Solving parametric polynomial systems

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Solving ???

$$\mathcal{E} = \{p_1, \ldots, p_r\}, \mathcal{F} = \{f_1, \ldots, f_l\}, \text{ with } p_i, f_i \in \mathbb{Q}[U, X]$$

$$U = U_1, \ldots, U_d \Rightarrow \textbf{parameters}$$

$$X = X_{d+1}, \ldots, X_n \Rightarrow \textbf{ indeterminates}$$

$$\mathcal{C} = \{x \in \mathbb{C}^n, p_1 = 0, \ldots, p_r = 0, f_1 \neq 0, \ldots, f_s \neq 0\}$$

$$\mathcal{S} = \{x \in \mathbb{R}^n, p_1 = 0, \ldots, p_r = 0, f_1 > 0, \ldots, f_s > 0\}$$

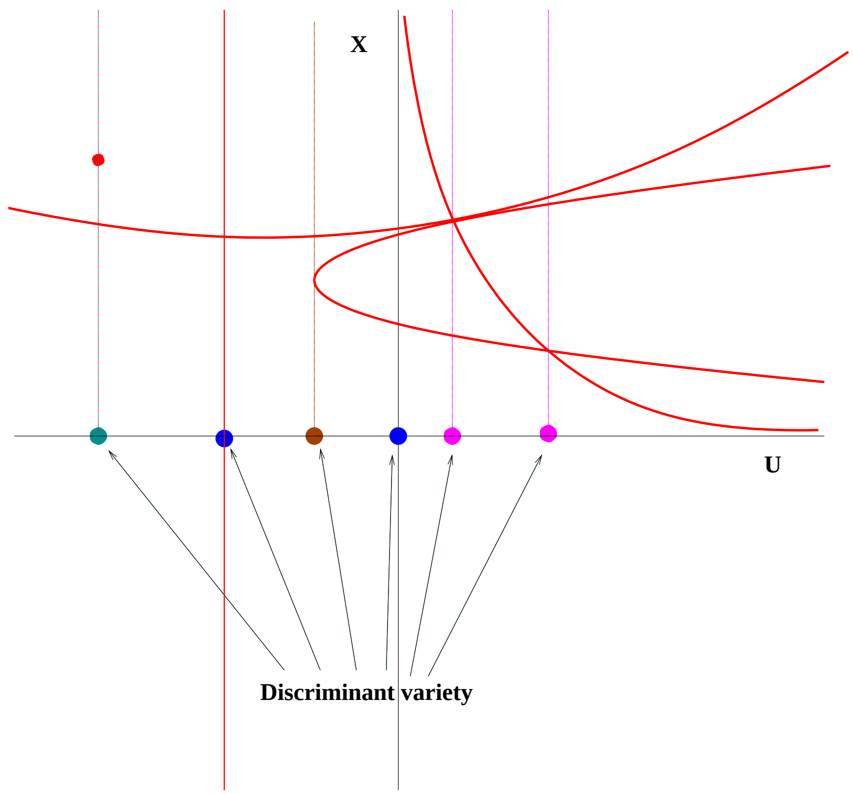
$\Pi_U : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ the canonical projection onto the parameters' space

- Exists parameters values u s.t. $\mathcal{C}_{|_{U=u}} \neq \emptyset$ or $\mathcal{S}_{|_{U=u}} \neq \emptyset$?
- Number of complex (resp. real) points of $\mathcal{C}_{|_{U=u}}$ or $\mathcal{S}_{|_{U=u}}$?
- "Simple" description of $\mathcal{C}$ or $\mathcal{S}$ wrt Π_U ?

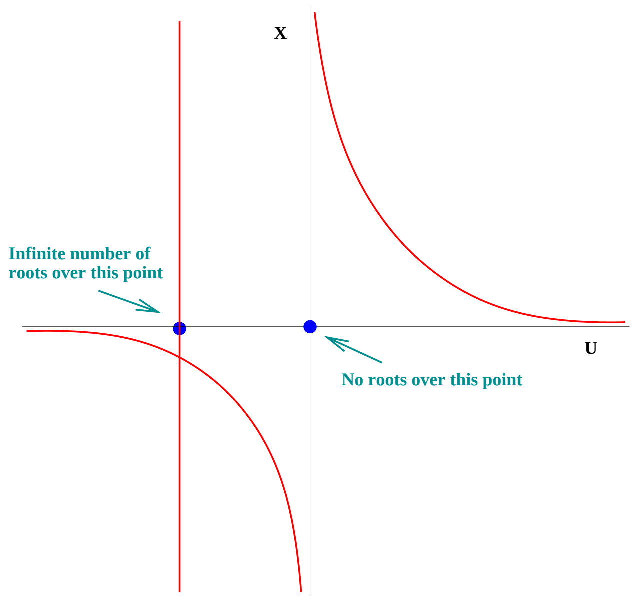
The study of $\mathcal{C}=\{x\in\mathbb{C}^n, p=0, f\neq 0, p\in\mathcal{E}, f\in\mathcal{F}\}$

If one wants (at least) to discuss the number of roots, one needs to characterize parameter's subsets $\mathcal{U}\subset\Pi_U(\mathcal{C})$ st $\#\left(\Pi_U^{-1}(u)\cap\mathcal{C}\right)$ is constant on $\mathcal{U}$.

"bad" parameters $(\mathcal{U}\cap\{\text{bad parameters}\}=\emptyset)$:



Points "going" to infinity :



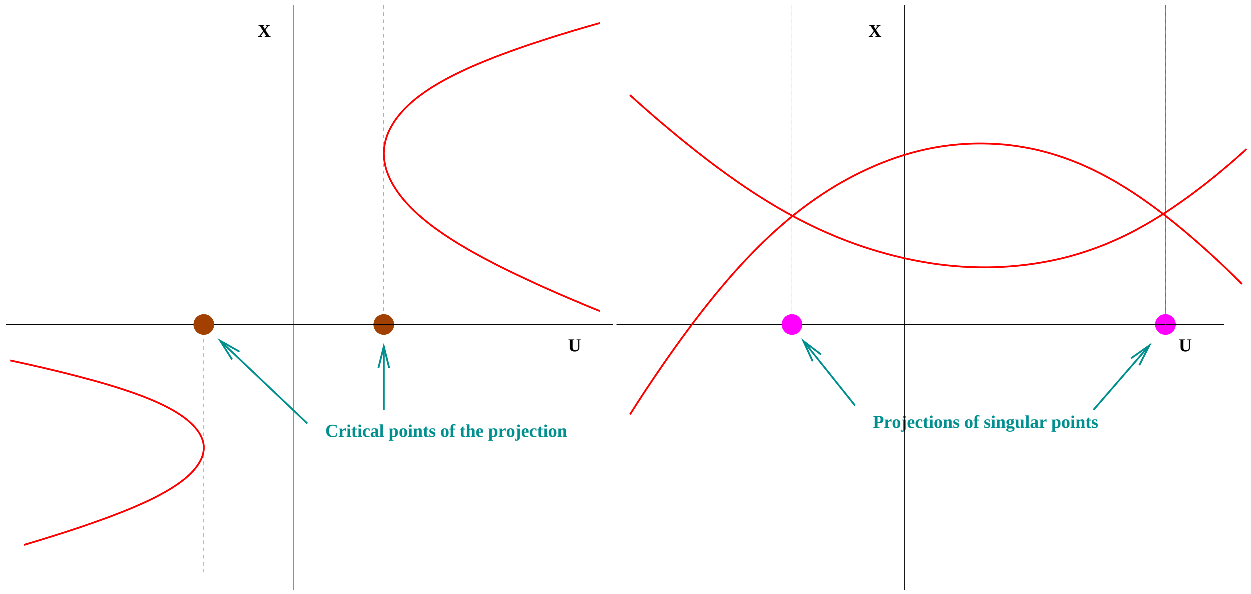
One needs to consider $\overline{\Pi_U(\mathcal{C})}$ ...

$O_\infty = \{u \in \overline{\Pi_U(\mathcal{C})}$, for any compact neighborhood $\mathcal{V} \ni u$, $\Pi_U^{-1}(\mathcal{V}) \cap \mathcal{C}$ is not compact$\}$.

Remark : $\overline{\Pi_U(\mathcal{C})} \setminus \Pi_U(\mathcal{C}) \subset O_\infty$.

$\mathcal{U}$ can not intersect properly O_∞ : $\mathcal{U} \cap O_\infty = \emptyset$ or $\mathcal{U}$

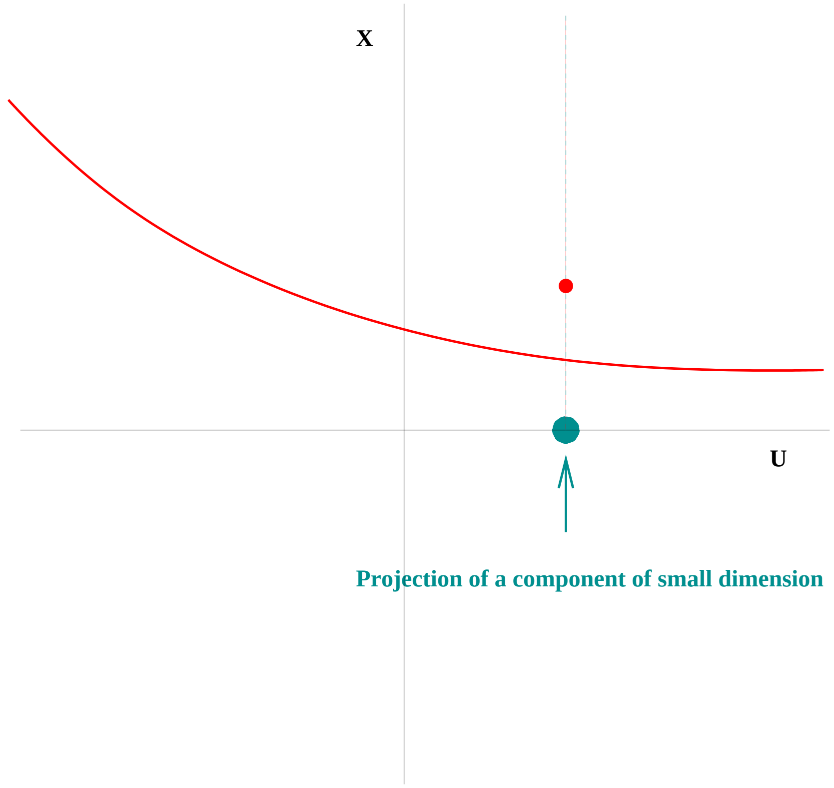
Critical points of the projection and singular points :



$$O_c = \{\text{critical points of } \Pi_U \text{ on } \mathrm{Reg}(\mathcal{C})\} \cup \{\text{singular points of } \mathcal{C}\}$$

$\mathcal{U}$ can not intersect O_c : $\mathcal{U} \cap O_c = \emptyset$

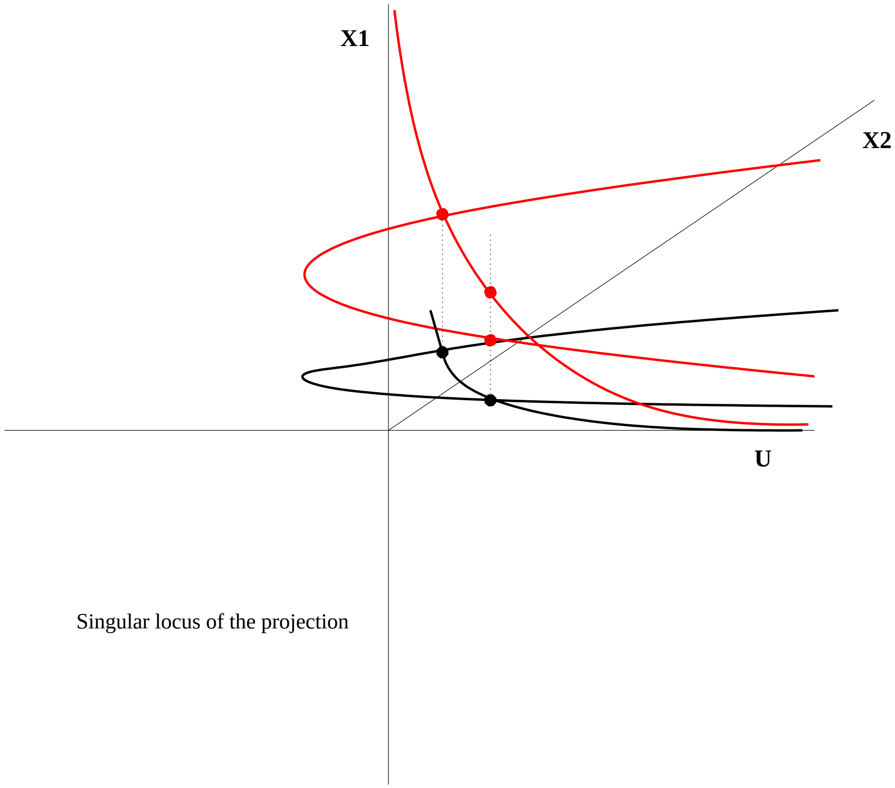
Components of small dimension



$O_{\mathrm{sd}} = \{$projection - by Π_U - of the components of "small" dimension$\}$

$\mathcal{U}$ can not intersect O_{sd} : $\mathcal{U} \cap O_{\mathrm{sd}} = \emptyset$

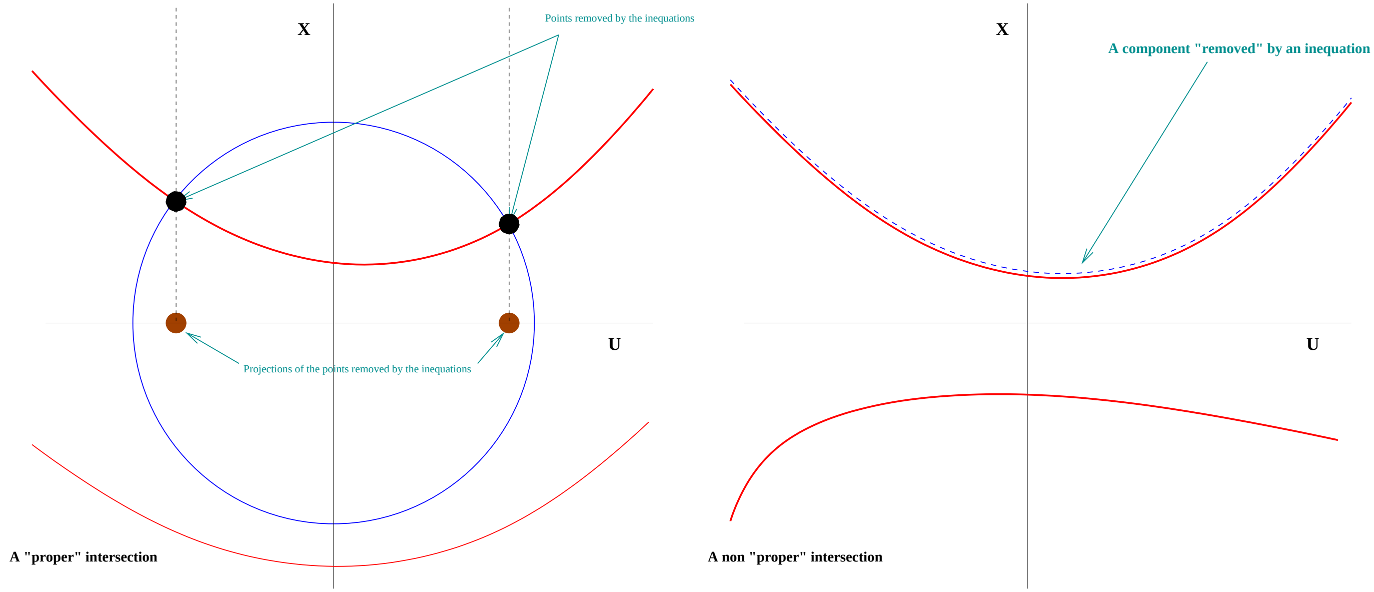
Singular locus of $\Pi_U(\mathcal{C})$



$O_{\text{sing}} = \text{Singular locus of } \Pi_U(\mathcal{C})$

$\mathcal{U}$ can not intersect O_{sing} : $\mathcal{U} \cap O_{\text{sing}} = \emptyset$

Inequations !



Study $\bar{\mathcal{C}} = \boldsymbol{V}(\langle\mathcal{E}\rangle : (\prod_{f\in\mathcal{F}} f)^\infty) = \overline{\boldsymbol{V}(\langle\mathcal{E}\rangle) \setminus \boldsymbol{V}(\prod_{f\in\mathcal{F}} f)}.$

$$O_\mathcal{F} = \{u \in \Pi_U(\mathcal{C}), \Pi_U^{-1}(u) \cap \boldsymbol{V}(\prod_{f\in\mathcal{F}} f) \cap \bar{\mathcal{C}}\}$$

$\mathcal{U}$ can not intersect properly O_f : $\mathcal{U} \cap O_\mathcal{F} = \emptyset$ or $\mathcal{U}$

Solving ???

Summary : if $\mathcal{U}$ is s.t. $u \longrightarrow \#\Pi_U^{-1}(u)$ is constant on $\mathcal{U}$, then $\mathcal{U}$ can not intersect properly $O_\infty \cup O_c \cup O_{\mathrm{sd}} \cup O_{\mathcal{F}}$ with

$O_\infty = \{u \in \overline{\Pi_{\mathcal{U}}(\bar{\mathcal{C}})}$, for any compact neightborhood $\mathcal{V} \ni u$, $\Pi_U^{-1}(\mathcal{V}) \cap \bar{\mathcal{C}}$ is not compact$\}$

$O_c = \{$critical values of $\Pi_U\} \cup \{$singular points of $\mathcal{C}\}$

$O_c =$ singular points of $\Pi_{\mathcal{U}}(\mathcal{C})$

$O_{\mathrm{sd}} =$ projection of the components of $\mathcal{C}$ of dimension less than $\dim(\Pi_U(\mathcal{C}))$

$O_{\mathcal{F}} = \{u \in \Pi_U(\bar{\mathcal{C}}), \Pi_U^{-1}(u) \cap \bar{\mathcal{C}} \cap \boldsymbol{V}(\prod_{f \in \mathcal{F}} f)$

Proposition : If $\mathcal{U} \subset \Pi_U(\mathcal{C})$ is any submanifold which do not meet $O_\infty \cup O_c \cup O_{\mathrm{sd}} \cup O_{\mathcal{F}}$, then $\Pi_U : \mathcal{C} \cap \Pi_U^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is a (analytic) covering.

In particular, the number of roots of $\mathcal{C}$ is constant over $\mathcal{U}$ and we have a "simple" description of $\mathcal{C}$ over $\mathcal{U}$.

$\Rightarrow$ Definition of "solving" a parametric system independent from any computational strategy.

With a computational point of view

Proposition $W_D = O_\infty \cup O_c \cup O_{\mathrm{sd}} \cup O_{\mathcal{F}}$ is Zariski closed.

$$1)\quad \begin{array}{ccccc} \mathbb{C}^d \times \mathbb{C}^{n-d} & \xrightarrow{\text{canonical}} & \mathbb{C}^d \times \mathbb{P}^{n-d} & \xrightarrow{\text{canonical}} & \mathbb{C}^d \\ \bar{\mathcal{C}}^z & \mapsto & \bar{\mathcal{C}}^p & & \\ & & \bar{\mathcal{C}}^p \cap H_\infty & \mapsto & O_\infty = \overline{O_\infty} \end{array}$$

2) $\overline{\Pi_U(\mathcal{C})} = \Pi_U(\mathcal{C}) \cup O_\infty$

3) $O_{\mathrm{xx}} \subset \overline{O_{\mathrm{xx}}} \subset O_{\mathrm{xx}} \cup O_\infty$

Remarks :

- $\Pi_U^{-1}(\Pi_U(\mathcal{C}) \setminus W_D) \cap \mathcal{C}$ is the set of "generic" resolution of $\mathcal{C}$ (the projection is a submanifold of dimension $\delta = \dim(\Pi_U(\mathcal{C}))$).

- "non generic parameters" (submanifolds of dimension $< \delta$) belong to W_D. They can be separately studied : $\mathcal{C} \cap W_D$ is "smaller' parametric system.

Complex Discriminant Varieties

Definition 1. *$W_D = O_\infty \cup O_c \cup O_{\mathrm{sd}} \cup O_{\mathcal{F}} \cup O_{\mathrm{sing}}$ is **the** minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.*

An algebraic variety W is a (large) discriminant variety of $\mathcal{C}$ w.r.t. Π_U iff:

- $W_D \subset W \subsetneq \overline{\Pi_U(\mathcal{C})}$
- $W = \overline{\Pi_U(\mathcal{C})}$ *iff* $\mathcal{C}_{|U=u}$ *is infinite or empty for almost all* $u \in \overline{\Pi_U(\mathcal{C})}$*;*

A D.V. is an algebraic variety W such that :

- $\Pi_U(\mathcal{C}) \setminus W = \cup_{i=1}^k \mathcal{U}_i$ is a finite union of submanifolds of dimension $\dim(\overline{\Pi_U(\mathcal{C})})$.
- $\Pi_U \colon \Pi_{\mathcal{U}}^{-1}(\mathcal{U}_i) \cap \mathcal{C} \longrightarrow \mathcal{U}_i$ is a (analytic) cover $\forall i$.

Discriminant Varieties in the Real case

If W_D is a minimal discriminant variety for $\mathcal{C}$ wrt Π_U, then either $W_D \cap \mathbb{R}^d$ is a (non necessarilly minimal) discriminant variety for $\mathcal{S}$ wrt Π_U or W_D contains $\Pi_U(\mathcal{S})$.

In the second case, we simply replace $\mathcal{S}$ by $\mathcal{S} \cap W_D$ and compute again (the dimension of the projection then decreases).

Note that this correspond tho the case where the dimension of the real counterpart of the main components (those of dimension δ whose projection is not contained in W_D) differ from the "complex" dimension.

To detect this : $\mathcal{S}_{|U=u}$ has no solutions $\forall u \in \overline{\Pi_U(\mathcal{C})} \cap \mathbb{R}^d \setminus W_D$

The "real" version of the minimal discriminant variety is a semi-algebraic set.

Discriminant Varieties in the Real case

Over each connected component of $\Pi_U(\mathcal{S}) \setminus W_D$:

- the number of real roots is constant;
- the sheets are locally diffeomorphic to the connected components;

For "solving" the initial problem, one needs to describe the connected components of $\Pi_U(\mathcal{C}) \cap \mathbb{R}^d \setminus W_D$ (we "eliminated" $n-d$ variables).

- Compute one point on each C.C. (Roy, Safey, TERA,..) + solving a zero-dimensional system : qualitative information.
- Compute a Cylindrical Algebraic Decomposition adapted to the polynomials defining the discriminant variety : full description.
- In practice : we use a "partial" CAD - avoid most projections as well as computations with algebraic numbers. In short : do not decompose W_D.

[optional] : For a full description : apply the algorithm on $\mathcal{S} \cap W_D$.

Examples of Large Discriminant Varieties

- Rational parametrizations : discriminant + leading coefficient of the univariate polynomial - depends on the choice of a generic change of variables;

- Cylindrical Algebraic decomposition (eliminate the indeterminates first) : the polynomials obtained after $n-d$ projections - depends on the order in which the projections are computed + consequences of the elimination variable by variable.

- Regular and separable triangular sets : leading coefficients in the main + discriminants in the main variable.

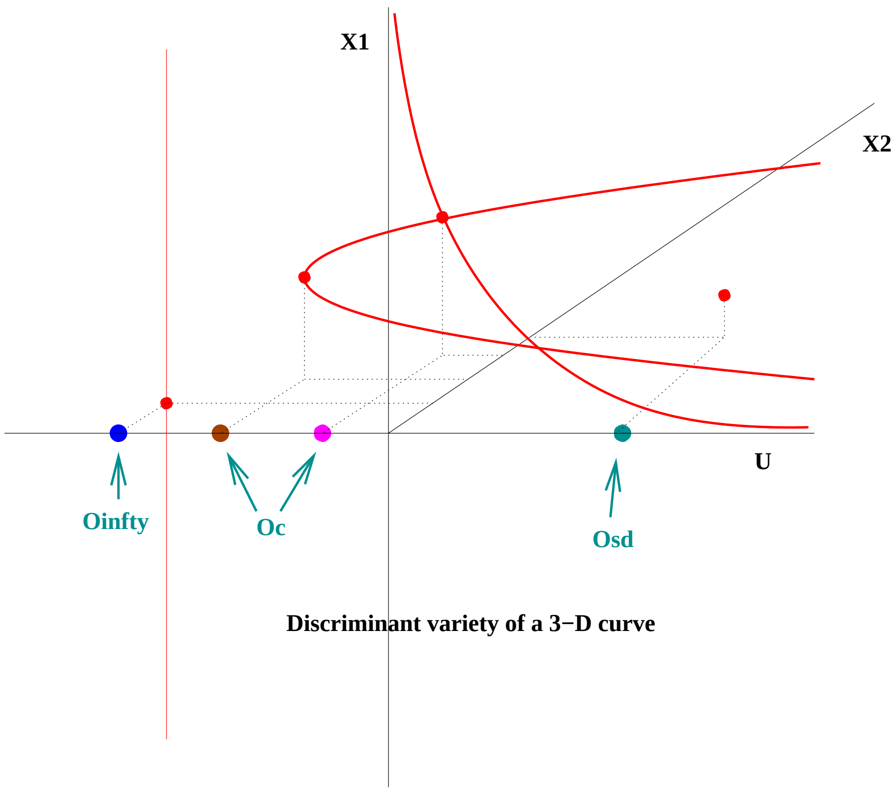
- (*) Vectors of multiplicities (Grigoriev and Vorobjov) : partition with much more elements (decomposes also the non generic solutions).

(*) $\Rightarrow \exists$ **a single exponential algorithm computing a large discriminant variety.**

A **challenge** : compute a minimal discriminant variety

Remark

When $n - d = 1$, the discriminant variety coincides with the zero set of the polynomials obtained after the first projection step in a CAD when starting with X_n. For $n-d>1$ it is smaller :



Discriminant variety of a 3−D curve

Well-behaved systems

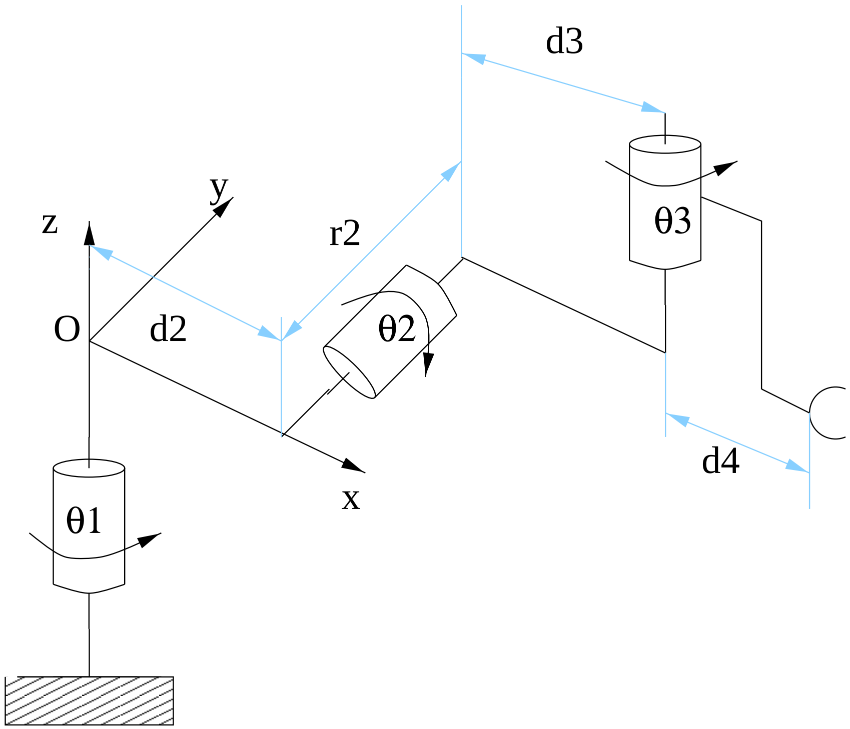
Most systems comming from applications (outside mathematics) are s.t. :

- for almost all $u \in \mathbb{R}^d$, the real roots of $\mathcal{E}_{|U=u} = 0$ can be computed (real roots) by a basic version of Newton's method

Most of them verify the following conditions (well-behaved systems)

- $\#\mathcal{E} = n - d$;
- $\overline{\Pi_U(\boldsymbol{V}(\langle\mathcal{E}\rangle))} = \mathbb{C}^d$;
- $\langle\mathcal{E}_{|U=u}\rangle \subset \mathbb{C}[X]$ is radical and zero-dimensional for allmost all $u \in \mathbb{C}^d$.

An example : cuspidal serial manipulators



End-user query : for which design parameters these kind of robot can change of posture without crossing a singularity ?

Cuspidal serial robots

Robotician query :

one can eliminate 2 angles θ_1 and θ_2, assume that $d_2 = 1$ without lost of generalty and show that the problem is **equivalent to deciding if a univariate polynomial p of degree 4 in $\tan(\frac{\theta_3}{2})$ has or not triple real roots**, its coefficients depending on 2 indeterminates ρ, z ($\rho^2 = x^2 + y^2$, x, y, z being the coordinates of the end-effector) and on 3 parameters d_4, d_3, r_2 (design parameters).

Our problem : decide for which values of $d_4 > 0, d_3 > 0$ and $r_2 > 0$ a system of 3 equations $\{p(d_4, d_3, r_2, z, \rho, T) = 0, \frac{\partial}{\partial T} p(d_4, d_3, r_2, z, \rho, T) = 0, \frac{\partial^2}{\partial T^2} p(d_4, d_3, r_2, z, \rho, T) = 0\}$ has real admissible solutions ($\rho > 0$).

$\overline{\Pi_U(\bar{\mathcal{C}})}$ and $\overline{O_{\mathcal{F}}}$

- $\langle\mathcal{E}\rangle : (\prod_{f\in\mathcal{F}} f)^\infty = \Big(\langle\mathcal{E}\rangle + \langle T\Big(\prod_{f\in\mathcal{F}} f\Big) - 1\rangle\Big) \cap \mathbb{Q}[U, X]$

- $\bar{\mathcal{C}} = \boldsymbol{V}(\langle\mathcal{E}\rangle : (\prod_{f\in\mathcal{F}} f)^\infty) = \overline{\boldsymbol{V}(\langle\mathcal{E}\rangle) \setminus (\cup_{f\in\mathcal{F}} \boldsymbol{V}(f))}$

Known result : If G is a Gröbner basis for any product of orderings $<_{U,X} = (<_U, <_X)$ with $U_i <_{U,X} X_j, \forall i, j$, then $G \cap \mathbb{Q}[U]$ is a Gröbner basis for $<_U$ of $\langle G\rangle \cap \mathbb{Q}[U]$.

In particular $\Pi_U(\boldsymbol{V}(G)) = \boldsymbol{V}(\langle G \cap \mathbb{Q}[U]\rangle)$ so that we can "efficiently" compute ideals I_Π and $I_{\mathcal{F}}$ such that $\boldsymbol{V}(I_\Pi) = \overline{\Pi_U(\bar{\mathcal{C}})}$ and $\boldsymbol{V}(I_{\mathcal{F}}) = \overline{O_{\mathcal{F}}}$.

In practice : $<_U$ and $<_X$ can be DRL orderings

Remark : valid for any parametric systems !

O_∞

G a Gröbner basis of $\langle \mathcal{E} \rangle$ wrt a DRL-block ordering $<_{U,X}$.

Theorem 2. *if* $\mathcal{E}_0 = G \bigcap \mathbb{Q}[U]$. *Then:*

- $\mathcal{E}_0$ *is a Gröbner basis of* $I \bigcap \mathbb{Q}[U]$ *w.r.t.* $<_U$;
- *Set* $\mathcal{E}_i^\infty = \mathcal{E}_0 \cup \mathcal{E}_i^\infty$ *for* $i = d+1 \ldots n$
- $\mathcal{E}_i^\infty$ *is a Gröbner basis of some ideal* $I_i^\infty \subset \mathbb{Q}[U]$ *w.r.t.* $<_U$;
- $W_\infty = \bigcup_{i=d+1}^{n} \boldsymbol{V}(I_i^\infty)$

Nothing to "compute" when G is known !

Remark : valid for any parametric systems

Computing $\overline{O_c}$ and $\overline{O_{\mathrm{sd}}}$

The main computational problems :

- Jacobian criteria are independant from the equations in the case of radical and equi-dimensional ideals.

  In such cases $\overline{O_c} = \boldsymbol{V}(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})) \cap \mathbb{Q}[U])(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})) \cap \mathbb{Q}[U])$.

- In the general case, $\boldsymbol{V}(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})) \cap \mathbb{Q}[U]) \cap \mathbb{Q}[U])$

  $\longrightarrow$ may give too much points (non radical ideals, embeeded components)

  $\longrightarrow$ may miss some points (components of small dimension)

  $\longrightarrow$ may be of same dimension than $\overline{\Pi_U(\mathcal{C})}$ (non radical ideals)

- We want to avoid as most as possible to compute a decomposition of the ideal into radical and/or equi-dimensional components (avoid also primary decompositions)

In the case of well-behaved systems

One can not suppose, <u>even in practice</u>, that $\langle \mathcal{E} \rangle$ is radical or equidimensional

artefacts from modelizations (from fractions to polynomials, changes of coordinates like $t = \tan(\alpha/2)$, etc.) often introduce primary but not prime components of arbitrary dimensions.

BUT in the case of well-behaved systems :

- the components of dimension $< n - d$ are "embeeded" components since $\#\mathcal{E} = n - d$ (in particular $O_{\mathrm{sd}} = \emptyset$).

- the projection of the zero set of components of dimension $> n - d$ are in O_∞

Theorem 3. *(Well-behaved systems)* $O_{\mathrm{sd}} = O_{\mathrm{sing}} = \emptyset$ *and :*

$$W_D = O_\infty \cup O_{\mathcal{F}} \cup \boldsymbol{V}(\langle \mathcal{E} \rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})) \cap \mathbb{Q}[U]) = O_\infty \cup O_{\mathcal{F}} \cup O_c$$

(use Krull's principal theorem and the hypothesis "$\langle \mathcal{E} \rangle \subset \mathbb{Q}(U)[X]$ is zero-dimensional")

Remarks

Let $\cap_{i=1}^{k} Q_i \cap_{i=k+1}^{k'} Q_i' \cap_{i=k'+1}^{k''} Q_i^{''}$ be a minimal primary decomposition of I with $\boldsymbol{V}(I) = \bar{\mathcal{C}}$ such that

- $\dim(Q_i) = \dim(Q_i \cap \mathbb{Q}[U]) = \dim(\Pi_U(\mathcal{C}))$;

- $\dim(Q_i') < \dim(\overline{\Pi_U(\mathcal{C})})$ and $\dim(Q_i' \cap \mathbb{Q}[U]) = \dim(Q_i')$;

- $Q_i^{''}$: all the other components;

Remark : $\cup_{i=k'+1}^{k''} \Pi_U(\boldsymbol{V}(Q_i'')) \subset O_\infty$;

Remark : for many applications one can ignore the Q_i';

ex : $\dim(\Pi_U(\boldsymbol{V}(I + \mathrm{Jac}_X^{n-\delta}(\mathcal{E}))) < \dim(\overline{\Pi_U(\mathcal{C})})$, $W = O_{\mathrm{sd}} \cup O_c \cup O_{\mathcal{F}} \cup O_{\mathrm{sing}} \cup \Pi_U(\boldsymbol{V}(I + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})))$ is a discriminant variety of $\overline{\boldsymbol{V}(\mathcal{C}) \setminus \cup_{i=k+1}^{k'} \boldsymbol{V}(Q_i')}$.

In practice

Assume the system is "well-behaved" and start the computation "as if".

Can test the hypothesis during the algorithm (avoids to perform a lot of costly operations before trying to solve the problem) : number of equations, I zero-dimensional in $Q(U)[X]$ (generically zero-dim ?), dimension of the elimination ideal of the Jacobian ideal (radical ?).

In any case, $O_{\mathcal{F}}, O_{\infty}, \overline{\Pi_U(\mathcal{C})}$ are definitively computed.

If O_{sd} is not detected to be empty (bad number of equations) : let the choice to the end-user, he can obtain a large discriminant variety of the "main" components of the system.

When $\mathbf{dim}(\boldsymbol{\Pi_U}(\boldsymbol{V}(\boldsymbol{I} + \mathbf{Jac}_{\boldsymbol{X}}^{\boldsymbol{n}-\boldsymbol{\delta}}(\boldsymbol{\mathcal{E}}))) < \mathbf{dim}(\boldsymbol{\Pi_U}(\boldsymbol{\mathcal{C}}))$ we get a large discriminant variety by replacing O_c by $\Pi_U(\boldsymbol{V}(I + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})))$ which is strict in the case of "well-behaved" systems.

When we can not avoid the decomposition

Let $\cap_{i=1}^{k} Q_i \cap_{i=k+1}^{k'} Q_i' \cap_{i=k'+1}^{k''} Q_i^{''}$ be a minimal primary decomposition of I with $\boldsymbol{V}(I) = \bar{\mathcal{C}}$ such that

- $\dim(Q_i) = \dim(Q_i \cap \mathbb{Q}[U]) = \dim(\Pi_U(\mathcal{C}))$;
- $\dim(Q_i') < \dim(\overline{\Pi_U(\mathcal{C})})$ and $\dim(Q_i' \cap \mathbb{Q}[U]) = \dim(Q_i')$;
- $Q_i^{''}$: all the other components;

We only need to compute $O_{\mathrm{sd}} \setminus O_\infty$ and $O_c \setminus O_\infty$ for the primary non prime components Q_i of I of dimension $\delta = \dim(\overline{\Pi_U(\mathcal{C})})$ with $\dim(\overline{\Pi_U(Q_i)}) = \dim(Q_i)$.

One challenge : computing efficiently $\sqrt{\cap_{i=1}^{k} Q_i}$ and $\left(\cap_{i=k+1}^{k'} Q_i'\right) \cap \mathbb{Q}[U]$

One idea : "extension"($\mathbb{Q}(U)[X]$)+localization by I_∞+"contraction"

(A new strategy for primary decomposition ?)

Back to the application

Number of real roots over the 2-D cells between two sheets

(d_3, r_2) : 2-D cells $\setminus$ d_4(sheets)	1	2	3	4	5	6	7
(1,1)	0	0	4	4	2	0	0
(1,2)	0	4	4	4	2	0	0
(1,3)	0	4	4	4	2	0	0
(1,4)	0	4	4	2	2	0	0
(1,5)	0	4	4	2	0	0	0
(2,1)	0	0	4	4	2	2	0
(2,2)	0	4	4	4	2	2	0
(2,3)	0	4	4	4	2	2	0
(2,4)	0	4	4	2	2	2	0
(3,1)	0	4	4	4	2	2	4
(3,2)	0	4	4	4	2	2	4
(3,3)	0	4	4	2	2	2	4
(4,1)	0	4	4	4	2	2	4
(4,2)	0	4	4	2	2	2	4
(5,1)	0	4	4	2	2	2	4

Conclusion

Discriminant Variety

Optimal object (at least in the complex case);

Efficiently computable for a large class of systems with existing tools;

Easy to implement : few basic black boxes - elimination of a block of variables, saturation of an ideal by one polynomial, zero-dimensional solving.

New : the complexity is simple exponential (algorithm + objects) for well-bahaved systems (G. Moroz - 2006).

Challenges :

- lazy decompositions of ideals knowing "I_∞" ??;
- other specifications than CAD for describing the cells of the complementary of an hypersurface ??;
- recusrsive use to describe the "non" generic solutions ??;
- back boxes for quantifier elimination (see D. Lazard's talk) : cell decomposition of basic semi-algebraic sets, ... ??