Gröbner Computations for the Ring of Proper Stable Rational Multivariate Functions

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Joint Work

- with my thesis-advisor Ulrich Oberst
- in the framework of the FWF-project

Constructive Multidimensional System Theory

Outline

Introduction

The Ring S of Proper Stable Rational Functions

Proper Stabilization

Gröbner Bases Computations in S

History

since 1985: stabilization of **discrete** multidimensional **transfer matrices**: N.K. Bose, J.P. Guiver, Z. Lin, A. Quadrat, S. Shankar, V.R. Sule, L. Xu, J.-Q. Ying, E. Zerz, et. al.

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Synthesis: **proper** stabilization of discrete and continuous IO **behaviours**

Basic Data

- an $\mathbb{C}$-algebra of operators $\mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_r]$
- a function space $\mathcal{F}$ with $\mathbb{C}[s]$-module structure

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Example: partial differential equations with constant coefficients

- $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R}^r, \mathbb{C})$
- $\mathbb{C}[s]$-module structure on $\mathcal{F}$ defined by $s_\rho \circ y := \frac{\partial y}{\partial z_\rho}$

Input/Output System

- matrices $P \in \mathbb{C}[s]^{k\times p},\ Q \in \mathbb{C}[s]^{k\times m}$ with
  $p = \mathrm{rank}(P) = \mathrm{rank}(P, -Q)$
  $\implies \exists_1 H \in \mathbb{C}(s)^{p\times m}$ with $PH = Q$
  $H =$ transfer matrix

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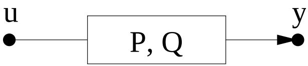
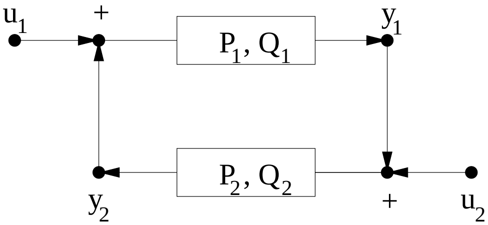
  $H =$ transfer matrix
- IO behaviour $\mathcal{B} := \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{(p+m)}, P \circ y = Q \circ u \right\}$ with
  input u and output y



Feedback System

IO systems $\mathcal{B}_i = \left\{ \begin{pmatrix} y_i \\ u_i \end{pmatrix} \in \mathcal{F}^{p+m}, P_i \circ y_i = Q_i \circ u_i \right\}, \; i = 1, 2$

$$\text{feedback}(\mathcal{B}_1, \mathcal{B}_2) := \left\{ \begin{pmatrix} y_1 \\ u_1 \\ u_2 \\ y_2 \end{pmatrix} \in \mathcal{F}^{2(p+m)}, \; \begin{array}{l} P_1 \circ y_1 = Q_1 \circ (u_1 + y_2) \\ P_2 \circ y_2 = Q_2 \circ (u_2 + y_1) \end{array} \right\}$$



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When does a compensator $\mathcal{B}_2$ exist, such that the feedback system

- is a stable IO system with input $\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$ and output $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and
- has a proper transfer matrix $H \in \mathbb{C}(s)^{(p+m)\times(p+m)}$?

The Multidimensional Degree – Definition

- $f \in \mathbb{C}[s]$
- $\deg_{s_\rho} :=$ Degree of f in $\mathbb{C}[s_1, \ldots, s_{\rho-1}, s_{\rho+1}, \ldots, s_r][s_\rho]$ for $\rho = 1, \ldots, r$
- $\deg(f) := \left(\deg_{s_1}(f), \ldots, \deg_{s_r}(f)\right) \in \mathbb{N}^r$ the multidimensional degree of f
- Extension to rational functions:
  $\mathbb{C}(s) \setminus \{0\} \longrightarrow \mathbb{Z}^r,\ \deg\left(\frac{a}{b}\right) := \deg(a) - \deg(b)$

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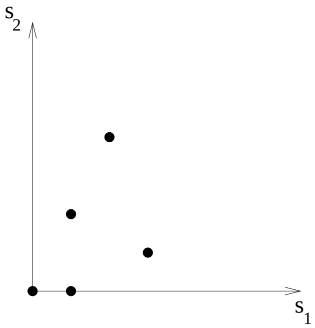
Important: $\deg$ not induced by monomial ordering $\Longrightarrow$ not suitable for Gröbner computations

The Multidimensional Degree – 1st Example

$$f = 1 + s_1 + 10 s_1 s_2{}^2 + 7 s_1{}^3 s_2 - 19 s_1{}^2 s_2{}^4 \in \mathbb{C}[s_1, s_2]$$



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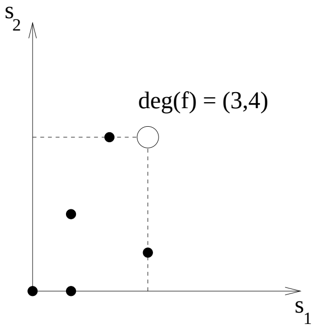
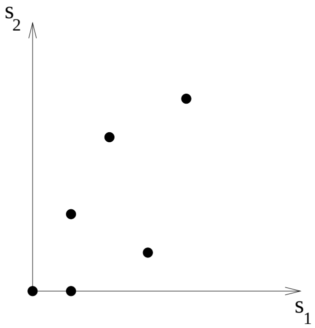


Possibility: $\deg(f) \notin \operatorname{supp}(f) := \{\mu,\ \operatorname{coeff}(f, s^\mu) \neq 0\}$

The Multidimensional Degree – 2nd Example

$$f = 1 + s_1 + 10 s_1 s_2{}^2 + 7 s_1{}^3 s_2 - 19 s_1{}^2 s_2{}^4 + 3 s_1{}^4 s_2{}^5 \in \mathbb{C}[s_1, s_2]$$



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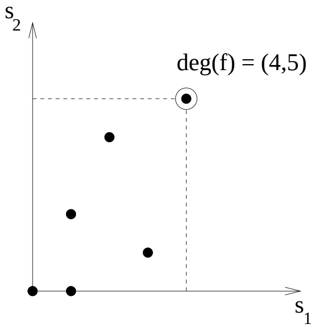

$f \in \mathbb{C}[s]$ component-wise unital (cw-unital) $:\Longleftrightarrow \deg(f) \in \operatorname{supp}(f)$

The Ring of Proper Rational Functions

Definition: The ring of proper rational functions

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where $\mathbb{C}[[s^{-1}]] =$ power series in $(s_1^{-1}, \ldots, s_r^{-1})$.

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Example: $\frac{1}{s_1 - \alpha} \in P$ for $\alpha \in \mathbb{C}$.

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Given: a decomposition $\mathbb{C}^r = \Lambda_1 \uplus \Lambda_2$, where

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  $\{t \in \mathbb{C}[s], V(t) \subseteq \Lambda_1\} =$
  set of stable polynomials
  $V(t) = \{\lambda \in \mathbb{C}^r,\ t(\lambda) = 0\}$

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Assumption: $\forall \rho = 1, \ldots, r\ :\ \mathrm{proj}_\rho(\Lambda_2) \neq \mathbb{C}$, where
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Corollary:

- p_ρ stable, cw-unital $\Longrightarrow \frac{1}{p_\rho} \in S$
- $p := (p_1, \ldots, p_r) \Longrightarrow \frac{1}{p^\mu} = \frac{1}{p_1^{\mu_1} \ldots p_r^{\mu_r}} \in S$ $\forall \mu \in \mathbb{N}^r$

The Ring S_1 and the Noetherianess of S

- $$S_1 := \left\{ \frac{a}{p^\mu},\ a \in \mathbb{C}[s],\ \deg(a) \leqslant \mu \right\} \subseteq S \text{ subring}$$

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- $T_1 := \left\{ \frac{t}{p^{\deg(t)}},\ t \text{ stable, cw-unital} \right\} \subseteq S_1$
- $S = S_{1 T_1}$ quotient ring $\Longrightarrow$ S is noetherian ring

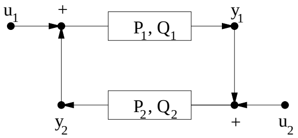
Proper Stabilization Theorem

Given: IO system

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathcal{F}^{(p+m)}, P_1 \circ y_1 = Q_1 \circ u_1 \right\} \text{ with}$$

associated row-module $W_1 := \mathbb{C}[s]^{1 \times k_1}(P_1, -Q_1)$

Equivalent: 1. $\exists$ compensator $\mathcal{B}_2$, such that $\mathrm{feedback}(\mathcal{B}_1, \mathcal{B}_2)$ is stable and has a proper transfer matrix



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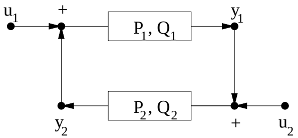

2. $\left(\mathbb{C}[s]^{1\times(p+m)} \cap \bigcap_{\lambda \in \Lambda_2} W_{1\mathfrak{m}_\lambda}\right)_T \cap S^{1\times(p+m)}$ is a direct summand of $S^{1\times(p+m)}$, where $\mathfrak{m}_\lambda = \{f \in \mathbb{C}[s], f(\lambda) = 0\}$

How to Check 2

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2. compute S-generators of $U_T \cap S^{1\times(p+m)}$
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Gröbner Bases Computations in S, part 1

Given: $\mathbb{C}[s]$-generating system of $U \subseteq \mathbb{C}[s]^{1\times l}$ submodule

Assumption: U is T-closed, i. e.

$$tu \in U,\ t \in T, u \in \mathbb{C}[s]^{1\times l} \Longrightarrow u \in U$$

Goal: S-generating system of $U_T \cap S^{1\times l}$

Compute via S_1:

$$\begin{aligned} U_1 &:= S_1^{1\times l} \cap \mathbb{C}[p, p^{-1}]U \\ &= \mathbb{C}[p^{-1}]^{1\times l} \cap \mathbb{C}[p, p^{-1}]U \\ &\subseteq \mathbb{C}[p^{-1}]^{1\times l} \text{ submodule} \end{aligned}$$

Gröbner Bases Computations in $\mathbb{C}[p, p^{-1}]$

Let $q = (q_1, \ldots, q_r)$ be additional variables,

$$\mathcal{I} := \sum_{\rho=1}^{r} \mathbb{C}[p, q](p_\rho q_\rho - 1) \subset \mathbb{C}[p, q] \text{ ideal}$$

Then

$$\begin{array}{ccc} \mathbb{C}[p, p^{-1}] & \cong & \mathbb{C}[p, q]/\mathcal{I} \\ p_\rho & \longleftrightarrow & \overline{p_\rho} \\ p_\rho^{-1} & \longleftrightarrow & \overline{q_\rho} \end{array}$$

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Use an elimination-termorder for p to construct a Gröbner basis for $\mathbb{C}[p^{-1}]^{1\times l} \cap \mathbb{C}[p, p^{-1}]U = U_1$

Gröbner Bases Computations in S, part 2

U is T-closed

$\Longrightarrow U_{1T_1} = U_T \cap S^{1\times l}$

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$\Longrightarrow$

S_1-generating system of $U_1 =$
S-generating system of $U_T \cap S^{1\times l}$

Summary

- short proof that S is noetherian

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- important step in checking the existence of a proper stable feedback system