Multidimensional behaviors: polynomial-exponential trajectories and linear exact modeling

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What is a multidimensional behavior?

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- what kind of signals?
- how many?
- how are they related?

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Formally:

- a signal set $\mathcal{A}$
- a signal number q
- a set $\mathcal{B} \subseteq \mathcal{A}^q$

In this talk: $\mathcal{B}$ ... smooth solution space of a linear, constant-coefficient system of PDE

$\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$ and

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$$

where $R \in \mathcal{D}^{g \times q}$ for $\mathcal{D} = \mathbb{K}[\partial_1, \ldots, \partial_n]$

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$\mathcal{B}$ autonomous $\Leftrightarrow$ there are no free variables in $\mathcal{B}$ $\Leftrightarrow$ no component of w is unconstrained by the system law $Rw = 0$ $\Leftrightarrow$ $\mathrm{rank}(R) = q$ [Oberst]

Algebraic characterization [Pommaret & Quadrat]:
the module $\mathcal{M} = \mathcal{D}^{1 \times q}/\mathcal{D}^{1 \times g}R$ is torsion
Here (constant coeff.) also equivalent: $\mathrm{ann}(\mathcal{M}) \neq 0$

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$$\mathcal{B} = \{w \in \mathfrak{D}'(\mathbb{R}, \mathbb{K}) \mid t^3\dot{w} + w = 0\} = 0$$

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There exists a smooth function u such that [Fröhler & Oberst]

$$(1 - t^2)^2\dot{y} = 2t(3 - 2t^2)y + u$$

has no solution $y \in \mathfrak{D}'(\mathbb{R}, \mathbb{K})$
Thus both u and y are constrained by the system law, although the module $\mathcal{M}$ is not torsion

Possible fixes: rational coeff, hyperfct / a.e. smooth fct

Poles

Let $\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}$ be autonomous, i.e.,

$$\text{rank}(R) = q$$

$\lambda \in \mathbb{C}^n$ is called a pole of $\mathcal{B}$ [Wood, Oberst et al.] $\Leftrightarrow$
$\mathcal{B}$ contains an exponential trajectory of frequency λ, i.e.,

$$\exists 0 \neq c \in \mathbb{C}^q : \quad w = c \exp_\lambda \in \mathcal{B}$$

where

$$\exp_\lambda(t) = \exp(\lambda_1 t_1 + \ldots + \lambda_n t_n)$$

for all $t \in \mathbb{R}^n$
(this is for $\mathbb{K} = \mathbb{C}$, appropriate modification for $\mathbb{K} = \mathbb{R}$)

Since $\partial_i \operatorname{exp}_\lambda = \lambda_i \operatorname{exp}_\lambda \Rightarrow$

$$Rc \operatorname{exp}_\lambda = R(\lambda) c \operatorname{exp}(\lambda)$$

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Algebraically speaking: $\lambda \in V = \mathcal{V}(\mathcal{F}_0(\mathcal{M})) = \mathcal{V}(\operatorname{ann}(\mathcal{M})) \subset \mathbb{C}^n$

$\mathcal{F}_0$ . . . 0-th Fitting ideal (generated by $q \times q$ minors of R)
V . . . the pole variety of $\mathcal{B}$

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Recall: by assumption, $\mathcal{B}$ autonomous, i.e., $V \neq \mathbb{C}^n$

Known from the 1d case: not sufficient to consider only exponential solutions, one has to admit polynomial-exponential solutions

$$w = p \,\mathsf{exp}_\lambda, \quad p \in \mathbb{C}[t_1, \ldots, t_n]^q$$

and their sums (here again $\mathbb{K} = \mathbb{C}$)

$\mathcal{B}$ has only polynomial-exponential trajectories $\Leftrightarrow$
$\mathcal{B}$ is finite-dimensional as a $\mathbb{K}$-vector space

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For 1d systems: finite-dimensional $\Leftrightarrow$ autonomous

In nd: finite-dimensional $\Rightarrow$ autonomous

but $\not\Leftarrow$, e.g. $\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{K}) \mid \partial_1 w = 0\}$
is autonomous, but not finite-dimensional

Finite-dimensional systems

$\mathcal{B}$ finite-dimensional $\Leftrightarrow$
$\exists \nu \in \mathbb{N}, A_i \in \mathbb{K}^{\nu \times \nu}, C \in \mathbb{K}^{q \times \nu}$ such that

$$w \in \mathcal{B} \quad \Leftrightarrow \quad \exists x \in \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{K}^{\nu}) : \left\{ \begin{array}{ccc} \partial_1 x & = & A_1 x \\ \vdots & & \vdots \\ \partial_n x & = & A_n x \\ w & = & Cx \end{array} \right.$$

Moreover, $A_i A_j = A_j A_i$ for all i, j

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Construction: companion matrices of zero-dim. module $\mathcal{M}$

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Construction: companion matrices of zero-dim. module $\mathcal{M}$

Then $w \in \mathcal{B} \Leftrightarrow$ for some $x_0 \in \mathbb{K}^\nu$:

$$w(t) = C \exp(A_1 t_1 + \ldots + A_n t_n) x_0 \text{ for all } t \in \mathbb{R}^n$$

If $\nu = \dim_{\mathbb{K}} \mathcal{B}$, this yields a basis of $\mathcal{B}$
Equivalent: $\bigcap_{\mu \in \mathbb{N}^n} \ker C A^\mu = 0$ (observability)

Algebraic characterization of finite-dimensional systems:

- module $\mathcal{M}$ has Krull dimension zero
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Caution: $\mathcal{D} = \mathbb{K}[\frac{d}{dt}, \sigma, \sigma^{-1}]$, σ ... shift operator
$\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{K})$

$$\mathcal{B} = \{w \in \mathcal{A} \mid \frac{dw}{dt} = 0\}$$

is finite-dimensional, but $\mathcal{M} = \mathcal{D}/\langle \frac{d}{dt} \rangle$ has Krull dim. 1

Reason: $(\sigma - 1)w = 0$ is analytic consequence of the system law, but not an algebraic consequence
"true" system module $\mathcal{D}/\langle \frac{d}{dt}, \sigma - 1 \rangle$ has Krull dim. 0

Polynomial-exponential trajectories of $\mathcal{B}$ w.r.t. a fixed pole
. . . local properties of $\mathcal{B}$ at λ

Decide whether total degree of polynomial part is bounded
(in 1d: always true, but not in nd)
multiplicity of a pole $\mu(\lambda) \in \mathbb{N} \cup \{\infty\}$ . . . $\mathbb{K}$-dim of poly part

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Geometrically: the multiplicity of λ is finite $\Leftrightarrow$
λ is an isolated point in the pole variety V $\Leftrightarrow$
$\lambda \notin \overline{V \setminus \{\lambda\}}$

Computationally: $(\mathbb{K} = \mathbb{C})$
Compute S with $(\mathcal{D}^{1\times g}R : \mathfrak{m}_\lambda^\infty) = \mathcal{D}^{1\times h}S$
Test whether $S(\lambda)$ has full column rank
If yes, then $\mu(\lambda) < \infty$ [Sturmfels, $q = 1$]

In fact: $\mu(\lambda) = \dim_{\mathbb{K}} \mathcal{M}_\lambda$ (localization)

Consider

$$\mathcal{B}_{\lambda,d} = \{w \mid Rw = 0, (\partial - \lambda)^{\mu} w = 0 \forall \mu \in \mathbb{N}^n : |\mu| = d\}$$

. . . polynomial-exponential trajectories with frequency λ and total degree of polynomial part $\leq d - 1$

$$\mathcal{B}_{\lambda} = \bigcup_{d \geq 0} \mathcal{B}_{\lambda,d}$$

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If $\mu(\lambda) < \infty$: $\exists d^*$: $\mathcal{B}_\lambda = \mathcal{B}_{\lambda,d^*}$

Thus, we can compute a basis of $\mathcal{B}_\lambda$

$\mu(\lambda) = \dim_{\mathbb{K}} \mathcal{B}_\lambda$

If $\mu(\lambda) = \infty$:

determine the growth of $\dim_{\mathbb{K}} \mathcal{B}_{\lambda,d}$ as $d \to \infty$

Result: for large d, $\dim_{\mathbb{K}} \mathcal{B}_{\lambda,d}$ is polynomial of degree $\dim(\mathcal{M}_\lambda)$

(whose first difference equals the Hilbert polynomial
in the homogeneous case)

Example: Cauchy-Riemann equations

$$\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2) \mid \partial_1 w_1 = \partial_2 w_2, \partial_1 w_2 = -\partial_2 w_1\}$$

$$R = \begin{bmatrix} \partial_1 & -\partial_2 \\ \partial_2 & \partial_1 \end{bmatrix}$$

$\mathcal{B}$ is autonomous

$\lambda \in \mathbb{C}^2$ pole $\Leftrightarrow \lambda_1^2 + \lambda_2^2 = 0$

e.g. $\lambda = (0,0)$ has multiplicity ∞

and $\dim_{\mathbb{R}} \mathcal{B}_{(0,0),d} = 2d$

Poles of non-autonomous systems

Given $\mathcal{B}$, let $\mathcal{B}_c$ be its controllable part
. . . the largest controllable subsystem of $\mathcal{B}$

Consider the poles of $\mathcal{B}/\mathcal{B}_c$ (which is autonomous)
. . . uncontrollable poles of $\mathcal{B}$ [Wood, Oberst et al.]

Algebraically: $\mathcal{B}/\mathcal{B}_c$ . . . torsion submodule of $\mathcal{M}$
[Pommaret & Quadrat]

Compare: Kalman decomposition of a state space system

So far: Autonomous system $\rightsquigarrow$ investigate the poly-exp solutions

Now: Inverse problem

Given poly-exp functions, find a system that may have generated them

. . . modeling from data / system identification

Model class: Systems given by linear PDE with complex coeff.

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Rw = 0\}, \quad R \in \mathcal{D}^{g \times q}, \quad \mathcal{D} = \mathbb{C}[\partial_1, \ldots, \partial_n]$$

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Modeling problem:

Given polynomial-exponential functions $w_1, \ldots, w_N$, that is,

$$w_i(t) = p_i(t) \exp_{\lambda_i}(t) \quad \text{for all } t \in \mathbb{R}^n$$

for some $p_i \in \mathbb{C}[t_1, \ldots, t_n]^q$ and $\lambda_i \in \mathbb{C}^n$

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find a model $\mathcal{B}^\star$ in the model class that

- explains the data, that is, $\mathcal{B}^\star \ni w_i \forall i$
- is as small as possible, that is,

$$\mathcal{B} \ni w_i \forall i \quad \Rightarrow \quad \mathcal{B} \supseteq \mathcal{B}^\star$$

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$\mathcal{B}^\star$ . . . most powerful unfalsified model (MPUM)
[Antoulas & Willems, $n = 1$]

Remark: there is no loss of generality in assuming that

$$w(t) = p(t)\,\mathsf{exp}_\lambda(t)$$

instead of

$$w(t) = \sum_{\text{finite}} p_i(t)\,\mathsf{exp}_{\lambda_i}(t)$$

since finite-dimensional systems satisfy

$$\mathcal{B} = \bigoplus_{\lambda \in V} \mathcal{B}_\lambda$$

and the MPUM is finite-dimensional, because it equals the span of the given functions and their derivatives

Construction of the MPUM

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$$\mathbb{C}[\partial_1, \ldots, \partial_n]/\mathfrak{m}_\lambda^{d+1} \cong \mathbb{C}^\delta$$

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Let $A_1, \ldots, A_n \in \mathbb{C}^{\delta \times \delta}$ be pairwise commuting matrices such that multiplication by ∂_i corresponds to multiplication by A_i
(companion matrices) $\mathrm{spec}(A_i) = \{\lambda_i\}$, $\exists z \neq 0 : A_i z = \lambda_i z \ \forall i$

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Then:

$$v \in \mathrm{span}_{\mathbb{C}}\{\partial^\mu w \mid \mu \in \mathbb{N}^n\} \Leftrightarrow \exists x_0 \in \mathbb{C}^\delta : v(t) = C\,\exp(\sum_{i=1}^{n} A_i t_i) x_0$$

where $C \in \mathbb{C}^{q \times \delta}$ is obtained from the coeff. of p by book-keeping

Example: $w(t_1, t_2) = 2 + t_1 - t_2$, $n = 2$, $q = 1$, $\lambda = 0$, $d = 1$

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$\mathbb{C}[\partial_1,\partial_2]/\langle\partial_1,\partial_2\rangle^2 \cong \mathbb{C}^3 :\ [1] \leftrightarrow e_1,\ [\partial_1] \leftrightarrow e_2,\ [\partial_2] \leftrightarrow e_3$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = [2, 1, -1]$$

$$C \exp(A_1 t_1 + A_2 t_2) = [2, 1, -1] \begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{bmatrix} = [w, \partial_1 w, \partial_2 w]$$

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$$C \exp(A_1 t_1 + A_2 t_2) = [2, 1, -1] \begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{bmatrix} = [w, \partial_1 w, \partial_2 w]$$

Thus
$v \in \mathrm{span}_{\mathbb{C}}\{w, \partial_1 w, \partial_2 w\} \Leftrightarrow v(t) = C \exp(A_1 t_1 + A_2 t_2) x_0$ for some x_0

So far:

$$v \in \mathcal{W} := \mathrm{span}_{\mathbb{C}}\{\partial^{\mu} w \mid \mu \in \mathbb{N}^n\} \Leftrightarrow \exists x_0 \in \mathbb{C}^{\delta} : v(t) = C \exp(\sum_{i=1}^{n} A_i t_i) x_0$$

But $\mathcal{W}$ must be contained in any model of w in the model class

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But $\mathcal{W}$ must be contained in any model of w in the model class

To show that $\mathcal{W} = \mathcal{B}^\star$, we need to find $R^\star$ such that

$$R^\star v = 0 \quad \Leftrightarrow \quad v \in \mathcal{W} \quad \Leftrightarrow \quad \exists x : \begin{cases} \partial_i x &= A_i x \text{ for } 1 \leq i \leq n \\ v &= Cx \end{cases}$$

This is possible due to the fundamental principle [Ehrenpreis, Malgrange, Palamodov] reflecting the injectivity of the $\mathcal{D}$-module $\mathcal{A}$

If $[X_1, \ldots, X_n, Y]$ is a $\mathcal{D}$-matrix whose rows generate the left kernel of

$$H = \begin{bmatrix} \partial_1 I - A_1 \\ \vdots \\ \partial_n I - A_n \\ C \end{bmatrix}$$

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Example:

$$H = \begin{bmatrix} \partial_1 & 0 & 0 \\ -1 & \partial_1 & 0 \\ 0 & 0 & \partial_1 \\ \partial_2 & 0 & 0 \\ 0 & \partial_2 & 0 \\ -1 & 0 & \partial_2 \\ 2 & 1 & -1 \end{bmatrix}$$

yields

$$R^\star = \begin{bmatrix} \partial_1 + \partial_2 \\ \partial_2^2 \end{bmatrix}$$

... MPUM of $w(t) = 2 + t_1 - t_2$

Note: vector space dimension 2, but model of size 3

We have constructed the MPUM of a single data trajectory

It takes the form $\exists x : \begin{cases} \partial_i x &= A_i x \text{ for } 1 \leq i \leq n \\ v &= Cx \end{cases}$

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It takes the form $\exists x : \begin{cases} \partial_i x &= A_i x \text{ for } 1 \leq i \leq n \\ v &= Cx \end{cases}$

From this, it is easy to construct the MPUM of several data trajectories $w_1, \ldots, w_N$ via

$$x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N)} \end{bmatrix} \quad A_i = \begin{bmatrix} A_i^{(1)} & & \\ & \ddots & \\ & & A_i^{(N)} \end{bmatrix} \quad C = [C^{(1)}, \ldots, C^{(N)}]$$

since the resulting model is equivalent to

$$v \in \mathcal{W}_1 + \ldots + \mathcal{W}_N = \mathrm{span}_{\mathbb{C}}\{\partial^\mu w_i \mid \mu \in \mathbb{N}^n, 1 \leq i \leq N\}$$

which must be contained in any model for $w_1, \ldots, w_N$ in the model class

Recursive update

Suppose that the MPUM representation $R^\star$ for $w_1, \ldots, w_N$ has already been constructed

Given $w_{N+1}(t) = p(t) \exp_\lambda(t)$, how should one adapt the model?

Recursive update

Suppose that the MPUM representation $R^\star$ for $w_1, \ldots, w_N$ has already been constructed

Given $w_{N+1}(t) = p(t)\,\text{exp}_\lambda(t)$, how should one adapt the model?

Step 1: Define $e := R^\star w_{N+1}$
Step 2: Let Γ be the MPUM of e
Step 3: Then $R^\star_{\text{new}} := \Gamma R^\star$ does it

Refinement: By successively adding $(\partial^{\mu} p) \cdot \exp_{\lambda}(t)$ to the model, Step 2 can be reduced to the case where e is purely exponential

$$e(t) = e_0 \exp_{\lambda}(t), \quad e_0 \text{ constant}, \quad \text{wlog } e_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \tilde{e}_0 \end{bmatrix}$$

Then the MPUM of e is

$$\Gamma = \begin{bmatrix} I & 0 & 0 \\ 0 & \partial_1 - \lambda_1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \partial_n - \lambda_n & 0 \\ 0 & -\tilde{e}_0 & I \end{bmatrix}$$

Example:

$$w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t_1}, w_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{t_2}, w_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t_1+t_2}$$

$$A_1 = \mathrm{diag}(0,1,0,1), A_2 = \mathrm{diag}(0,0,1,1), C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

$$R^\star = \begin{bmatrix} -\partial_2^2 + \partial_2 & 0 \\ 0 & \partial_2^2 - \partial_2 \\ -\partial_1 + 1 & \partial_2 - 1 \\ \partial_2 & \partial_1 \end{bmatrix}, \quad \mathsf{dim}_{\mathbb{C}}(\mathsf{MPUM}) = 4$$

Additional data trajectory: $\quad w_5 = \begin{bmatrix} t_1 + t_2 \\ t_1 \end{bmatrix}$

$$R^\star_{\mathsf{new}} = \begin{bmatrix} \partial_1 & & & & \\ \partial_2 & & & & \\ & 1 & & & \\ & & 1 & & \\ -1 & & & 1 & \\ -2 & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \partial_1 & \\ & & \partial_2 & \\ & & & 1 \end{bmatrix} R^\star$$

$$\mathsf{dim}_{\mathbb{C}}(\mathsf{MPUM}_{\mathsf{new}}) = 6$$

Minimality issues

$$\{v \mid v(t) = C\exp(A_1 t_1 + \ldots + A_n t_n)x_0 \text{ for some } x_0 \in \mathbb{C}^\delta\}$$

$A_i \in \mathbb{C}^{\delta\times\delta}$ pairwise commuting, $C \in \mathbb{C}^{q\times\delta}$

δ . . . size of the representation $(A_1, \ldots, A_n, C)$

representation minimal $\Leftrightarrow$ $\nexists$ representation of smaller size $\Leftrightarrow$

$$\bigcap_{\mu} \ker(C A_1^{\mu_1} \cdots A_n^{\mu_n}) = \{0\}$$

(observability)

Direct method (linear algebra) to reduce any given representation to minimality (Kalman decomposition)

Alternative characterization of minimality:

$(A_1, \ldots, A_n, C)$ minimal $\Leftrightarrow$

$$\ker \begin{bmatrix} \lambda_1 I - A_1 \\ \vdots \\ \lambda_n I - A_n \\ C \end{bmatrix} = \{0\}$$

for all $\lambda \in \mathrm{spec}(A_1, \ldots, A_n) := \{\lambda \in \mathbb{C}^n \mid \exists z \neq 0 : A_i z = \lambda_i z \forall i\}$
(Hautus test)

Advantage here: $\mathrm{spec}(A_1, \ldots, A_n)$ is known

Iterative method (linear algebra) reduces any given representation to minimality

Example:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = [2, 1, -1]$$

$$\lambda_1 = \lambda_2 = 0, \quad \text{rank} \begin{bmatrix} A_1 \\ A_2 \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} = 2 < 3$$

in accordance with $\dim_{\mathbb{C}}(\text{MPUM}) = 2$

Minimal representation:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad C = [2, 1]$$

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- combination of computer algebra & linear algebra
  model reduction