Logarithmic $\mathcal{D}$-modules

J.M. Ucha

Special Semester on Gröbner Bases and related methods

Linz. May 16, 2006

The rings

- $R = \mathbf{C}[x_1, \ldots, x_n]$

- $\mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$ convergent power series at 0.

- $W = R\langle \partial_1, \ldots, \partial_n \rangle$ and $\mathcal{D} = \mathcal{O}\langle \partial_1, \ldots, \partial_n \rangle$, the rings extensions generated by the relations

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i r - r \partial_i = \frac{\partial r}{\partial x_i},$$

for $r \in R, \mathcal{O}$ respectively.

Gröbner bases for these rings were developed by Briançon-Maisonobe and F.J. Castro (1984).

Motivation

Problem (Z. Mebkhout, 1996. Still open): Obtain a constructive proof of

$$Ext^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) = 0$$

for a plane curve $f = 0$.

It is useful a presentation of $Ann_{\mathcal{D}}(1/f)$ because

$$\mathcal{O}[1/f] \simeq \mathcal{D} \cdot f^{\alpha} \simeq \mathcal{D}/Ann_{\mathcal{D}}(f^{\alpha}),$$

where α is the smallest integer root of the *Bernstein-Sato* polynomial of f (Bernstein 1972, Björk 1979).

Algorithmic computation of $Ext^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O})$ is in general a difficult task...

Example.- If $f = x^2 + y^3$ you have to prove that for every $\varphi \in \mathcal{O}$ there exist $h_1, h_2 \in \mathcal{O}$ such that

$$(-3y^2\partial_x + 2x\partial_y,\ 3x\partial_x + 2y\partial_y + 5) \bullet \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \varphi.$$

This case is easy. Try instead $f = x^4 + y^5 + xy^4$.

Logarithmic $\mathcal{D}$-modules

Logarithmic $\mathcal{D}$-modules (initiated by Calderón-Narváez) has appeared in a natural way to treat the following problems:

- Presenting $\mathcal{O}[1/f] \simeq Ann_{\mathcal{D}}(f^{\alpha}) \Rightarrow$

- Logarithmic Comparison Theorem.

Presenting $\mathcal{O}[1/f]$

There are algorithms based in Gröbner bases computations in PBW algebras (Oaku-Takayama '1999 and Briançon-Maisonobe '2002) to obtain:

- $Ann_{\mathcal{D}[s]}(f^s) \to Ann_{\mathcal{D}}(f^\alpha)$
- The Bernstein-Sato polynomial (Oaku '1997). More generally Bernstein-Sato ideals.

Direct computation: Great expectations with the *synergy*:

- `slimgb` (Brickenstein-Levandovskyy '2005) in Singular.

- Briançon-Maisonobe algorithm.

Especially in the calculation of Bernstein-Sato ideals (success with two transversal cuspids!) Of course, there are intractable interesting examples.

Logarithmic $\mathcal{D}$-modules produce natural *approximations* to obtain $Ann_{\mathcal{D}[s]}(f^s)$ and $Ann_{\mathcal{D}}(f^\alpha)$.

Logarithmic derivations

Let $D \equiv (f = 0)$ be a divisor (hypersurface) in $X := \mathbf{C}^n$.

K. Saito '1980 introduced the complex $\Omega^\bullet(\log D)$ of holomorphic differential forms with logarithmic poles along D.

For $P \in D$ a vector field $\delta \in Der(\mathcal{O}_P)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_P$. The $\mathcal{O}_P$-module of logarithmic derivations is denoted by $Der(-\log D)_P$.

Idea: If $\delta(f) = af$ then $(\delta + a)(1/f) = 0$

The (left) ideal of $\mathcal{D}$

$$Ann^{(1)}_{\mathcal{D}}(1/f) = \langle \delta + a | \delta \in Der(-\log f), \delta(f) = af \rangle$$

—the ideal generated by operators of order one in the derivatives of the annihilating ideal— is a natural approximation of $Ann_{\mathcal{D}}(1/f)$ if $\alpha = 1$.

For the general case, consider

$$\langle \delta - \alpha a | \delta \in Der(-\log f), \delta(f) = af \rangle.$$

Computation of $Ann_{\mathcal{D}}^{(1)}(1/f)$

Given $f \in R$,

$$(a_1\partial_1 + \cdots + a_n\partial n) \cdot (f) = af \Leftrightarrow$$

$$(a_1, \ldots, a_n, -f) \in Syz\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f\right).$$

For $D \equiv (f = 0)$ and $P \in D$, the computation of $Der(-\log f)_P$ is a commutative computation *valid for the analytical setting*.

Non-logarithmic approximations

Instead of $Ann_{\mathcal{D}}^{(1)}(1/f)$ (order one in the derivatives) you can consider elements that annihilate $1/f$ of any order $l \geq 1$ in the derivatives with commutative calculations (Tajima),

$$Ann_{\mathcal{D}}^{(1)}(1/f) \subseteq Ann_{\mathcal{D}}^{(2)}(1/f) \subseteq \cdots \subseteq Ann_{\mathcal{D}}(1/f).$$

Problem: Is there any bound b such that

$$Ann_{\mathcal{D}}^{(b)}(1/f) = Ann_{\mathcal{D}}(1/f)$$

(say, for any plane curve $f = 0$)?

Returning to the initial problem

Problem ($\star$): When does the equality

$$Ann_{\mathcal{D}}^{(1)}(1/f) = Ann_{\mathcal{D}}(1/f)$$

hold?

Our *debut* on this subject was (Castro-Ucha '2001):

Theorem. *For a plane curve $D \equiv (f = 0)$, $Ann^1_{\mathcal{D}}(1/f) = Ann_{\mathcal{D}}(1/f)$ if and only if f is quasi-homogeneous.*

LCT come on stage

In dimension 2 (only) the following conditions are equivalent:

1) f is quasi-homogeneous

2) D is *Euler homogeneous* (there is a χ with $\chi(f) = f$).

3) D is *Locally Quasi-Homogeneous (LQH)* (for all $P \in D$ there exists a system of local coordinates $(V; x_1, \ldots, x_n)$ centered at P such that $D \cap V$ has a strictly weighted homogeneous defining equation with respect to $(x_1, \ldots, x_n)$)

4) ... "*LCT holds for $f = 0$*" (Calderón-Castro-Narváez-Mond '2002)

Free divisors

On the other hand, every plane curve is a *free* divisor: D is free at $P \in D$ if $Der(-\log D)_P$ is a free $\mathcal{O}_P$-module.

By Saito's criterion $D \equiv (f = 0) \subset \mathbf{C}^n$ is free at P if and only if there exist n vector fields $\delta_i = \sum_{j=1}^n a_{ij}\partial_j \in Der(-\log D)_P$, such that $\det(a_{ij}) = uf$ for a unit $u \in \mathcal{O}_P$

Example.- $D \equiv (f = xyz(x+y)(x+z) = 0)$ is free at 0 because

$$\begin{vmatrix} 0 & 2xy-2xz-5xy-3y^2-2yz & 5xz+2yz+3z^2 \\ x^2+2xz & xy+2yz & -4xz-3z^2 \\ x & y & z \end{vmatrix} = 25f$$

Let us denote by i_D the inclusion morphism $\Omega^\bullet(\log D) \hookrightarrow \Omega^\bullet(\star D)$.

Thm.- If D is a locally quasi-homogeneous *free* divisor then the morphism i_D is a quasi-isomorphism, i.e. i_D induces an isomorphism on cohomology. (Castro-Narváez-Mond '1996)

As Grothendieck's Comparison Theorem proves that the last complex calculates the cohomology of the complement of D in X, we say that *Logarithmic Comparison Theorem (LCT) holds for a divisor* D if the morphism i_D is a quasi-isomorphism.

Following the clues, we proved (Castro-Ucha '2002)

Thm.- If $D \equiv (f = 0)$ is a LQH *free* divisor then $Ann_{\mathcal{D}}^{(1)}(1/f) = Ann_{\mathcal{D}}(1/f)$.

It was a crucial fact that LQH free divisors are *Koszul-free* and then of *Spencer type*: this means that the complex

$$\mathcal{D} \otimes_{\mathcal{O}} \wedge^{\bullet} Der(\log D) \to M^{\log D} \to 0$$

is a (locally) free resolution of

$$M^{\log D} = \mathcal{D}/\langle Der(-\log D)\rangle \simeq (\mathcal{D}/Ann_{\mathcal{D}}^{(1)}(1/f))^{\star},$$

and that these $\mathcal{D}$-modules are holonomic.

Torrelli's gamble: LCT holds for $D \equiv (f = 0)$ if and only if $Ann_{\mathcal{D}}^{(1)}(1/f) = Ann_{\mathcal{D}}(1/f)$.

Torrelli himself has proved the conjecture for Koszul-free divisors and is true for (Spencer) free divisors (Castro-Ucha '2004). Moreover, if LCT holds for a free divisor D then:

• D is of Spencer type (Calderón-Narváez '2005).

• -1 is the smallest root of the Bernstein-Sato polynomial (Castro-Ucha '2004).

Recent advances Some advances have been obtained in this context of LCT and free (and non-free divisors):

1) $Ann_W^{(1)}(1/f)$ and arrangement of hyperplanes.

2) Extending the family of free divisors for which LCT holds.

Arrangements of hyperplanes

We have a method for testing if $Ann_W^{(1)}(1/f) = Ann_W(1/f)$ for arrangements of hyperplanes.

It is based in a combinatorial description of the characteristic cycle of the $\mathcal{D}$-module R_f, due to Àlvarez-Montaner, García–López and Zarzuela (Àlvarez-Montaner, Castro, Ucha '2005 submitted).

Arrangements like

$$(f = xyzt(x+y)(x+z)(x+t)(y+z)(y+t)(z+t)$$
$$(x+y+z)(x+y+t)(x+z+t)(y+z+t)(x+y+z+t) = 0)$$

turn out to verify Walther's conjecture.

Recent advances

2) LCT dos not imply LQH for $n \geq 3$ and there are Euler homogeneous free divisors that does not verify LCT. It is a conjecture that Euler homogeneous is a necessary condition for LCT (true in dimension 3, Granger-Schulze '2005 preprint).

The divisor $D \equiv (xy(x+y)(xz+y) = 0) \subset \mathbf{C}^3$ is not LQH and verifies LCT (it is only a computation!).

Thm.- Let us suppose that $D \equiv (f = 0) \subset \mathbf{C}^n$ is a free Spencer divisor and -1 is the smallest integer root of its Bernstein-Sato polynomial.

If for every $P \in D$ we have a $\chi = w_1 x_1 \partial_1 + \cdots + w_n x_n \partial_n$ such that $\chi(f) = f$ with some $w_i > 0$ then LCT holds for D.

Key.- Explicit computation of Ext groups.