Two association schemes on 40 and 64 points: A supplement to the paper by Bannai-Bannai-Bannai.

Matan Ziv-Av
(Jointly with Mikhail Klin)

Department of Mathematics
Ben-Gurion University of the Negev

Beer Sheva, Israel

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1. Introduction

A few weeks ago we became acquainted with a preliminary version of a paper [1] by Eiichi, Etsuko and Hideo Bannai. In this paper two association schemes are investigated.

The scheme $\mathfrak{M}_1$ is a Schurian imprimitive scheme with 4 classes of valency 4, 8, 3, 24. It has automorphism group ${}^{(1)}G$ of order 1920. This group ${}^{(1)}G$ is a subgroup of index 2 in the wreath product $S_5 \wr S_2$ of order $2^5 \cdot 5!$. ${}^{(1)}G$ has well-known faithful permutation actions of degree 10, 16 and 32. In particular, action of degree 16 corresponds to the automorphism group of the famous Clebsch graph.

The scheme $\mathfrak{M}_2$ on 64 points is a Schurian imprimitive scheme with 3 classes of valency 7, 14, 42. Its automorphism group ${}^{(2)}G$ has order $2^6 \cdot 336 = 21504$ and can be described as a semidirect product $(\mathbb{Z}_4)^3 \rtimes (L(3,2) \times \mathbb{Z}_2)$.

2. Scheme on 40 points

2.1 Initial description

The 40 points association scheme with 4 classes was introduced in [4] via its geometric representation in $\mathbb{R}^{10}$. This scheme is related to a possible universally optimal code in $\mathbb{R}^{10}$. We denote it $\mathfrak{M}_1$. This is an imprimitive association scheme with two kinds of blocks (in systems of imprimitivity) of size 4 and 8.

Its intersection numbers are given by:

$A_1^2 = 3A_0 + 2A_1$ $\qquad A_2^2 = 4A_0 + 4A_1$

$A_3^2 = 8A_0 + 2A_2 + 2A_4$

$A_4^2 = 24A_0 + 16A_1 + 18A_2 + 12A_3 + 14A_4$

$A_1A_2 = 3A_2$ $\qquad A_1A_3 = A_4$

$A_1A_4 = 3A_3 + 2A_4$ $\qquad A_2A_3 = A_3 + A_4$

$A_2A_4 = 3A_3 + 3A_4$

$A_3A_4 = 8A_1 + 6A_2 + 6A_3 + 4A_4$

[1] contains a proof of the uniqueness of this scheme (up to isomorphism) provided that the parameters are as above.

Using GAP, found that $G = Aut(\mathfrak{M}_1)$ is a rank 5 transitive permutation group of order 1920, with sub-degrees 1,4,8,3,24. Using generators of G with the aid of COCO we found that $\mathfrak{M}_1$ has (besides the obvious mergings corresponding to imprimitive SRG's $10 \circ K_4$ and $5 \circ K_8$) a primitive merging with 2 classes. One of the classes is rank 3 SRG, with the parameters (40,12,2,4) with automorphism group of order 51840. This graph corresponds to the symplectic generalized quadrangle of order 3.

The automorphism group G of $\mathfrak{M}_1$ was identified with the aid of GAP. It turns out that it is a subgroup of index 2 in the wreath product $S_5 \wr S_2$ of order $2^5 5!$.

This information, which was obtained with the use of computer, served for us as a source for a computer free explanation of $\mathfrak{M}_1$ which is given below. Due to exceptional properties of $\mathfrak{M}_1$, we hope that this explanation is of a certain independent interest.

2.2 Auxiliary structures

2.2.1 Configuration 8_3

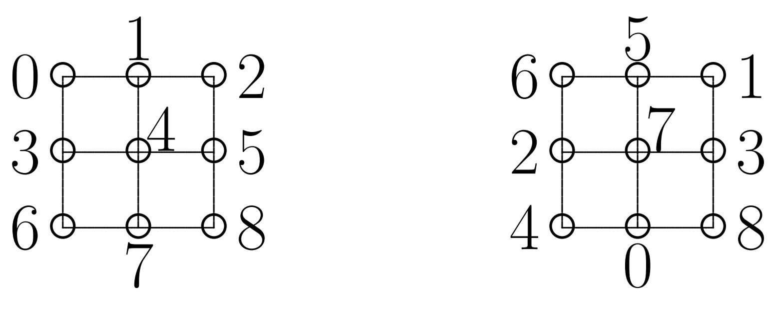
First we describe axiomatics of an incidence structure 8_3. It has 8 points and 8 blocks, each block contains 8 points. Any two points are incident together to at most one block.

Proposition 1 Point graph of 8_3 is isomorphic to $\overline{4 \circ K_2}$.

Proposition 2 Configuration 8_3 is unique up to isomorphism.

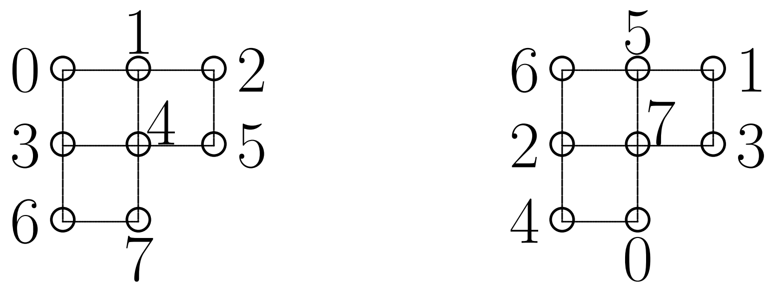
We will consider a few different models for 8_3 which seem to be isomorphic.

Model 1: (Punctured affine plane of order 3). We take the affine plane of order 3 as follows:



Remove from it one point (in our case 8) and all the

4 lines (blocks) through it. What remains is 8_3.



Model 2: (non-zero vectors of $(\mathbb{Z}_3)^2$). We take the 8 non-zero vectors of $(\mathbb{Z}_3)^2$ as points. Every equation of the form $ax + by = 1$ (a or b is not zero) defines a block of 3 points $(x, y) \in (\mathbb{Z}_3)^2$. This incidence structure is 8_3. Another definition of the blocks in this model is: $U + v$, where U is a dimension 1 subspace, and $v \notin U$. This definition gives immediately:

Proposition 3: $H = Aut(8_3) \cong GL(2, 3)$

2.2.2 Point graph of 8_3

The point graph of 8_3 is $\overline{4 \circ K_2}$, with automorphism group $S_4 \wr S_2$ of order $4!2^4 = 384$. Since $|H| = 48$, we get

Proposition 4 The same copy of $\overline{4 \circ K_2}$ is the point graph of 8 different copies of 8_3.

Two copies of 8_3 that have the same point graph are called *commonly antipodal* copies.

Two blocks like 025 and 134 in the 8_3 configuration:

$$\{025, 047, 134, 156, 357, 036, 127, 246\}$$

(that is, contain ends of the same 3 edges of a $4 \circ K_4$ graph) are called *antipodal blocks*.

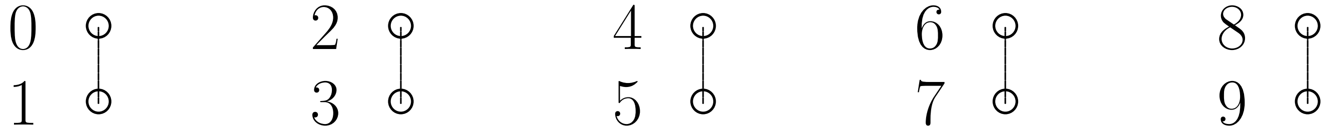
Proposition 5

a) There is exactly one copy of 8_3 with a prescribed point graph, that includes a prescribed pair of non antipodal blocks (which agree with this graph).

b) Two commonly antipodal copies of 8_3 either have in common a pair of antipodal blocks, or do not have blocks in common.

2.2.3 Embedding of $4 \circ K_2$ *to* $5 \circ K_2$

Let us take a graph $5 \circ K_2$:



as our canonical graph. A block from now on means a triplet of independent vertices of this graph. Let Ω be the set of all copies of 8_3 that have a subgraph of this graph as the complement of their point graph. $|\Omega| = 40$.

There are 40 pairs of antipodal blocks, and any pair is in exactly 4 copies of 8_3. Let $\mathfrak{B}$ be the set of pairs of antipodal blocks. A pair of antipodal blocks and a copy of 8_3 are incident if the blocks forming the pair are blocks of the copy of 8_3.

2.3 Structure GQ(3)

The incidence structure $I = (\Omega, \mathfrak{B})$ is a model of $GQ(3)$, that is a $PG(4, 4, 1)$. It is easy to see from construction that $K = R = 4$. For $T = 1$, we use Proposition 5.

Up to isomorphism, there are two models of $GQ(3)$, which are dual to each other [7], $W(3)$ and $Q(4, 3)$. They are distinguishable by the fact that $W(3)$ contains spreads, but $Q(4, 3)$ does not.

Proposition 6 Our incidence structure I is isomorphic to $W(3)$.

To find a spread, we can take the 10 pairs of antipodal blocks such that each of them includes one of the 10 blocks in $\left\{ {\{0,2,4,6,8\} \atop 3} \right\}$.

2.4 Color graph

We'll define 4 relations on Ω as follows: a pair of configurations (a, b) belongs to:

R_1 if they have the same point graph and they have a common block.

R_2 if they do not have the same point graph and do have a common block.

R_3 if they have the same point graph and do not have a common block.

R_4 if they do not have the same point graph and do not have a common block.

$\mathfrak{M}' =< \Delta, R_1, R_2, R_3, R_4 >$ is the centralizer algebra of $S_5 \wr S_2$, and therefore an association scheme.

By direct counting, we see that the intersection numbers of this scheme are the same as those of $\mathfrak{M}_1$ from [1], and therefore, by the uniqueness, the schemes are isomorphic.

2.5 $Aut(\mathfrak{M}')$

The automorphism group of $5 \circ K_2$ is $S_5 \wr S_2$. This automorphism group does not act faithfully on Ω, since the permutation $(0,1)(2,3)(4,5)(6,7)(8,9)$ acts as the identity map. Any other permutation acts non identically on Ω.

Since the order of a stabilizer of a point in this scheme is at most 48, it follows that the induced action of $S_5 \wr S_2$ on Ω is $Aut(\mathfrak{M}')$.

3. Scheme on 64 points

3.1 Initial description D. de Caen and E. R. van Dam have introduced in [2] an infinite series of imprimitive association schemes with 2^{4t+2} points and 5 classes. Such association scheme has many mergings, in particular, the one with classes $R_1 = R'_1 \cup R'_2$, $R_2 = R'_3 \cup R'_4$ and $R_3 = R'_5$.

Later on, Cohn ([4]) considering a potential universally optimal spherical code in $\mathbb{R}^{14}$ became again interested in the above mentioned merging (denoted $\mathfrak{M}_2$) with 3 classes for a particular case $t = 1$.

Let us recall definition of this association scheme: The set X is the set of ordered pairs of elements of $\mathbb{F}_8$, $X = \mathbb{F}_8 \times \mathbb{F}_8$. If $(\alpha, x), (\beta, y) \in X$ then:

$((\alpha, x), (\beta, y)) \in R_0 \iff \alpha = \beta$ and $x = y$.

$((\alpha, x), (\beta, y)) \in R_1 \iff \alpha \neq \beta$ and $x = y$.

$((\alpha, x), (\beta, y)) \in R_2 \iff x \neq y$ and either $\alpha + \beta = (x + y)^3$ or $\alpha + \beta = xy(x + y)$.

R_3 is the complement of $R_0 \cup R_1 \cup R_2$.

Its intersection numbers are given by the following algebraic relations:

$$\begin{aligned} A_1^2 &= 7A_0 + 6A_1 \\ A_2^2 &= 14A_0 + 2A_1 + 4A_3 \\ A_3^2 &= 42A_0 + 30A_1 + 24A_2 + 28A_3 \\ A_1A_2 &= A_2 + 2A_3 \\ A_1A_3 &= 6A_2 + 5A_3 \\ A_2A_3 &= 12A_1 + 12A_2 + 8A_3 \end{aligned}$$

Bannai, Bannai and Bannai became interested in this scheme, giving in [1] its geometric representation in the unit sphere in $\mathbb{R}^{14}$. They proved that this scheme $\mathfrak{M}_2$ is uniquely characterized by its parameters. Their proof essentially using the Gram matrix corresponding to geometric realization.

We are interested in investigating properties of $Aut(\mathfrak{M}_2)$ and to get in this way an alternative description of $\mathfrak{M}_2$.

3.2 Computer results and first observations

Using GAP we found that the automorphism group $G = Aut(\mathfrak{M}_2)$ has order $2^6 \cdot 336 = 21504$. Moreover, we discovered that G has a normal subgroup, which is regular and isomorphic to $(\mathbb{Z}_4)^3$. For arbitrary point of $\mathfrak{M}_2$, its stabilizer in G is isomorphic to a group H of order 336. H has 4 orbits on points of $\mathfrak{M}_2$, of lengths 1, 7, 14, 42. In other words, association scheme $\mathfrak{M}_2$ is Schurian scheme with 3 classes. Group H acts faithfully on suborbits of length 14 and 42, while action of H on 7 points is not-faithful and is similar to the automorphism group of Fano plane, that is to simple group PGL(3,2) of order 168.

3.3 Merging of cyclotomic scheme over $(\mathbb{Z}_4)^3$

Group $G = Aut((\mathbb{Z}_4)^3) = GL(3, \mathbb{Z}_4)$ has order $86016 = 2^{12} \cdot 3 \cdot 7$. G has one conjugacy class of subgroups of order 7. Therefore, we can use the uniquely defined semidirect product $(\mathbb{Z}_4)^3 \rtimes \mathbb{Z}_7$. The association scheme

$$< (\mathbb{Z}_4)^3, (2 - orb((\mathbb{Z}_4)^3 \rtimes \mathbb{Z}_7, (\mathbb{Z}_4)^3) >$$

has 9 classes. This is a *cyclotomic scheme* over $(\mathbb{Z}_4)^3$.

This association scheme has 4 isomorphic mergings to a scheme of 3 classes: leaving the symmetric class of valency 7, merging together two antisymmetric classes to obtain a class of valency 14, and merging the other 6 classes to obtain a class of valency 42. We denote one of those fusion schemes by $\mathfrak{M}'_2$.

Proposition 7 $\mathfrak{M}'_2 \cong \mathfrak{M}_2$.

The structure constants of $\mathfrak{M}'_2$ are the same as for $\mathfrak{M}_2$, and uniqueness of $\mathfrak{M}_2$ was proved in [1].

4 Concluding remarks

- We found that using information (uniqueness, automorphism groups), which is not easily available without use of a computer can finally help us to find different constructions for the investigated schemes, and provide computer free proofs of those facts.
- It is not clear if $\mathfrak{M}_1$ can be embedded into an infinite series of interesting objects.
- Regarding $\mathfrak{M}_2$, the cyclotomic model may be extended to higher dimensions.
- Other constructions of $\mathfrak{M}_2$ were found, including:
  - construction of the automorphism group as a semidirect product of $(\mathbb{Z}_3)^4$ and $L(3,2) \times \mathbb{Z}_2$;
  - a model based on Fano plane;
  - construction from the imprimitivity set.
- We were informed by Eiichi Bannai that Kanat

Abdukhalikov reached independently many similar results about $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and has some interesting ideas about generalization of $\mathfrak{M}_2$.

Acknowledgments

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