Solution of difference equations for Feynman Integrals

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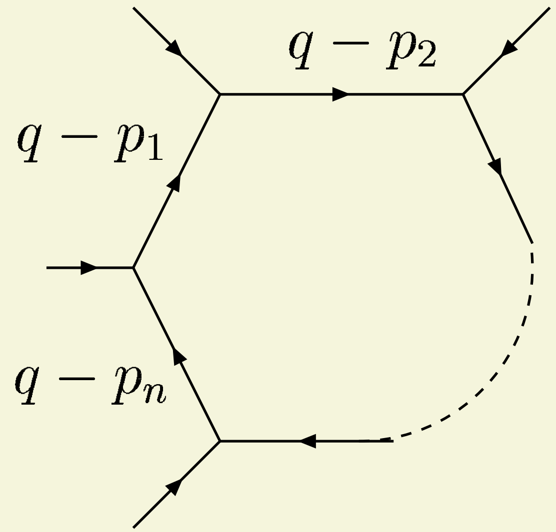
Outline of Talk:

- Difference equations and their solution for one-loop integrals
- Solution of difference equation for two-loop sunrise diagram (arXiv:hep-ph/0603227)
- Concluding remarks

We consider scalar one-loop integrals depending on $n - 1$ external momenta:

$$I_n^{(d)} = \int \frac{d^d q}{[i\pi^{d/2}]} \prod_{j=1}^{n} \frac{1}{(q-p_j)^2 - m_j^2 + i\epsilon.},$$

The convention for the momenta are given in Fig.



A generalized recurrence relation connecting integrals $I_n^{(d)}$ with different space-time dimensions is:

$$(d-n+1)G_{n-1}I_n^{(d+2)} - 2\Delta_n I_n^{(d)} = \sum_{k=1}^{n} (\partial_k \Delta_n)\mathbf{k}^- I_n^{(d)},$$

where

$$\partial_j \equiv \partial / \partial m_j^2,$$

and

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix}, \quad G_{n-1} = -2^n \begin{vmatrix} (p_1-p_n)(p_1-p_n) & \ldots & (p_1-p_n)(p_{n-1}-p_n) \\ (p_1-p_n)(p_2-p_n) & \ldots & (p_2-p_n)(p_{n-1}-p_n) \\ \vdots & \vdots & \vdots \\ (p_1-p_n)(p_{n-1}-p_n) & \ldots & (p_{n-1}-p_n)(p_{n-1}-p_n) \end{vmatrix}$$

$$Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2.$$

In the sum k^- means that the k-th factor (propagator) is removed from the integral.

Assuming that we know expressions for $n-1$ point functions then the above equation is inhomogeneous first order difference equation with respect to d.

By the redefinition

$$I_n^{(d)} = \frac{1}{\Gamma\left(\frac{d-n+1}{2}\right)} \left(\frac{\Delta_n}{G_{n-1}}\right)^{\frac{d}{2}} \overline{I}_n^{(d)}$$

we obtain the simpler equation

$$\overline{I}_n^{(d+2)} = \overline{I}_n^{(d)} + \frac{\Gamma\left(\frac{d-n+1}{2}\right)}{2\Delta_n} \left(\frac{G_{n-1}}{\Delta_n}\right)^{\frac{d}{2}} \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(d)}.$$

We can parameterize d as

$$d = 2l - 2\varepsilon,$$

where l is integer and ε small.

Then the solution of the equation for $\overline{I}_n^{(d)}$ can be written as

$$\overline{I}_n^{(2l-2\varepsilon)} = \sum_{r=0}^{l} \frac{\Gamma\left(\frac{2r-1-2\varepsilon-n}{2}\right)}{2\Delta_n} \left(\frac{G_{n-1}}{\Delta_n}\right)^{r-1-\varepsilon} \sum_{k=1}^{n} (\partial_k \Delta_n) \mathbf{k}^- I_n^{(2r-2-2\varepsilon)} + \tilde{b}_n(\varepsilon),$$

By shifting the summation index $r \to r + l + 1$, changing the solution can be rewritten in a 'covariant' w.r.t. d form:

$$I_n^{(d)} = b_n(\varepsilon) - \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- I_n^{(d+2r)}.$$

where $(a)_r \equiv \Gamma(r+a)/\Gamma(a)$ is the Pochhammer symbol.

b_n can be determined from the asymptotic behavior of $I_n^{(d)}$ for $d \to \infty$ or by setting up a differential equation for it. This term depends on the kinematic domain.

Example: 2-point function

Expression for $I_2^{(d)}$ with lines labeled by i, j includes two one-fold sums over tadpole integrals $I_1^{(d)}$,

$$I_1^{(d)}(m_i) = -\Gamma\left(1 - \frac{d}{2}\right)(m_i^2)^{\frac{d-2}{2}}.$$

Substituting I_1 into general expression gives

$$\frac{2\lambda_{ij} I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} = b_2 + \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_j^2 G_1}{\lambda_{ij}}\right)^r + \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(-\frac{m_i^2 G_1}{\lambda_{ij}}\right)^r,$$

where

$$G_1 = -4p_{ij}^2.$$

and the infinite series can be represented as hypergeometric functions, i.e.

$$\sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} z^r = {}_2F_1\left[\begin{array}{c} 1, \frac{d-1}{2}; \\ \frac{d}{2}; \end{array} z \right]$$

Boundary constant b_2 can be obtained from the asymptotic behavior of I_2 at $d \to \infty$ or from differential equation.

$$\frac{2\lambda_{ij} I_2^{(d)}}{\Gamma\left(1-\frac{d}{2}\right)} = b_2 \quad + \quad \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \; {}_2F_1\left[1, \frac{d-1}{2}, \frac{d}{2}, \left(-\frac{m_j^2 G_1}{\lambda_{ij}}\right)\right]$$

$$+ \quad \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \; {}_2F_1\left[1, \frac{d-1}{2}, \frac{d}{2}, \left(-\frac{m_i^2 G_1}{\lambda_{ij}}\right)\right]$$

3-point function

In complete analogy to $I_2^{(d)}$ the 3-point function can be evaluated. We have to sum over 2-point functions, which are represented in terms of hypergeometric functions ${}_2F_1$. The result for $I_3^{(d)}$ reads

$$\frac{\lambda_{ijk}}{\Gamma\left(2-\frac{d}{2}\right)} I_3^{(d)} = b_3 + \theta_{ijk}\,\partial_k\lambda_{ijk} + \theta_{kij}\,\partial_j\lambda_{ijk} + \theta_{jki}\,\partial_i\lambda_{ijk},$$

where

$$b_3 = 2^{\frac{3}{2}}\pi\,\sqrt{-g_{ijk}}\,r_{ijk}^{\frac{d-2}{2}},$$

provided an extremum of h_3 ($G_2 < 0$) occurs inside the integration region of the Feynman parameters. Otherwise $b_3 = 0$. For $\lambda_{ij} \neq 0$ we have

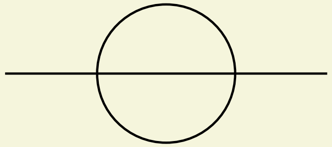
$$\lambda_{ij}\,\theta_{ijk} = -\left[\frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}+\frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}}\right] r_{ij}^{\frac{d-2}{2}} \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-2}{2}\right)}{4\Gamma\left(\frac{d-1}{2}\right)}\,{}_2F_1\left[\begin{array}{c} 1,\frac{d-2}{2};\\ \frac{d-1}{2}; \end{array} \frac{r_{ij}}{r_{ijk}}\right]$$

$$+ \frac{(m_i^2)^{\frac{d-2}{2}}}{2(d-2)} \frac{\partial_j\lambda_{ij}}{\sqrt{1-\frac{m_i^2}{r_{ij}}}} F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_i^2}{r_{ijk}},\frac{m_i^2}{r_{ij}}\right)$$

$$+ \frac{(m_j^2)^{\frac{d-2}{2}}}{2(d-2)} \frac{\partial_i\lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}} F_1\left(\frac{d-2}{2},1,\frac{1}{2},\frac{d}{2};\frac{m_j^2}{r_{ijk}},\frac{m_j^2}{r_{ij}}\right).$$

To our knowledge there exists no simpler hypergeometric representation of the $3-$point function for d dimensions in the literature.

Difference and differential equations for the sunrise integral



$J_3^{(d)}$

The generic two-loop self-energy type diagram in d dimensional Minkowski space with three equal mass propagators is given by the integral:

$$J_3^{(d)}(\nu_1,\nu_2,\nu_3) \equiv \int\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2-m^2)^{\nu_1}((k_1-k_2)^2-m^2)^{\nu_2}((k_2-q)^2-m^2)^{\nu_3}}.$$

For integer values of ν_j the integrals (11) can be expressed in terms of only three basis integrals $J_3^{(d)}(1,1,1)$, $J_3^{(d)}(2,1,1)$ and $J_3^{(d)}(0,1,1) = (T_1^{(d)}(m^2))^2$ where

$$T_1^{(d)}(m^2) = \int \frac{d^d k}{[i\pi^{\frac{d}{2}}]} \frac{1}{k^2-m^2} = -\Gamma\left(1-\frac{d}{2}\right) m^{d-2}.$$

The relation connecting $d-2$ and d dimensional integrals $J_3^{(d)}(\nu_1,\nu_2,\nu_3)$:

$$\begin{aligned} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3) &= \nu_1\nu_2 J_3^{(d)}(\nu_1+1,\nu_2+1,\nu_3) \\ &+ \nu_1\nu_3 J_3^{(d)}(\nu_1+1,\nu_2,\nu_3+1) + \nu_2\nu_3 J_3^{(d)}(\nu_1,\nu_2+1,\nu_3+1). \end{aligned}$$

At $\nu_1=\nu_2=\nu_3=1$ and $\nu_1=2,\ \nu_2=\nu_3=1$ we obtain two equations. Use the recurrence relations to simplify their r.h.s. Shifting $d \rightarrow d+2$ give two more relations.

They are used to exclude $J_3^{(d)}(2,1,1)$ from one of the relations, so that we obtain a difference equation for the master integral $J_3^{(d)}(1,1,1) \equiv J_3^{(d)}$:

$$\begin{aligned} 12z^3(d+1)(d-1)(3d+4)(3d+2) \qquad & J_3^{(d+4)} \\ -4m^4(1-3z)(1-42z+9z^2)z(d-1)d \qquad & J_3^{(d+2)} \\ -4m^8(1-z)^2(1-9z)^2 \qquad & J_3^{(d)} \\ = 3z[(z+1)(27z^2+18z-1)d^2-4z(1+9z)d-48z^2]m^{2d+2} \qquad & \Gamma\left(-\frac{d}{2}\right)^2, \end{aligned}$$

where

$$z = \frac{m^2}{q^2}.$$

The integral $J_3^{(d)}$ satisfies also a second order differential equation. Taking the second derivative of $J_3^{(d)}$ with respect to mass gives

$$\frac{d^2}{dm^2\,dm^2} J_3^{(d)}(1,1,1) = 6J_3^{(d)}(2,2,1) + 6J_3^{(d)}(3,1,1).$$

By using recurrence relations integrals on the r.h.s can be reduced to the same three basis integrals. Using

$$J_3^{(d)}(2,1,1) = \frac{1}{3}\,\frac{d}{dm^2} J_3^{(d)}(1,1,1)$$

we obtain:

$$2(1-z)(1-9z)z^2 \frac{d^2 J_3^{(d)}}{dz^2} - z[9z^2(d-4) + 10z(d-2) + 8 - 3d]\frac{dJ_3^{(d)}}{dz}$$
$$+ (d-3)[z(d+4) + d - 4]J_3^{(d)} = 12zm^{(2d-6)}\Gamma^2\left(2 - \frac{d}{2}\right).$$

The differential equation were used to find the momentum dependence of arbitrary periodic constants in the solution of the difference equation.

Solution of the dimensional recurrency

Difference equation is a second order inhomogeneous equation with polynomial coefficients in d. The full solution of this equation is given by:

$$J_3^{(d)} = J_{3p}^{(d)} + \widetilde{w}_a(d) J_{3a}^{(d)} + \widetilde{w}_b(d) J_{3b}^{(d)},$$

where $J_{3p}^{(d)}$ is a particular solution of the equation, $J_{3a}^{(d)}$, $J_{3b}^{(d)}$ is a fundamental system of solutions of the associated homogeneous equation and $\widetilde{w}_a(d)$, $\widetilde{w}_b(d)$ are arbitrary periodic functions of d satisfying relations:

$$\widetilde{w}_a(d+2) = \widetilde{w}_a(d), \qquad \widetilde{w}_b(d+2) = \widetilde{w}_b(d).$$

The order of the polynomials in d of the associated homogeneous difference equation can be reduced by making the substitution

$$J_3^{(d)} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)} \overline{J}_3^{(d)}.$$

The homogeneous equation for $\overline{J}_3^{(d)}$ takes the simpler form

$$\frac{16z^3}{27m^8(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+4)} - \frac{2(1-3z)(1-42z+9z^2)zd}{27m^4(1-z)^2(1-9z)^2}\,\overline{J}_3^{(d+2)} - \frac{(3d-2)(3d-4)}{36}\,\overline{J}_3^{(d)} = 0.$$

Putting

$$d = 2k - 2\varepsilon, \quad y^{(k)} = \rho^{-k}\,\overline{J}_3^{(2k-2\varepsilon)},$$

we transform equation to a standard form

$$A\rho^2 y^{(k+2)} + (B + C\,k)\rho y^{(k+1)} - (\alpha + k)(\beta + k)y^{(k)} = 0,$$

where

$$A = \frac{16z^3}{27m^8(1-z)^2(1-9z)^2}, \quad B = \frac{4\varepsilon}{27}\frac{(1-3z)(1-42z+9z^2)z}{m^4(1-z)^2(1-9z)^2},$$

$$C = -\frac{B}{\varepsilon}, \quad \alpha = -\varepsilon - \frac{1}{3}, \quad \beta = -\varepsilon - \frac{2}{3},$$

and ρ is for the time being, an arbitrary constant.

In order to get homogeneous equation into a more convenient form, we will define three parameters ρ, x and γ by the equations

$$A\rho^2 = x(1-x), \;\; B\rho = \gamma - (\alpha+\beta+1)x, \;\; C\rho = 1-2x.$$

These have the solution

$$\begin{aligned} x &= \frac{1-2C\rho}{2} = \frac{(1-9z)^2}{(1+3z)^3} = \frac{q^2(q^2-9m^2)^2}{(q^2+3m^2)^3}, \\ \rho &= \frac{1}{\sqrt{4A+C^2}} = \frac{27}{4}\frac{m^4(1-z)^2(1-9z)^2}{z(1+3z)^3} = \frac{27}{4}\frac{m^2(q^2-m^2)^2(q^2-9m^2)^2}{(q^2+3m^2)^3}, \\ \gamma &= B\rho + (\alpha+\beta+1)x = -\varepsilon, \end{aligned}$$

and the equation can accordingly be written in the form

$$x(1-x)y^{(k+2)} + [(1-2x)k + \gamma - (\alpha+\beta+1)x]y^{(k+1)} - (\alpha+k)(\beta+k)y^{(k)} = 0.$$

it can be transformed to the equation with linear in k coefficients by rescaling $y^{(k)}$

$$y^{(k)} = \Gamma(\alpha+k)\widetilde{y}^{(k)} \quad \text{or} \quad y^{(k)} = \Gamma(\beta+k)\widetilde{y}^{(k)}.$$

The fundamental system of solutions of homogeneous equation consist of two functions. In the case when $|1-x| < 1$ (large q^2) the solutions are

$$
\begin{aligned}
y_1^{(k)} &= (-1)^k \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{\Gamma(\alpha+\beta-\gamma+k+1)} \;{}_2F_1(\alpha+k,\beta+k,\alpha+\beta-\gamma+k+1;1-x), \\
y_2^{(k)} &= \frac{\Gamma(\alpha+\beta-\gamma+k)}{(1-x)^k} \;{}_2F_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1-k;1-x).
\end{aligned}
$$

Once we know the solutions of the homogeneous equation a particular solution $J_{3p}^{(d)}$ can be obtained by using Lagrange's method of variation of parameters.

The argument of the Gauss' hypergeometric function is related to the maximum of the Kibble cubic form:

$$
\Phi(s,t,u) = stu - (s+t+u)m^2(m^2+q^2) + 2m^4(m^2+3q^2),
$$

provided that $s+t+u = q^2+3m^2$. The maximal value $\Phi_{\max} = \frac{1}{27}\, q^2(q^2-9m^2)^2$ occurs at $s=t=u=\frac{1}{3}\,(q^2+3m^2)$ and we see that the kinematical variable (1) can be written as

$$
x = \frac{\Phi(s,t,u)}{stu}\Big|_{s=t=u=\frac{1}{3}\,(q^2+3m^2)} .
$$

This observation may be useful in finding the characteristic variable in the general mass case.

Explicit analytic expression for $J_3^{(d)}$

To find the full solution we assume that q^2 is large. The solution of the associated homogeneous difference equation will be of the form

$$
\begin{aligned}
J_{3,h}^{(d)} &= w_1(z)\frac{\Gamma\left(\frac{d}{2}-\frac{1}{3}\right)\Gamma\left(\frac{d}{2}-\frac{2}{3}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)}\rho^{\frac{d}{2}}e^{i\pi\frac{d}{2}}\;{}_2F_1\left[\begin{matrix}\frac{d}{2}-\frac{1}{3},\frac{d}{2}-\frac{2}{3};\\ \frac{d}{2};\end{matrix}\;1-x\right] \\
&+ w_2(z)\frac{\Gamma^2\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{3d}{2}-3\right)\Gamma\left(\frac{d-1}{2}\right)}\frac{\rho^{\frac{d}{2}}}{(1-x)^{\frac{d}{2}}}\;{}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3};\\ 2-\frac{d}{2};\end{matrix}\;1-x\right].
\end{aligned}
$$

The arbitrary periodic functions $w_1(z)$ and $w_2(z)$ can be determined either from the $d\to\infty$ asymptotics or using the differential equation. From differential equation we obtain two simple equations

$$
z(1-z)(1+3z)(1-9z)\frac{dw_1(z)}{dz}-2(1+6z-39z^2)w_1(z)=0,
$$

$$
z(1+3z)(1-9z)\frac{dw_2(z)}{dz}+3(1-z)w_2(z)=0.
$$

Solutions of equations

$$w_1(z) = \frac{\kappa_1 z^2(1+3z)^2}{(1-9z)^2(1-z)^2}, \quad w_2(z) = \frac{\kappa_2 z^3}{(1+3z)(1-9z)^2}.$$

Integration constants $\kappa_1,\ \kappa_2$ we fix from the first two terms of the large momentum expansion of $J_3^{(d)}$

$$J_3^{(d)} = m^{2-4\varepsilon}\Gamma^2(1+\varepsilon)\left[\frac{\Gamma(-1+2\varepsilon)\Gamma^3(1-\varepsilon)}{z\Gamma^2(1+\varepsilon)\Gamma(3-3\varepsilon)}(-z)^{2\varepsilon} + \frac{6\Gamma^2(-\varepsilon)}{\Gamma(3-2\varepsilon)}(-z)^{\varepsilon}\right] + O(z).$$

The application of Lagrange's method of finding a particular solution gives

$$J_{3p}^{(d)} = \frac{3zm^{2d-6}}{(1+\sqrt{z})^2}\Gamma^2\left(1-\frac{d}{2}\right)F_2\left(1,\frac{1}{2},\frac{d-1}{2},\frac{d}{2},d-1;\sqrt{z}R,R\right),$$

where

$$R = \frac{4\sqrt{z}}{(1+\sqrt{z})^2},$$

and F_2 is the Appell function:

$$F_2(\alpha,\beta,\beta',\gamma,\gamma';x,y) = \sum_{k,l=0}^{\infty}\frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_k(\gamma')_l}\frac{x^k\,y^l}{k!\,l!}, \quad |x|+|y|<1.$$

Collecting all contributions, setting $d = 4 - 2\varepsilon$, applying Euler transformation for the first ${}_2F_1$ function we obtain

$$
\begin{aligned}
J_3^{(d)} &= \frac{6\Gamma^2(-\varepsilon)\Gamma^2(1+\varepsilon)(-z)^{\varepsilon}(1-z)^{2-2\varepsilon}}{m^{4\varepsilon-2}\,\Gamma(3-2\varepsilon)(1+3z)}\, {}_2F_1\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}\,; \\ 2-\varepsilon\,; \end{matrix} \ \frac{27(1-z)^2 z}{(1+3z)^3}\right] \\
&+ \frac{\Gamma(-1+2\varepsilon)\Gamma^3(1-\varepsilon)(-z)^{2\varepsilon}(1-9z)^{2-2\varepsilon}}{m^{4\varepsilon-2}\,\Gamma(3-3\varepsilon)z(1+3z)}\, {}_2F_1\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}\,; \\ \varepsilon\,; \end{matrix} \ \frac{27(1-z)^2 z}{(1+3z)^3}\right] \\
&+ \frac{3zm^{2-4\varepsilon}}{(1+\sqrt{z})^2}\Gamma^2(-1+\varepsilon)\, F_2\left(1, \frac{1}{2}, \frac{3}{2} - \varepsilon, 2-\varepsilon, 3-2\varepsilon; \sqrt{z}R, R\right).
\end{aligned}
$$

The use of dimensional recurrences was essential to obtain this result!

Integral representation convenient for the ε expansion of ${}_2F_1$:

$$
{}_2F_1\left[\begin{array}{cc} \frac{1}{3},\frac{2}{3}\,; & \dfrac{27(1-z)^2 z}{(1+3z)^3} \\ 2-\varepsilon\,; & \end{array}\right]
= \frac{(1+3z)}{(1-z)} \frac{\Gamma(2-\varepsilon)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\varepsilon\right)} \int_0^1 \frac{du}{\sqrt{u}} [(1-u)(1-wu)(1-zu)]^{\frac{1}{2}-\varepsilon}
$$

Integral representation for Appell's F_2 function

$$
F_2\left(1,\frac{3}{2}-\varepsilon,\frac{1}{2},3-2\varepsilon,2-\varepsilon,R,\sqrt{z}R\right)
= \frac{2\Gamma(3-2\varepsilon)}{\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}(1+\sqrt{z})^2 \int_0^1 \frac{dt\,[t(1-t)]^{\frac{1}{2}-\varepsilon}}{(4zt+1-z+L)}\; {}_2F_1\left[\begin{array}{cc} 1,\varepsilon\,; & \dfrac{4tz+1-z-L}{4tz+1-z+L} \\ 2-\varepsilon\,; & \end{array}\right]
$$

where

$$
L=\sqrt{(4zt-1-z)^2-4z}.
$$

This integral representation can be used for the ε expansion of the F_2 function.

The imaginary part of $J_3^{(d)}$ on the cut comes from the two ${}_2F_1$ functions:

$$\mathrm{Im} J_3^{(d)} = \frac{-4z\,\pi^2\sqrt{3\pi}m^{2-4\varepsilon}}{\Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma\left(2-\varepsilon\right)\left(1+3z\right)}\left[\frac{(1-9z)^2}{108z^2}\right]^{1-\varepsilon} {}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}\,; & \frac{(1-9z)^2}{(1+3z)^3}\\ 2-\varepsilon\,; & \end{matrix}\right].$$

At $d=4$ for the imaginary part we verify the known result.

Using explicit formula we find the on-threshold value of the integral:

$$\begin{aligned} J_3^{(d)}\Big|_{q^2=9m^2} &= \frac{\Gamma^2(\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\,{}_3F_2\left[\begin{matrix}1,-1+2\varepsilon,\frac{3}{2}-\varepsilon\,; & -\frac{1}{3}\\ \frac{1}{2}+\varepsilon,2-\varepsilon\,; & \end{matrix}\right] \\ &= \frac{\Gamma^2(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)}\left\{-\frac{3}{2\varepsilon^2}+\frac{9}{4\varepsilon}+\frac{75}{8}-\frac{8\pi}{\sqrt{3}}+O(\varepsilon)\right\}. \end{aligned}$$

The analytic expression was not known. The first several terms in the ε expansion are in agreement with the result of Davydychev and Smirnov.

Several remarks about solution of dimensional recurrency

- For the fi rst time analytic expression for the sunrise diagram was found
- The differential equation is Heun equation with four regular singular points, located at $q^2 = 0, m^2, 9m^2, \infty$. In general reduction of the Heun equation to the hypergeometric equation is a complicated mathematical problem
- The associated homogeneous difference equation for $J_3^{(d)}$ is simple, and admits reduction to a hypergeometric type of equation with linear coeffi cients.
- This is a general situation. Kinematical singularities of Feynman integrals are located on complicated manifolds. In the case when the differential equations are of the fi rst order there are no problems to solve them. However, to solve a second or higher order differential equations in general will be a problem because of complicated structure of the kinematical

singularities.

- Singularities of Feynman integrals are poles in $1/(d-n)$ with integer n. This has been used for an evident rescaling of the integral by ratios of Γ functions which allowed us to reduce the order of the polynomial coeffi cients in the difference equation.

Concluding remarks

- Calculating Feynman integrals by solving difference equations w.r.t. d provides new powerful tool for fi nding analytical representation of the integral.
- Dimensional recurrences can be used for analytical as well as numerical evaluation of master integrals.
- Techniques for analytical solution of the higher order difference equations are needed!
- Classifi cation of solvable cases analogous to differential equations i.e. analog of Kamke's book on differential equations is needed also for difference equations!