Generalized factorization of PDEs: A tool for finding their closed-form solutions, $dim \geq 2$

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Outline

A teaser: $dim = 2$, $ord = 2$, 1773

$dim = 2$, $ord \geq 3$, 2005

refrain: Gröbner bases, Gröbner bases, ...

$dim \geq 3$, $ord = 2$, 1901–2006

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Other interesting partial results

General theory of factorization of an arbitrary single LPDO

General approach: Abelian categories

Algorithmic problems

Acknowledgment

A teaser: solvable non-factorizable LPDEs, *dim* $= 2$, *ord* $= 2$

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$$u = c_0 F + c_1 F' + \ldots + c_n F^{(n)} + d_0 G + d_1 G' + \ldots + d_{n+1} G^{(n+1)}$$

with definite $c_i(x, y)$, $d_i(x, y)$ and $F(x)$, $G(y)$ — two arbitrary functions.

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How are solutions and factorizations of LPDEs related?

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It has the complete explicit solution (S.Ts., ISSAC'2005):

$$\begin{cases} u_1 = 2e^y G(x) + e^x (3F(y) + F'(y)) + \exp \frac{x+y}{2} H(x-y), \\ u_2 = e^y G'(x) + 2e^x F'(y) - 2u_1, \\ u_3 = D_x u_1 + 3u_1 - 2(e^y G'(x) + 2e^x F'(y)), \end{cases}$$

where $F(y)$, $G(x)$ and $H(x-y)$ are three arbitrary functions of one variable each.

Technology (Ts., ISSAC'2005): generalized Laplace transformations

For this system the transformation is:

$$\begin{cases} \overline{u}_1 = u_1, \\ \overline{u}_2 = u_2 + 2u_1, \\ \overline{u}_3 = ((D_x + D_y)u_1 - u_1 - 2u_2 - 4u_1). \end{cases}$$

The transformed system:

$$\begin{cases} D_x \overline{u}_3 = \overline{u}_3, \\ D_y \overline{u}_2 = 2\overline{u}_3 + \overline{u}_2, \\ (D_x + D_y)u_1 = \overline{u}_3 + 2\overline{u}_2 + u_1. \end{cases}$$

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Alternative technology (F.Schwarz, 2005):

Transform the system into Janet (Gröbner) base, with term order: LEX, $u_3 > u_2 > u_1, x > y$:

$$u_{1,xxy} - u_{1,xx} + u_{1,xyy} - 3u_{1,xy} + 2u_{1,x} - u_{1,yy} + 2u_{1,y} - u_1 = 0,$$
$$u_{2,y} + 3u_2 - 2u_{1,x} + 8u_1 = 0,$$
$$u_{2,x} - u_2 - \tfrac{1}{2}u_{1,xx} - \tfrac{1}{2}u_{1,xy} + 3u_{1,x} + \tfrac{1}{2}u_{1,y} - \tfrac{5}{2}u_1 = 0,$$
$$u_3 + 2u_2 - u_{1,x} + u_1 = 0.$$

The first equation *factors* (!!):

$$D_x^2 D_y - D_x^2 + D_x D_y^2 - 3D_x D_y + 2D_x - D_y^2 + 2D_y - 1$$

$$= (D_x + D_y - 1)(D_y - 1)(D_x - 1).$$

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Conjecture: For *constant-coefficient* systems this Gröbner basis technology is equivalent to the generalized Laplace technology.

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$dim = 3$, $ord = 2$, 1901–2006

U.Dini. *Sopra una classe di equazioni a derivate parziali di second'ordine con un numero qualunque di variabli*. Atti Acc. Lincei. Mem. Classe fis., mat., nat. (5) 4, 1901, p. 121–178.

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An example:

$$Lu = (D_x D_y + x D_x D_z - D_z)u = 0.$$

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It has a complete solution, obtained using Dini's procedures:

$$u = \int \Big(v\,dx + (D_y + xD_z)v\,dz \Big) + \theta(y),$$

where $v = \int \phi(x, xy - z)\,dx + \psi(y, z)$.

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How this solution was obtained?

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Now we can find v, and the u from (1).

$dim = 3$, $ord = 2$: general result

Theorem

Let $L = \sum_{i+j+k\leq 2} a_{ijk}(x, y, z) D_x^i D_y^j D_z^k$ have factorizable principal symbol: $\sum_{i+j+k=2} a_{ijk}(x, y, z) D_x^i D_y^j D_z^k = \hat{S}_1 \hat{S}_2$ (mod lower-order terms) with generic (non-commuting) first-order LPDO $\hat{S}_1$, $\hat{S}_2$. Then there exist two Dini transformations $L_{(1)}$, $L_{(-1)}$ of L.

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Proof.

One can represent L in two possible ways:
$L = \hat{S}_1 \hat{S}_2 + \hat{T} + a(x, y, z) = \hat{S}_2 \hat{S}_1 + \hat{U} + a(x, y, z)$
with some first-order LPDO $\hat{T}$, $\hat{U}$. We will consider the first one obtaining a transformation of L into $L_{(1)}$.

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Answer: when $[(\hat{S}_2 + \beta), (\hat{V} + b)]$ is a linear combination of $(\hat{S}_2 + \beta)$ and$(\hat{V} + b)$ alone.

Proof (cont.)

Then $Lu = (\hat{S}_1 + \alpha)\underbrace{(\hat{S}_2 + \beta)u}_{v} + \hat{V}u + bu = 0 \Longleftrightarrow$

$$\begin{cases} (\hat{S}_2 + \beta)u = v, \\ (\hat{V} + b)u = -(\hat{S}_1 + \alpha)v. \end{cases}$$

So $[(\hat{S}_2 + \beta), (\hat{V} + b)]u = \Big((\hat{S}_2 + \beta)(\hat{S}_1 + \alpha) + (\hat{V} + b)\Big)v$

When $[(\hat{S}_2 + \beta), (\hat{V} + b)]u$ can be transformed into an expression involving *only* v?

Answer: when $[(\hat{S}_2 + \beta), (\hat{V} + b)]$ is a linear combination of $(\hat{S}_2 + \beta)$ and$(\hat{V} + b)$ alone.

One can check that this is possible to do choosing $\alpha(x, y, z)$ and $\beta(x, y, z)$ appropriately (for *generic* $\hat{S}_i$, $\hat{V}$).

$dim = 3$, $ord = 2$: chains of Dini transformations

$$\ldots \leftarrow \quad L_{(-2)} \quad \leftarrow \quad L_{(-1)} \quad \leftarrow \quad L \quad \rightarrow \quad L_{(1)} \quad \rightarrow \quad L_{(2)} \quad \rightarrow \ldots$$

Outline

Other interesting partial results:

J. Le Roux. *Extensions de la méthode de Laplace aux équations linéaires aux derivées partielles d'ordre supérieur au second*. Bull. Soc. Math. de France, 27:237–262, 1899.

Laura Pisati. *Sulla estensione del metodo di Laplace alle equazioni differenziali lineari di ordine qualunque con due variabili indipendenti*. Rend. Circ. Matem. Palermo, 1905, t. 20, p. 344–374.

Louise Petrén. *Extension de la méthode de Laplace aux équations* $\sum_{i=0}^{n-1} A_{1i} \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n} A_{0i} \frac{\partial^i z}{\partial y^i} = 0$. Lund Univ. Arsskrift. Bd. 7, Nr. 3, pages 1–166, 1911.

dim = *ord* > 2:

C. Athorne. *A* $\mathbf{Z}^2 \times \mathbf{R}^3$ *Toda system*. Phys. Lett. A, 206:162–166, 1995.

SYSTEMS WITH FINITE-DIMENSIONAL SOLUTION SPACE:

Z. Li, F. Schwarz and S.P. Tsarev. *Factoring systems of linear PDEs with finite-dimensional solution spaces*. J. Symbolic Computation, 36:443–471, 2003.

Min Wu. *On Solutions of Linear Functional Systems and Factorization of Modules over Laurent-Ore Algebras*, PhD. thesis, Beijing, 2005.

"CLASSICAL" FACTORIZATION:

D. Grigoriev and F. Schwarz. *Factoring and solving linear partial differential equations*. Computing, 73:179–197, 2004.

F.Winkler, E.Shemyakova, 2006.

Outline

Approach 1: Rings/modules (S.Ts., 1998)

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3) Algorithms?

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(the same even for multivariate polynomials!)

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- If a LPDO of order n is solvable then its symbol splits into n linear factors.

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Algebraically: $M \cdot P = N \cdot L$.

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$$S : \begin{cases} L_{11}u_1 + \ldots + L_{1s}u_s = 0, \\ \cdots \\ L_{k1}u_1 + \ldots + L_{ks}u_s = 0, \end{cases}$$

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Problem: chains are infinite....

The solution: Serre-Grothendieck factorcategory!

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For a given (say, determined) system of L.P.D.E. take the subcategory $\mathcal{S}_{n-2}$ of (overdetermined) systems with solution space parameterized by functions of at most $n-2$ variables. Then in the factorcategory $\mathcal{S}/\mathcal{S}_{n-2}$ ascending chains are finite!

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3) Is there an algorithm to solve *a first-order* linear PDE with rational coefficients in $dim = 3$ ($dim = 2$ seems to be solved)?

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2) Anyone who will give comments :-)