Differential Gröbner bases technique for calculating Feynman diagrams

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Outline of Talk:

- Introduction - radiative corrections and Feynman integrals
- Generalized recurrence relations for Feynman integrals
- Basics of differential Gröbner basis technique
- One-loop examples of DGB technique
- DGB for two-loop propagators
- Bypassing kinematical singlarities
- Concluding remarks

Introduction

The main tool to calculate physical quantities in particle physics: perturbation theory:

$$\sigma(\{p_j^2\},\{m_k^2\}) = \sum_k g^k M_k$$

where g is small parameter.

Examples

QED: Anomalous magnetic moment of the electron ($\alpha = e^2/(4\pi) \equiv 1/137$):

$$a_e^{theor} = \frac{1}{2}\frac{\alpha}{\pi} - 0.328478965\left(\frac{\alpha}{\pi}\right)^2 + 1.181241456\left(\frac{\alpha}{\pi}\right)^3 - 1.17283\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

QCD: e^+e^- annihilation cross section:

$$\sigma(e^+e^- \to hadrons) = \frac{4\pi\alpha^2}{3E_{c.m}^2}\left(\sum_q \epsilon_q^2\right)$$

$$\times\left[1 + \frac{\alpha_s}{\pi} + (1.9857 - 0.1153 n_f)\left(\frac{\alpha_s}{\pi}\right)^2 + \ldots\right]$$

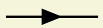
The method to calculate coefficients in the expansion was proposed by Feynman.

- Specify the Lagrangian (QED, QCD, Standard Model, . . .):

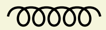
$$L(\Phi) = L_{kinetic}(\Phi) + L_{interaction}(\Phi)$$

- define expressions for propagators and vertices

  $L_{kinetic}$ - defines propagators (lines):

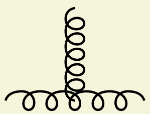


$$\frac{1}{k^2 - m_j^2}$$



$$\frac{g_{\mu\nu} - \alpha \frac{k_\mu k_\nu}{k^2}}{k^2 - m_j^2}$$

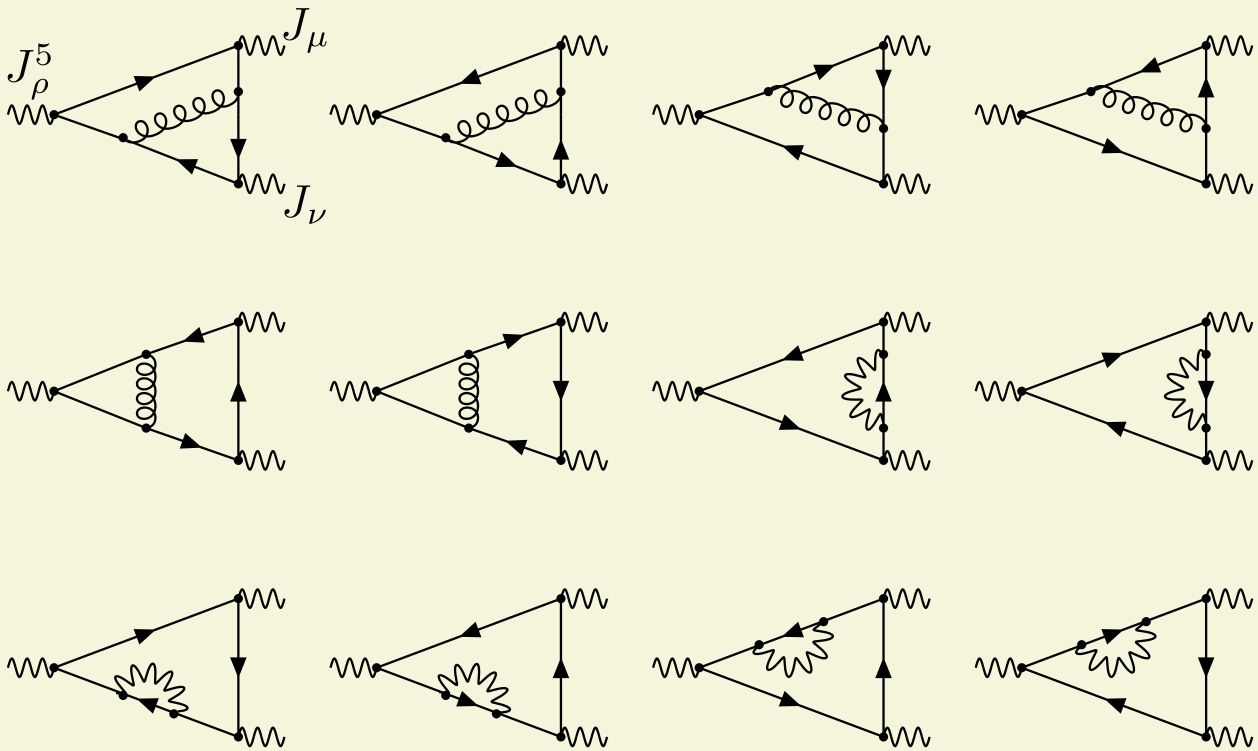
  $L_{interaction}$ - defines vertices:



$$g[(p-q)_\mu g_{\nu\rho} + (r-p)_\nu g_{\mu\rho} + (q-r)_\rho g_{\mu\nu}]$$

- generate all possible diagrams needed for the specified process
- calculate all Feynman integrals
- add contributions from all relevant diagrams to obtain coefficients M_k in perturbation expansion.

Example: Two-loop QCD diagrams contributing to $< AVV >$ correlator



Each vertex or line corresponds to a factor in Feynman diagram

Problems:

- UV and IR infinities $=>$ dimensional regularization (change number of space-time dimensions $4 \to d = 4 - 2\varepsilon$, ε - small).
- Expressions for Feynman diagrams are long $=>$ computer algebra systems (Schoonschip, Reduce, Form)
- Too many diagrams $=>$ automatic generation (Qgraf, FeynArts,. . .)
- each diagram can be combination of many integrals $=>$ integration by parts relations
- individual integrals are complicated $=>$ Mellin- Barns technique, differential equation method, dimensional recurrences, . . .

Feynman diagrams will be combinations of integrals of the form (momentum representation):

$$G^{(d)}(\{s_j\},\{m_k\}) = \int d^dk_1 \ldots \int d^dk_L \frac{k_{1\mu} \ldots k_{L\rho}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$

where

$$\overline{k}_{j\mu} = \sum_{n=1}^{L} \omega_{jn} k_{n,\mu} + \sum_{m=1}^{E} \eta_{jm} q_{m,\mu}$$

Not all of these integrals are independent. Integrals with very different ν_j can be expressed in terms of restricted number of scalar integrals, called master integrals:

$$G^{(d)}(\{s_j\},\{m_k\}) = \sum_{k=1}^{n_b} \frac{R(d,\{m_k\},\{q_iq_j\})}{S(d,\{m_k\},\{s\})} B_k(d,\{m_k\},\{s\})$$

where R and S are polynomials in d, masses m_k^2 and scalar products of external momenta q_iq_j and B_k are master or bases integrals.

The complexity of the calculation a diagrams is related to the

- number of multifold d dimensional integrations $\equiv$ number of loops -L
- number of external momenta E and number of masses - N

Most frequently used technique to reduce Feynman integrals to 'master' integrals is:

Integration By Parts method:

F.V. Tkachov, Phys.Lett. **100B** (1981) 65;

K.G. Chetyrkin and F.V. Tkachov, Nucl.Phys.**192** (1981) 159.

Use relation

$$\int d^d k_1 \ldots \int d^d k_L \frac{\partial}{\partial k_{j\mu}} \frac{k_{j\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0$$

differentiate with respect to momentum k:

$$\frac{\partial}{\partial k_{j\mu}} k_{j\nu} = g_{\mu\nu}, \quad \frac{\partial}{\partial k_{j\mu}} \frac{1}{(\overline{k}_1^2 - m_1^2)^{\nu_1}} = -2\nu_j \frac{(k_{j\mu} - q_{j\mu})}{(\overline{k}_1^2 - m_1^2)^{\nu_1+1}}$$

represent scalar products of momenta in terms of scalar factors standing in the denominator and external momenta:

$$k_1 q_1 = \frac{1}{2}\{[(k_1 + q_1)^2 - m_1^2] - [k_1^2 - m_1^2] - q_1^2\}.$$

This procedure leads to a set of equations connecting integrals with different shifts of parameters ν_k: known as integration by parts relations

$==>$ Problem of irreducible numerators

Another approach is possible. Use relation

$$\int d^dk_1 \ldots \int d^dk_L \frac{\partial}{\partial k_{j\mu}} \frac{k_{k\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0$$

differentiate with respect to momentum k, represent tensor integrals emerging after differentiation in terms of integrals with shifted dimension

$$\int d^dk_1 \ldots \int d^dk_L \frac{k_{1\mu} \ldots k_{N\nu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} =$$

$$T(q, \partial, \mathbf{d}^+) \int d^dk_1 \ldots \int d^dk_L \frac{1}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$

where $\mathbf{d}^+ G^{(d)} = G^{(d+2)}$ and $\partial_j = \frac{\partial}{\partial m_j^2}$ A general formula for $T(q, \partial, \mathbf{d}^+)$ was given by O.T. ,Phys.Rev. **D54** .

Parametric representation of Feynman integrals It is useful for numerical calculations of integrals and proving general theorems about arbitrary Feynman integrals. In the momentum representation for arbitrary integral

$$G^{(d)}(\{s_i\},\{m_k\}) = \int \frac{d^dk_1}{\pi^{d/2}} \ldots \frac{d^dk_L}{\pi^{d/2}} \frac{1}{[(k_1-p_1)^2-m_1^2]^{\nu_1} \ldots [(k_L-p_N)^2-m_M^2]^{\nu_N}}$$

use formula

$$\frac{1}{(k^2-m^2+i\epsilon)^{\nu}} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^{\infty} d\alpha \alpha^{\nu-1} \exp[i\alpha(k^2-m^2+i\epsilon)],$$

and perform Gaussian integration over loop momenta k_r

$$\int d^dk \exp[i(Ak^2+2(pk))] = i\left(\frac{\pi}{iA}\right)^{\frac{d}{2}} \exp\left[-\frac{ip^2}{A}\right].$$

The final result

$$G^{(d)}(\{s_i\},\{m_k\}) = C(\nu_1,..\nu_N) \int_0^{\infty} d\alpha_1 \ldots \int_0^{\infty} d\alpha_N \alpha_1^{\nu_1-1} \ldots \alpha_N^{\nu_N-1}$$

$$\times \frac{1}{D(\alpha)^{d/2}} \exp\left[i\left(\frac{Q(\alpha,\{s_k\})}{D(\alpha)} - \sum_{l=1}^{N} \alpha_l (m_l^2 - \imath\epsilon)\right)\right]$$

where $D(\alpha)$ is homogeneous polynomial of degree L and $Q(\alpha,\{s_k\})$ of degree $L+1$ in α.

From the parametric representation for any L-loop multi-leg Feynman integral a relation was derived:

$$G^{(d-2)} = (-1)^L D(\partial) G^{(d)}$$

where $D(\partial)$ is differential operator of order L and

$$\partial_j = \frac{\partial}{\partial m_j^2}.$$

This relation is due to simple structure of the integrand in d and masses m_k^2:

$$\frac{\partial}{\partial m_k^2} \rightarrow \alpha_k$$

$$D(\partial) \rightarrow D(\alpha) \rightarrow d \rightarrow d+2,$$

$$\mathrm{Examples:} \qquad D(\partial) = \partial_1 + \partial_2, \quad Q(\alpha) = \alpha_1\alpha_2 q^2 \quad \mathrm{one-loop\ \ self-energy}$$

$$D(\partial) = \partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3, \;\; Q(\alpha) = q^2\alpha_1\alpha_2\alpha_3, \quad \mathrm{two-loop\ \ sunrise}$$

Examples of generalized recurrence relations

The one-loop propagator type diagram with massive particles:

$$I^{(d)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2) = \int \frac{d^dk_1}{[i\pi^{d/2}]} \frac{1}{(k_1^2-m_1^2)^{\nu_1}[(k_1-q)^2-m_2^2]^{\nu_2}}.$$

In this case $D(\alpha)=\alpha_1+\alpha_2$, and therefore

$$\begin{aligned} I^{(d-2)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2) &= -(\partial_1+\partial_2)I^{(d)}_{\nu_1\nu_2}(q^2,m_1^2,m_2^2) \\ &= -\nu_1 I^{(d)}_{\nu_1+1\ \nu_2}(q^2,m_1^2,m_2^2) - \nu_2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2,m_1^2,m_2^2). \end{aligned}$$

We can get another recurrence relation connecting integrals with different d. From the identity:

$$\int d^dk_1 \frac{\partial}{\partial k_{1\mu}} \left[\frac{(k_1+q)_\mu}{(k_1^2-m_1^2)^{\nu_1}[(k_1-q)^2-m_2^2]^{\nu_2}} \right] \equiv 0,$$

we obtain:

$$\begin{aligned} &\nu_1 \int \frac{d^dk_1}{[i\pi^{d/2}]} \frac{(qk_1)}{(k_1^2-m_1^2)^{\nu_1+1}[(k_1-q)^2-m_2^2]^{\nu_2}} \\ &= \left(\frac{d}{2}-\nu_1\right) I^{(d)}_{\nu_1\nu_2} - \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1} - \nu_1 m_1^2 I^{(d)}_{\nu_1+1\ \nu_2} + \nu_2(q^2-m_1^2) I^{(d)}_{\nu_1\ \nu_2+1}. \end{aligned}$$

The integral with the scalar product (qk_1) can be written in terms of scalar integrals with shifted d:

$$\int \frac{d^dk_1}{[i\pi^{d/2}]} \frac{(qk_1)}{(k_1^2 - m_1^2)^{\nu_1+1}[(k_1-q)^2 - m_2^2]^{\nu_2}} = \nu_2 q^2 I^{(d+2)}_{\nu_1+1\ \nu_2+1}.$$

Inserting this expression into previous equation we obtain:

$$\nu_1\nu_2 q^2 I^{(d+2)}_{\nu_1+1\ \nu_2+1} - \left(\frac{d}{2} - \nu_1\right) I^{(d)}_{\nu_1\nu_2} + \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1} + \nu_1 m_1^2 I^{(d)}_{\nu_1+1\ \nu_2} - \nu_2(q^2 - m_1^2) I^{(d)}_{\nu_1\ \nu_2+1} \equiv 0.$$

In addition to the above relations two more relations can be obtained from the traditional method of integration by parts:

$$2\nu_2 m_2^2 I^{(d)}_{\nu_1\ \nu_2+1} + \nu_1 I^{(d)}_{\nu_1-1\ \nu_2+1} + \nu_1(m_1^2 + m_2^2 - q^2) I^{(d)}_{\nu_1+1\ \nu_2} - (d - 2\nu_2 - \nu_1) I^{(d)}_{\nu_1\nu_2} = 0,$$

$$\nu_1 I^{(d)}_{\nu_1+1\ \nu_2-1} - \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1} - \nu_1(m_1^2 - m_2^2 + q^2) I^{(d)}_{\nu_1+1\ \nu_2} - \nu_2(m_1^2 - m_2^2 - q^2) I^{(d)}_{\nu_1\ \nu_2+1}$$

$$+ (\nu_2 - \nu_1) I^{(d)}_{\nu_1\nu_2} = 0.$$

At $m_1 = 0,\ m_2 = m$, the relations connecting integrals $I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2)$ with different d are:

$$\nu_1\nu_2 q^2 I^{(d+2)}_{\nu_1+1\ \nu_2+1}(q^2, 0, m^2) - \left(\frac{d}{2} - \nu_1\right) I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2) + \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1}(q^2, 0, m^2) - \nu_2 q^2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2, 0, m^2) \equiv 0,$$

$$I^{(d-2)}_{\nu_1\nu_2}(q^2, 0, m^2) + \nu_1 I^{(d)}_{\nu_1+1\ \nu_2}(q^2, 0, m^2) + \nu_2 I^{(d)}_{\nu_1\ \nu_2+1}(q^2, 0, m^2) \equiv 0.$$

The integral $I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2)$ is proportional to the Gauss hypergeometric function :

$$I^{(d)}_{\nu_1\nu_2}(q^2, 0, m^2) = (-1)^{\nu_1+\nu_2} \frac{\Gamma(\nu_1 + \nu_2 - \frac{d}{2})\Gamma(\frac{d}{2} - \nu_1)}{(m^2)^{\nu_1+\nu_2-\frac{d}{2}}\Gamma(\frac{d}{2})\Gamma(\nu_2)}\, {}_2F_1\left[\begin{array}{ll} \nu_1, \nu_1 + \nu_2 - \frac{d}{2}\,; & \dfrac{q^2}{m^2} \\ \frac{d}{2}\,; & \end{array}\right].$$

There are fi fteen relations of Gauss between contiguous functions ${}_2F_1$:

$${}_2F_1\,(a \pm 1, b \pm 1, c \pm 1, x)$$

In our case $\quad a = \nu_1,\ \ b = \nu_1 + \nu_2 - d/2,\ \ c = d/2$.

Substituting explicit result into IBP relations we fi nd they reproduce only six relations of Gauss. The reason - parameter c of ${}_2F_1$ in IBP relations does not change, therefore all corresponding relations cannot be reproduced.

Generalized recurrence relations give new relations for Feynman integrals!!!

They extend number of recurrency parameters: $\{\nu_j\} \rightarrow \{\nu_j, d\}$

Dimensionality relations can be used for:

- calculating tensor integrals
- finding bases of master integrals without kinematical singularities
- evaluating master integrals

Generalized recurrence relations connect integrals with different powers of propagators and also integrals with different dimensionality d. It is easy to write down a big number of integration by parts and generalized recurrence relations.

How to use these relations? What is the number of master integrals? Is there minimal number of relations which is enough to reduce all integrals to master integrals?

There is mathematical theory answering to these questions. This is

Theory of Gröbner bases.

The key element of these theory

Buchberger algorithm.

Differential Gröbner basis technique for Feynman diagrams

1. **O.V. Tarasov** *“Reduction of Feynman graph amplitudes to a minimal set of basic integrals”* **,**

Acta Physica Polonica, v B29 (1998) 2655

2. **O. V. Tarasov,** *“Computation of Groebner bases for two-loop propagator type integrals,”*

Talk at ACAT-2003

Nucl. Instrum. Meth. A 534 (2004) 293 [arXiv:hep-ph/0403253].

Gröbner Basis is a nice set of recurrence relations or differential relations for Feynman integrals allowing to reduce large (in principle infinite) number of integrals in terms of finite number of integrals

Main steps of the algorithm:

- Tensor integrals express in terms of scalar ones with shifted space-time dimension

$$I_{\mu\nu\ldots} = T_{\mu\nu\ldots}(\partial, \mathbf{d}^+) I$$

- Scalar integrals with dots on lines represent as derivatives w.r.t. masses

$$\int \frac{d^dk_1 \ldots d^dk_L}{\ldots(k_1^2 - m_1^2)^{\nu_1}((k_1-p_1)^2 - m_2^2)^{\nu_2}\ldots}$$
$$= \frac{1}{(\nu_1-1)!(\nu_2-1)!} \frac{\partial^{\nu_1}}{\partial(m_i^2)^{\nu_1}} \frac{\partial^{\nu_2}}{\partial(m_j^2)^{\nu_2}}$$
$$\times \int \frac{d^dk_1 \ldots d^dk_L}{\ldots(k_1^2 - m_i^2)((k_1-p_1)^2 - m_j^2)\ldots} |_{m_i^2=m_1^2, m_j^2=m_2^2} ,$$

- For scalar integrals with different number of lines write down generalized recurrence relations and transform them into a system of differential equations by replacing

$$\frac{1}{(k_1^2 - m_j^2)^{r+1}} = \frac{1}{r!} \frac{\partial^r}{\partial (m_j^2)^r} \frac{1}{(k_1^2 - m_j^2)}$$

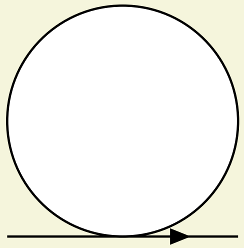
- Find differential Gröbner basis for this overdetermined system. One can use Maple, Mathematica or other computer systems.

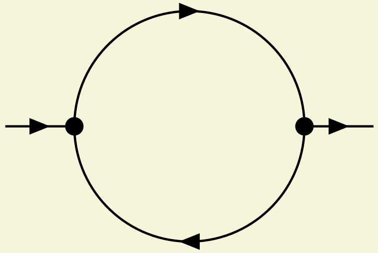
- Use relations from the Gröbner basis to reduce all possible integrals (i.e. higher order derivatives) in terms of fixed finite number of basic integrals (i.e. lower order derivatives)

- To reduce integrals $G^{(d+2j)}$ with shifted space-time dimension use relation:

$$G^{(d-2)} = D(\partial_j) G^{(d)}$$

Example of DGB for one-loop integrals





The basis for tadpole integral consists of two relations:

$$\partial_i T_i^{(d)} = \frac{d-2}{2m_i^2} T_i^{(d)}, \qquad T_i^{(d+2)} = -\frac{2m_i^2}{d} T_i^{(d)}$$

where

$$\partial_j = \frac{\partial}{\partial m_j^2} \quad \text{and} \quad T_i^{(d)} = \frac{1}{i\pi^{(d/2)}} \int \frac{d^d k_1}{k_1^2 - m_i^2}.$$

The Gröbner basis for propagator type integral consists of three relations:

$$2\lambda_{ij} \partial_i I_{2,ij}^{(d)} = (3-d)(\partial_i \lambda_{ij}) I_{2,ij}^{(d)} - \frac{\partial \lambda_{ij}}{\partial p_{ij}} \frac{(d-2)}{2m_i^2} T_i^{(d)} + 2(d-2) T_j^{(d)},$$

$$2\lambda_{ij} \partial_j I_{2,ij}^{(d)} = (3-d)(\partial_j \lambda_{ij}) I_{2,ij}^{(d)} - \frac{\partial \lambda_{ij}}{\partial p_{ij}} \frac{(d-2)}{2m_j^2} T_j^{(d)} + 2(d-2) T_i^{(d)},$$

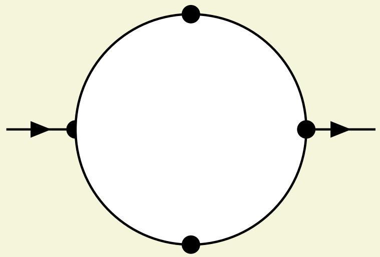
$$(d-1)g_{ij}I_{2,ij}^{(d+2)} = 2\lambda_{ij}I_{2,ij}^{(d)} + (\partial_i\lambda_{ij})T_j^{(d)} + (\partial_j\lambda_{ij})T_i^{(d)},$$

where

$$\lambda = -p_{ij}^2 - m_i^4 - m_j^4 + 2p_{ij}m_i^2 + 2p_{ij}m_j^2 + 2m_i^2m_j^2,$$

$$g_{ij} = -4p_{ij} = -4(p_i - p_j)^2.$$

Application of DGB to an integral



$$\frac{1}{i\pi^{d/2}}\int\frac{d^dk_1}{(k_1^2-m^2)^2((k_1-p_1)^2-m^2)^2}=\partial_i\partial_j\ I_{2,ij}^{(d)}\Big|_{m_i^2=m_j^2=m^2},$$

$$I_{2,ij}^{(d)}=\frac{1}{i\pi^{d/2}}\int\frac{d^dk_1}{(k_1^2-m_i^2)((k_1-p_1)^2-m_j^2)}.$$

$$\partial_i I_{2,ij}^{(d)}=f_1(m_i^2,m_j^2)I_{2,ij}^{(d)}+r_1(m_i^2,m_j^2),$$

$$\partial_j I_{2,ij}^{(d)}=f_2(m_i^2,m_j^2)I_{2,ij}^{(d)}+r_2(m_i^2,m_j^2),$$

$$\partial_j T_j^{(d)}=t_jT_j^{(d)}.$$

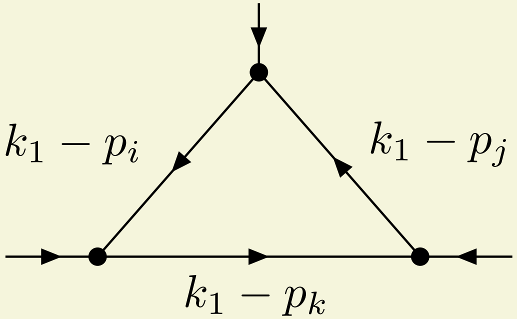
$$\begin{aligned}\partial_j\partial_i I_{2,ij}^{(d)} &= \partial_j[f_1(m_i^2,m_j^2)I_{2,ij}^{(d)}+r_1(m_i^2,m_j^2)]\\ &= (\partial_j f_1)I_{2,ij}^{(d)}+f_1\partial_j I_{2,ij}^{(d)}+\partial_j r_1\\ &= [\partial_j f_1+f_1f_2]I_{2,ij}^{(d)}+f_1r_2+\partial_j r_1\end{aligned}$$

$$r_1 = \frac{1}{2\lambda_{ij}}\left[-\frac{\partial\lambda_{ij}}{\partial p_{ij}}\frac{(d-2)}{2m_i^2}T_i^{(d)} + 2(d-2)T_j^{(d)}\right] = f_3 T_j^{(d)} + r_3 T_i^{(d)},$$
$$r_2 = f_4 T_i^{(d)} + r_4 T_j^{(d)}.$$

$$\begin{aligned}
\partial_j r_1 &= \partial_j[f_3 T_j^{(d)} + r_3 T_i^{(d)}] \\
&= (\partial_j f_3) T_j^{(d)} + f_3 \partial_j T_j^{(d)} + (\partial_j r_3) T_i^{(d)} \\
&= [(\partial_j f_3) + f_3 t_j] T_j^{(d)} + T_i^{(d)} \partial_j r_3.
\end{aligned}$$

$$\begin{aligned}
\partial_j \partial_i I_{2,ij}^{(d)} = [\partial_j f_1 + f_1 f_2] I_{2,ij}^{(d)} &+ [f_1 f_4 + \partial_j r_3] T_i^{(d)} \\
&+ [r_4 f_1 + (\partial_j f_3) + f_3 t_j] T_j^{(d)}
\end{aligned}$$

Vertex type integrals



The GB for 1-loop vertex integrals consists of 3 differential relations:

$$2\lambda_{ijk}\partial_i I^{(d)}_{3,ijk} = \frac{4-d}{2}(\partial_i\lambda_{ijk})I^{(d)}_{3,ijk} - 2(d-3)\left[\frac{p_{ij}}{\lambda_{ij}}\frac{\partial\lambda_{ijk}}{\partial y_{ik}}\,I^{(d)}_{2,ij}\right.$$
$$\left.+\frac{p_{ik}}{\lambda_{ik}}\frac{\partial\lambda_{ijk}}{\partial y_{ij}}\,I^{(d)}_{2,jk} + \frac{2p_{jk}}{\lambda_{jk}}\frac{\partial\lambda_{ijk}}{\partial y_{ii}}\,I^{(d)}_{2,jk}\right] + (d-2)\left[\frac{(\partial_j\lambda_{ijk})}{8m_k^2\lambda_{ik}}T^{(d)}_k\right.$$
$$\left.+\frac{1}{4m_i^2}\left(\frac{\partial_k\lambda_{ik}}{\lambda_{ik}}\frac{\partial\lambda_{ijk}}{\partial y_{ij}} + \frac{\partial_j\lambda_{ij}}{\lambda_{ij}}\frac{\partial\lambda_{ijk}}{\partial y_{ik}}\right)T^{(d)}_i + \frac{(\partial_k\lambda_{ijk})}{8m_j^2\lambda_{ij}}T^{(d)}_j\right],$$

$+2$ other relations by cyclic permutations

One dimensional recurrency relation

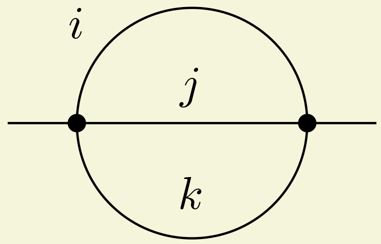
$$(d-2)g_{ijk}I^{(d+2)}_{3,ijk} = 2\lambda_{ijk}I^{(d)}_{3,ijk} + (\partial_i\lambda_{ijk})I^{(d)}_{2,jk} + (\partial_j\lambda_{ijk})I^{(d)}_{2,ik} + (\partial_k\lambda_{ijk})I^{(d)}_{2,ij}.$$

where

$$\begin{aligned}
\lambda_{ijk} = & \quad 2(p_{jk}+p_{ik}-p_{ij})(m_i^2m_j^2+p_{ij}m_k^2) \\
& + \quad 2(p_{ik}+p_{ij}-p_{jk})(p_{jk}m_i^2+m_k^2m_j^2) \\
& + \quad 2(p_{jk}+p_{ij}-p_{ik})(m_k^2m_i^2+p_{ik}m_j^2) \\
& - \quad 2m_j^4p_{ik}-2p_{ij}m_k^4-2m_i^4p_{jk}-2p_{ij}p_{ik}p_{jk},
\end{aligned}$$

$$g_{ijk} = \quad 2p_{ij}^2-4(p_{ik}+p_{jk})p_{ij}+2(p_{ik}-p_{jk})^2.$$

Differential Gröbner bases for propagator integrals



$\equiv J_{3,ijk}^{(d)}$

Differential GB for $J_{3,ijk}^{(d)}$ integrals consists of 8 relations:

- 3 relations for $\partial_i \partial_j , \quad i \neq j$
- 3 relations for ∂_j^2
- 1- relation for $J_{3,ijk}^{(d+2)}$ in terms of d- dimensional integrals.
- 1- relation for $\partial_j J_{3,ijk}^{(d+2)}$ in terms of d dimensional integrals.

Explicit expressions for the Gröbner bases are:

$$2D_{ijk}\partial_i\partial_j J_{3,ijk}^{(d)} = 2h_{ijk}\partial_i J_{3,ijk}^{(d)} + 2h_{jik}\partial_j J_{3,ijk}^{(d)} + 4m_k^2\sigma_{ijk}\partial_k J_{3,ijk}^{(d)}$$

$$+ \frac{(d-2)^2}{4m_i^2 m_j^2}[m_i^2\phi_{jik}T_j^{(d)}T_k^{(d)} + m_j^2\phi_{ijk}T_i^{(d)}T_k^{(d)} - 2\rho_{ijk}T_i^{(d)}T_j^{(d)}],$$
$$+ \frac{1}{2}(3d-8)(d-3)\phi_{kji}J_{3,ijk}^{(d)}$$

$$2m_i^2 D_{ijk}\partial_i^2 J_{3,ijk}^{(d)} = m_i^2 S_{ijk}\partial_i J_{3,ijk}^{(d)} + m_j^2 S_{jik}\partial_j J_{3,ijk}^{(d)} + m_k^2 S_{kij}\partial_k J_{3,ijk}^{(d)}$$
$$+ (d-4)D_{ijk}\partial_i J_{3,ijk}^{(d)} - (3d-8)(d-3)\rho_{ijk}J_{3,ijk}^{(d)}$$

$$+ \frac{(d-2)^2}{4}[\phi_{ijk}T_j^{(d)}T_k^{(d)} + \phi_{jik}T_i^{(d)}T_k^{(d)} + \phi_{kij}T_i^{(d)}T_j^{(d)}],$$

and 2 dimensional recurrency relations:

$$J_{3,ijk}^{(d+2)} = w_i \partial_i J_{3,ijk}^{(d)} + w_j \partial_j J_{3,ijk}^{(d)} + w_k \partial_k J_{3,ijk}^{(d)} + w_0 J_{3,ijk}^{(d)} + t_0,$$

$$\partial_i J_{3,ijk}^{(d+2)} = w_i^{(1)} \partial_i J_{3,ijk}^{(d)} + w_j^{(1)} \partial_j J_{3,ijk}^{(d)} + w_k^{(1)} \partial_k J_{3,ijk}^{(d)} + w_0^{(1)} J_{3,ijk}^{(d)} + t_1,$$

where $t_0,\ t_1$ are tadpole contributions and $w_j^{(k)},\ w_i$ are ratios of polynomials in momentum and masses.

Polynomial coefficients are

$$
\begin{aligned}
D_{ijk} &= [q^2-(m_i+m_j+m_k)^2][q^2-(m_i+m_j-m_k)^2] \\
&\quad [q^2-(m_i-m_j+m_k)^2][q^2-(m_i-m_j-m_k)^2], \\
\rho_{ijk} &= -q^6+3q^4(m_i^2+m_j^2+m_k^2) \\
&\quad -q^2[3(m_i^4+m_j^4+m_k^4)+2(m_i^2m_j^2+m_i^2m_k^2+m_j^2m_k^2)]+m_i^6+m_j^6+m_k^6 \\
&\quad -m_i^2(m_j^4+m_k^4)-m_j^2(m_i^4+m_k^4)-m_k^2(m_i^4+m_j^4)+10m_i^2m_j^2m_k^2, \\
\phi_{ijk} &= 4[q^4+2q^2(m_i^2-m_j^2-m_k^2)+(m_j^2-m_k^2)^2+m_i^2(2m_j^2+2m_k^2-3m_i^2)], \\
\sigma_{ijk} &= -\frac{1}{4}(d-3)[\phi_{ijk}+\phi_{jik}+2\phi_{kij}], \\
h_{ijk} &= -\frac{1}{2}(d-3)[m_k^2\phi_{ijk}+2m_i^2\phi_{kij}-2\rho_{ijk}], \\
S_{ijk} &= -(d-3)[m_k^2\phi_{jik}+m_j^2\phi_{kij}-4\rho_{ijk}].
\end{aligned}
$$

Bypassing kinematical singularities via other dimensions

In many interesting for physical application cases Gram determinants are zero. For example, in the considered case

$$D_{ijk} = [q^2-(m_i+m_j+m_k)^2][q^2-(m_i+m_j-m_k)^2]$$
$$[q^2-(m_i-m_j+m_k)^2][q^2-(m_i-m_j-m_k)^2]=0.$$

At the final stage of the application of the differential bases technique it cause problems because Gram determinants are standing in denominators!!!.

We found quite unexpected solution of this problem.

Key idea to solve the problem : choose another bases of master integrals- transform d dimensional integrals to $d-2$ dimensional integrals. In case of $J^{(d)}_{3,ijk}$ integrals:

$$12(d-1)q^2\partial_i^2 J^{(d+2)}_{3,ijk} = -2(3\Delta_{jk6}+m_i^2(6m_j^2+6m_k^2-2q^2-m_i^2))\partial_i J^{(d)}_{3,ijk}$$
$$+16m_k^2(q^2-m_k^2)\partial_k J^{(d)}_{3,ijk}+16m_j^2(q^2-m_j^2)\partial_j J^{(d)}_{3,ijk}$$
$$-2((7d-17)q^2+(d-3)(m_i^2-5m_2^2-5m_k^2))J^{(d)}_{3,ijk}-2(d-2)T^{(d)}_j T^{(d)}_k$$
$$+(d-2)(1-3\frac{u_{jk6}}{m_i^2})T^{(d)}_i T^{(d)}_k+(d-2)(1-3\frac{u_{kj6}}{m_i^2})T^{(d)}_i T^{(d)}_j.$$

$$6q^2(d-2)\partial_i\partial_j J^{(d+2)}_{3,ijk} = 2m_i^2(q^2-m_i^2-3m_j^2+3m_k^2)\partial_i J^{(d)}_{3,ijk}$$
$$+2m_j^2(q^2-3m_i^2-m_j^2+3m_k^2)\partial_j J^{(d)}_{3,ijk}-4m_k^2(q^2-m_k^2)\partial_k J^{(d)}_{3,ijk}$$
$$+2((d-2)q^2+(d-3)(m_i^2+m_j^2-2m_k^2))J^{(d)}_{3,ijk}$$
$$+2(d-2)T^{(d)}_i T^{(d)}_j-(d-2)T^{(d)}_k(T^{(d)}_i+T^{(d)}_j).$$

$$\Delta_{jk6} = m_j^4+m_k^4+q^4-2q^2m_j^2-2q^2m_k^2-2m_k^2m_j^2.$$

- There is no Gram determinant in the denominators at the r.h.s !!!
- The number of terms at the right hand sides of the above relations decreased: - 24 in the first case and - 40 in the second case (to be compared with 382 and 629 terms in appropriate recurrence relations and 162 and 214 terms in the original DGB).
- Reduction to the generic dimension d can be done by using relations between $d-2$ and d dimensional integrals given at the previous transparencies.
- Differential GB transforming d dimensional into $d-2$ dimensional integrals can be derived for any type of integrals!

Concluding remarks

- Differential Gröbner bases technique turns out to be useful tool: calculation of two-loop vertex diagrams with massless particles and arbitrary external momenta were done using this technique
- Fast methods and software is needed for reducing higher derivatives by using Gröbner bases to bases set of functions and their derivatives. Relevant command in Maple dsubs works very slowly!
- It would be nice to fi nd another method to bypass kinematical singularities!
- Effi cient methods and packages are badly needed for fi nding differential Gröbner bases.