Finiteness Issues on Differential Standard Bases

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- An *ordinary differential ring* $\mathcal{R}$ is a commutative ring with a derivative operator δ: $\delta(a+b) = \delta a + \delta b$; $\delta(ab) = \delta ab + a\delta b$.
- $\Theta := \{\delta^k : k \geq 0\}$.
- An ideal I of $\mathcal{R}$ is *differential* iff $\delta I \subset I$.
- $[F]$ denotes the differential ideal generated by F.
- $\mathcal{F}$ is a differential *field of constants* of characteristic zero.
- $\mathcal{F}\{y\} := \mathcal{F}[y, \delta y, \delta^2 y, \ldots]$ — a ring of differential polynomials.
- $y_i := \delta^i y$.
- $\mathbb{M}$ — the set of all differential monomials.
- $\mathrm{lm}_{\prec} f$ — the *leading monomial* of a polynomial $f \notin \mathcal{F}$ w.r.t. $\prec$.

A differential ideal can have no finite system of differential generators.

Example.

Let $\mathcal{F}\{y\}$ be the ordinary ring of diferential polynomials.

Then, the sequence of differential ideals

$$[y^2] \subset [y^2, y_1^2] \subset \cdots \subset [y^2, \ldots, y_i^2] \subset \cdots \subset \mathcal{F}\{y\}$$

is an infinite strictly increasing sequence.

Admissible orderings

An *admissible ordering* on the set of differential monomials $\mathbb{M}$ must satisfy the following axioms:

- $M \prec N \implies MP \prec NP \quad \forall\, M, N, P \in \mathbb{M}$;
- $1 \preceq P \quad \forall\, P \in \mathbb{M}$;
- $y_i \prec y_j \iff i < j$.

These properties are sufficient to guarantee that any admissible ordering well orders $\mathbb{M}$ (Zobnin, 2003).

Examples: **lex**, **deglex**, **wt-lex**, **degrevlex**, **wt-revlex**, ...

Any monomial ordering can be specified by an $m \times (k+1)$ *monomial matrix* $\mathcal{M}$ with real entries and lexicographically positive columns such that $\mathrm{Ker}_{\mathbb{Q}}\, \mathcal{M} = \{0\}$:

$$\mathcal{M}\begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_k \end{pmatrix} \prec_{\mathrm{lex}} \mathcal{M}\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_k \end{pmatrix} \iff y_0^{\alpha_0} \dots y_k^{\alpha_k} \prec y_0^{\beta_0} \dots y_k^{\beta_k}.$$

Definition 1. A set of monomial matrices $\{\mathcal{M}_k\}$ is called *concordant* if the matrix $\mathcal{M}_{k-1}$ can be obtained from $\mathcal{M}_k$ by deleting the rightmost column and then by deleting a row of zeroes, if such a row exists.

Theorem. Any admissible ordering on differential monomials can be specified by a concordant set of monomial matrices or, equivalently, by an infinite monomial matrix.

Examples of admissible orderings

Lex

$$\begin{pmatrix}\mathbf{1}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{1}\\ 1 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 1 & \mathbf{0}\\ 1 & 0 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{0}\\ 0 & 1 & 0 & \mathbf{0}\\ 1 & 0 & 0 & \mathbf{0}\end{pmatrix} \dots$$

$$\begin{pmatrix} \dots & \dots & \dots & \dots \\ & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

DegLex

$$\begin{pmatrix}\mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & \mathbf{1}\\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \begin{pmatrix}1 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 1 & \mathbf{0}\end{pmatrix}, \begin{pmatrix}1 & 1 & 1 & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1}\\ 0 & 0 & 1 & \mathbf{0}\\ 0 & 1 & 0 & \mathbf{0}\end{pmatrix} \dots$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \end{pmatrix}$$

Examples of orderings (ctd.)

DegRevLex

$$(\mathbf{1}), \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} 1 & 1 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & \mathbf{1} \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \ldots$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \ldots \\ & 1 & 1 & 1 & \ldots \\ & & 1 & 1 & \ldots \\ & & & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

WtRevLex

$$(\mathbf{1}), \begin{pmatrix} 1 & \mathbf{2} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} 1 & 2 & \mathbf{3} \\ 0 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \mathbf{4} \\ 0 & 1 & 1 & \mathbf{1} \\ 0 & 0 & 1 & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \ldots$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots \\ & 1 & 1 & 1 & \ldots \\ & & 1 & 1 & \ldots \\ & & & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

$\boldsymbol{\delta}$-lexicographic and $\boldsymbol{\beta}$-orderings

The ordering is called δ-lexicographic if $\mathrm{lm}_{\prec}\,\delta M = \mathrm{lm}_{\mathrm{lex}}\,\delta M$ for any monomial M.

Example. The orderings **lex**, **deglex** and **wt-lex** are δ-lexicographic.

If, in contrast, all summands in $\delta^k M$ are compared inverse lexicographically then we call $\prec$ a *β-ordering*.

Example. **Degrevlex** and **wt-degrevlex** are β-orderings.

$\boldsymbol{\delta}$-fixedness

Definition 2. An admissible ordering $\prec$ is *δ-fixed* if

$$\forall\, f \in \mathcal{F}\{y\} \setminus \mathcal{F} \qquad \exists\, M \in \mathbb{M}; \quad \exists\, k_0, r \in \mathbb{N}:$$

$$\mathrm{lm}_{\prec}\, \delta^k f = M y_{r+k} \quad \text{for all} \quad k \geq k_0.$$

Example. Any δ-lexicographic ordering is δ-fixed.

Concordance with quasi-linearity

Let $\prec$ be an admissible ordering.

A polynomial $f \in \mathcal{F}\{x\} \setminus \mathcal{F}$ is *$\prec$-quasi-linear* if $\deg \operatorname{lm}_{\prec} f = 1$.

Example. $f = y_1 + y_0^2$ is quasi-linear w.r.t. **lex**, but not **deglex**.

We say that $\prec$ is *concordant with quasi-linearity* if the derivative of any $\prec$-quasi-linear polynomial is quasi-linear too.

Example. **Lex**, **deglex**, **degrevlex** are concordant with quasi-linearity, as well as any δ-lexicographic ordering.

Differential standard bases

Fix an admissible ordering $\prec$. Consider a differential ideal I of $\mathcal{F}\{x\}$.

A set $G \subset I$ is a *differential standard basis* of I if ΘG is an algebraic Gröbner basis of I in $\mathcal{F}[y_0, y_1, y_2, \ldots]$ (possibly, infinite).

If we know finite DSB of a differential ideal I, we can algorithmically test the membership to this ideal:

Example. Any linear ideal has a **finite** differential standard basis.

Unfortunately, differential standard bases are often **infinite**:

Example. The ideal $[y^n]$, $n \geq 2$, does not have finite DSB w.r.t. **lex**.

But it has a finite DSB (consisting only of y^n) w.r.t. any β-ordering (e.g., *degrevlex*)!

Finiteness criterion

Let I be a proper differential ideal of $\mathcal{F}\{y\}$.

Necessary condition. For a **$\boldsymbol{\delta}$-fixed** ordering $\prec$

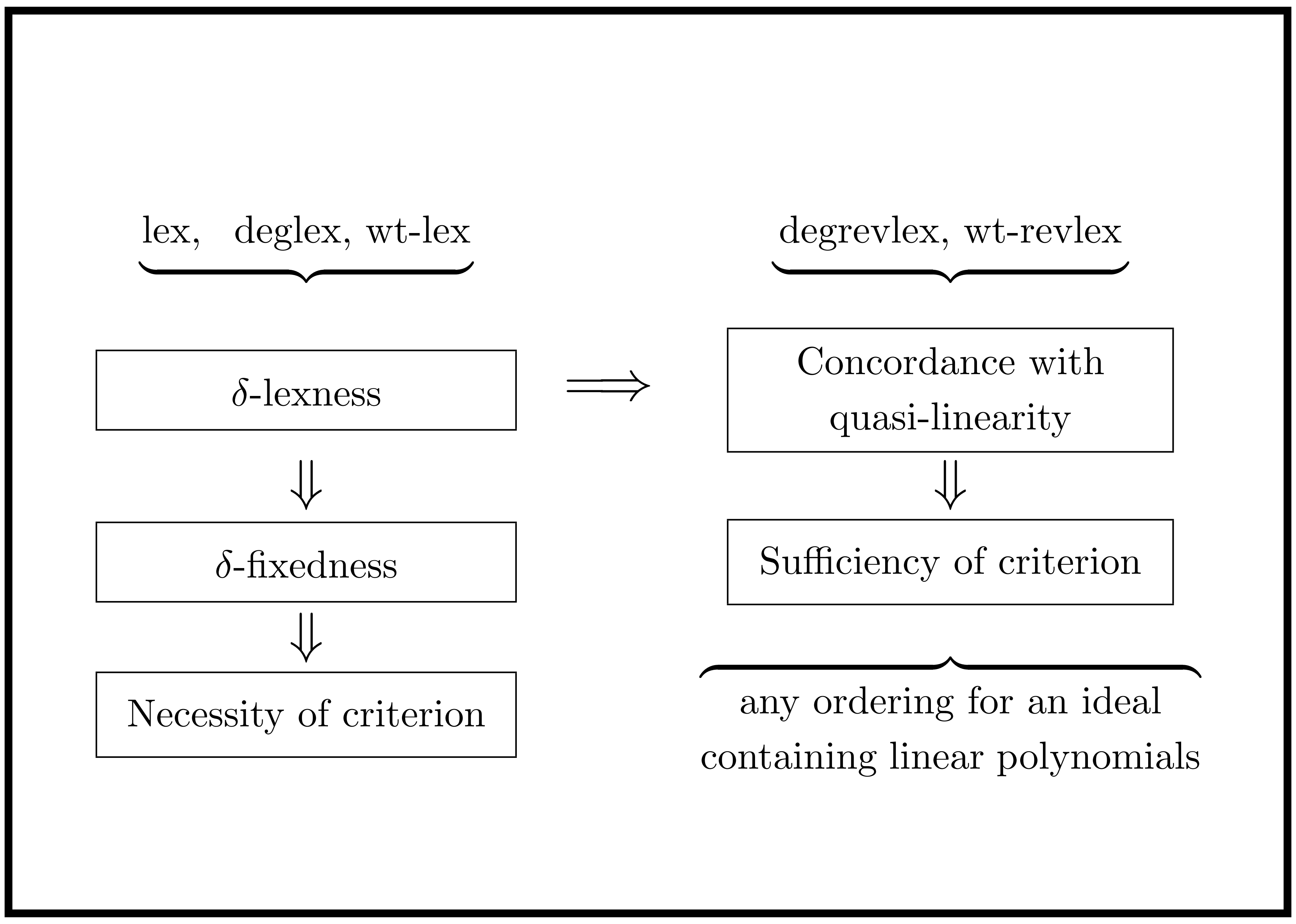
I has a finite DSB w.r.t. $\prec$	$\Rightarrow$	I contains a $\prec$-quasi-linear polynomial

Sufficient condition.

For a **concordant with quasi-linearity** ordering $\prec$

I has a finite DSB w.r.t. $\prec$	$\Leftarrow$	I contains a $\prec$-quasi-linear polynomial

Corollary. For **$\boldsymbol{\delta}$-lexicographic** orderings the condition is necessary and sufficient.


lex, deglex, wt-lex
δ-lexness
δ-fixedness
Necessity of criterion
degrevlex, wt-revlex
Concordance with quasi-linearity
Sufficiency of criterion
any ordering for an ideal containing linear polynomials

Corollaries

<u>GENERALIZATIONS OF G. CARRÀ FERRO'S THEOREMS</u>:

Corollary. Let $\prec$ be **δ-fixed**.
If the degree of each monomial in $f_1, \ldots, f_n$ is greater than 1 then $[f_1, \ldots, f_n]$ has no finite DSB w.r.t. $\prec$.

Corollary. Let $\prec$ be **strictly δ-stable**. The reduced DSB of $[f]$ w.r.t. $\prec$ consists of f itself $\iff$ f is $\prec$-quasi-linear.

<u>KEY ROLE OF **lex**</u>:

A DSB w.r.t. a **δ-fixed** ordering is finite	$\Rightarrow$	A **lex** DSB is also finite

Improved Ollivier process

Implementation in Maple: http://shade.msu.ru/~difalg/DSB.

Input:
$F \subset \mathcal{F}\{y\}$, a finite set of polynomials;
$\prec$, a δ-fixed admissible ordering
that is concordant with quasi-linearity.

Output:
Reduced differential standard basis of $[F]$ if it is finite.
Otherwise the process does not stop.

Improved Ollivier process (ctd.)

$G := F; \quad H := \varnothing;$
$s := \max_{f \in F} \operatorname{ord} f; \quad k := 0;$
repeat
  $G_{old} := \varnothing;$
  while $G \neq G_{old}$ **do**
    $H :=$ **Diff Complete** $(G, s + k);$
    $G_{old} := G;$
    $G :=$ **ReducedGröbnerBasis** $(H, \prec);$
  end do;
  $k := k + 1;$
until $G \subset \mathcal{F}$ or G contains a quasi-linear polynomial;
return DiffAutoreduce $(G, \prec);$

Finite bases: an example

Fix the **pure lexicographic** ordering.

Consider the DSB of the ideals $[y_1^n + y],\ n \geq 3$:

- $\boldsymbol{y_1^n} + y_0$;
- $n\,\boldsymbol{y_0}\,\boldsymbol{y_2} - y_1^2$;
- $n\,\boldsymbol{y_1^{n-2}}\,\boldsymbol{y_2^2} + y_2 = y_2\,(n\,y_1^{n-2}\,y_2 + 1)$;
- $\boldsymbol{y_3} - n(n-2)\,y_1^{n-3}\,y_2^3$.

The DSB are finite, since $[y_1^n + y]$ contains a quasi-linear polynomial.

By the way, one can prove that these ideals are radical.

Ideal of separants

For a differential ideal I let $S_I := \{S_h \mid h \in I, h \notin \mathcal{F}\} \cup \{0\}$.
Proposition.

- S_I is a (non-differential) ideal in $\mathcal{F}[y, y_1, y_2, \ldots]$. It is called the ideal of separants of I.
- $S_I = 1$ iff I contains a quasi-linear polynomial.
- For any differential polynomial $f \in \mathcal{F}\{y\} \setminus \mathcal{F}$ we have

$$[f] + (S_f) \subset S_{[f]} \subset [f] : S_f^{\infty} + (S_f).$$

Finite DSB and radical ideals

Let $\prec$ be a δ-fixed and concordant with quasi-linearity ordering.

If **ord** $\boldsymbol{f = 0}$ then the following are equivalient:

- $[f]$ has a finite DSB w.r.t. $\prec$;
- $[f]$ is radical;
- f is square-free.

Example.
For $f = ay + b$, where $a, b \in \mathcal{F}$, $[f]$ has a finite lex-DSB $\{f\}$, while $[y^2]$ has not.

Let $f = \sum_{i=0}^{d} Q_i(y) y_1^i \in \mathcal{F}[y, y_1]$ be a **first order** diff. polynomial.

Let $S_f = \sum_{i=1}^{d} i Q_i(y) y_1^{i-1}$ be the separant of f.

The ideal $[f]$ has a finite DSB iff

- $[f] : S_f^{\infty} + (S_f) = 1$, and
- $Q_2 \in \sqrt{(Q_0, Q_1)}$, and
- (Q_0, Q_1^2) is square-free.

Kolchin proved (1941) that in these cases the ideal $[f]$ is radical.
We conjecture that in the contrary case $[f]$ is not radical
(we proved it in most subcases).

Example.
Let $f_{m,n} = (y_1 + 1)^m - c\,y^n, \quad c \in \mathcal{F},\ c \neq 0.$
Then $[f_{m,n}]$ is radical and has a finite lex-DSB iff $m \nmid n$.

In this case $[f_{m,n}]$ contains a quasi-linear polynomial of order $[\frac{n}{m}] + 3$.

For higher orders and for non-principal differential ideals the theorem does not work:

Example.
Consider $f = (y_2 + 1)^2 + y$. We have $\operatorname{ord} f = 2$. The ideal $[f]$ is radical, but has no finite lex-DSB.

Example.
The ideal $[y^2, y_1]$ has a finite lex-DSB, but it is not radical.

Other orderings: a conjecture

Conjecture (M. V. Kondratieva, A. Zobnin).
A proper ideal I has a finite DSB w.r.t.
a concordant with quasi-linearity β-ordering $\prec$ iff either

- I contains a $\prec$-quasi-linear polynomial, or
- $I = [f^p]$, where f is $\prec$-quasi-linear and $p \geq 1$.

The sufficiency ($\Longrightarrow$) is easy to prove

The necessity ($\Longleftarrow$) is still an open problem.

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