyl algebra", *Journal of the London Mathematical Society*, 18 (1978), 429-442.

[26] D. C. Youla, P. F. Pickel, "The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices", *IEEE Transactions on Circuits and Systems*, 31 (1984), 513-517.

Parametrizing orthonormal wavelets by moments

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Over the last two decades wavelets have become a fundamental tool in many areas of applied mathematics and engineering ranging from signal and image processing to numerical analysis, see for example [8]. In this talk, we discuss parametrizations of filter coefficients of scaling functions and compactly supported orthonormal wavelets with several vanishing moments. We introduce the first discrete moments of the filter coefficients as parameters and use symbolic computation and in particular Gröbner bases to solve the resulting parametrized polynomial equations.

We first outline Daubechies [3] construction of orthonormal wavelets based on scaling functions and the related multiresolution analysis. A scaling function satisfies a dilation equation

$$\phi(x) = \sum_{k=0}^{N} h_k \phi(2x - k)$$

given by a linear combination of (real) filter coefficients h_k and dilated and translated versions of the scaling function.

Conditions on the scaling functions imply, using the dilation equation, constraints on the filter coefficients. Orthonormality of the integer translates of the scaling function implies that the number of filter coefficients is even and gives quadratic equations for the filter coefficients and vanishing moments of the associated wavelet linear constraints. If the filter coefficients satisfy the equations for orthonormality and the normalization $\sum h_k = 2$, then there exists a unique normalized solution of the dilation equation. For almost all such scaling functions

$$\psi(x) = \sum_{k=0}^{N} (-1)^k h_{N-k} \phi(2x - k)$$

is an orthonormal wavelet.

Vanishing moments of the associated wavelet, that is, $\int x^n \psi(x) = 0$, are related to several properties of the scaling function and wavelet, for example to the approximation order and smoothness. Daubechies wavelets [3] have the maximal number of vanishing moments for a fixed number of filter coefficients and so there are only finitely many solutions.

Parametrizing all possible filter coefficients that correspond to compactly supported orthonormal wavelets has been studied by several authors. In all parametrizations the filter coefficients are expressed in terms of trigonometric functions and there is no natural interpretation of the angular parameters for the resulting scaling function. Furthermore, one has to solve transcendental constraints to find wavelets with more than one vanishing moment.

In the proposed parametrization we omit one vanishing moment condition and introduce the first discrete moments, $m_n = \sum h_k k^n$, of the filter coefficients as parameters. The discrete moments can be expressed in terms of the (continuous) moments of the scaling function, $M_n = \int x^n \phi(x)$, and thus have a natural interpretation. Moreover, we can use the fact that even moments are determined by odd up to the number of vanishing moments [7].

We solve the resulting parametrized polynomial equations for the filter coefficients using symbolic computation and for the more involved equations in particular Gröbner bases. Gröbner bases were introduced by Buchberger in [1]. Applications of Gröbner bases to the design of wavelets and filter coefficients are for example discussed in [2, 4, 5].

After computing and illustrating several examples we outline some applications. Finally, we demonstrate a MATLAB package to compute with parametrized wavelets and discuss possible extensions of our approach. For further details and references we refer to [6].

References

[1] B. Buchberger. *An Algorithm for Finding the Bases Elements of the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal (German)*. PhD thesis, University of Innsbruck, 1965.

[2] F. Chyzak, P. Paule, O. Scherzer, A. Schoisswohl, and B. Zimmermann. The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval. *Experiment. Math.*, 10(1):67–86, 2001.

[3] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41(7):909–996, 1988.

[4] J. Lebrun and I. Selesnick. Gröbner bases and wavelet design. *J. Symbolic Comput.*, 37(2):227–259, 2004.

[5] H. Park. Optimal design of synthesis filters in multidimensional perfect reconstruction FIR filter banks using Gröbner bases. *IEEE Trans. Circuits Systems I Fund. Theory Appl.*, 49(6):843–851, 2002.

[6] G. Regensburger. Parametrizing compactly supported orthonormal wavelets by discrete moments. Technical Report 2005-23, SFB F013, 2005. `http://www.sfb013.uni-linz.ac.at/`.

[7] G. Regensburger and O. Scherzer. Symbolic computation for moments and filter coefficients of scaling functions. *Ann. Comb.*, 9(2):223–243, 2005.

[8] G. Strang and T. Nguyen. *Wavelets and filter banks.* Wellesley-Cambridge Press, Wellesley, MA, 1996.

Gröbner Computations in the Ring of Multivariate Proper Stable Rational Functions

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The ring of multivariate proper stable rational functions plays an important role in the proper stabilization of discrete or continuous multidimensional input/output systems by feedback. In this talk we prove, that this ring is a noetherian factorial domain, to which the Gröbner bases theory, suitably adapted, can be applied. This, in turn, is an important tool to check the proper stabilizability of an input/output system and to construct proper stabilizing compensators.

The approach to these problems, and the mathematics behind solving them, have been strongly influenced by and worked out in collaboration with U. Oberst, the author's Ph.D.-advisor.

Applications of Grobner Bases in Synthesis of Multidimensional Control Systems

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Stabilization, asymptotic tracking and disturbance rejection or regulation are basic and important requirements for feedback control system synthesis. The purpose of this presentation is to show the possibility on applications of Gröbner bases in synthesis of multidimensional feedback control systems.

Multidimensional behaviors: polynomial-exponential trajectories and linear exact modelling

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We study linear systems of partial differential equations with complex coefficients. First, we investigate the space of polynomial-exponential solutions belonging to a fixed frequency. In particular, the finite-dimensionality of this vector space is characterized, and, in the case of finite dimension, a basis is constructed. In the case where the dimension is infinite, the asymptotic growth of the dimensions with respect to a degree bound is shown to be polynomial; the connection with the Hilbert-Samuel function is discussed.

In the second part of the talk, the problem of linear exact modeling is studied. In other words, given a finite number of polynomial-exponential signals, one seeks the smallest system admitting these solutions. We present both a direct and a recursive method for constructing this "most powerful unfalsified model", and we address minimality issues.

The algorithmic side of these questions involves the manipulation of polynomial modules, such as the calculation of quotients, saturations, companion matrices, dimensions, and localizations (all of which can be performed using Groebner bases), and the combination of these methods with tools from computational linear algebra.