Using Gröbner Bases for Solving Linear 2Pt Boundary Value Problems

Special Semester on Gröbner Bases
Workshop on Symbolic Analysis
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Outline of the Talk

Cooperation Prof. Buchberger—Prof. Engl within SFB F013.

Importance of boundary conditions for practical usage of differential equations:

> *Lions: «God made the differential equations, but the devil made the boundary conditions.»*

Simple starting point: Linear two-point boundary value problems (briefly BVPs).

- How to **see** them. (On the operators level.)
- How to **solve** them. (By Gröbner bases.)
- How to **factor** them. (Through Stieltjes conditions.)
- How to **divide** them. (In the Mikusiński style.)

Parts (1) and (2): Doctoral thesis → JSC.

Parts (3) and (4): Ongoing joint research with Georg Regensburger.

Two-Point Boundary Value Problems

Given $f \in C^\infty[a, b]$, find $u \in C^\infty[a, b]$ such that:

$$(\#) \quad \begin{array}{l} T\,u = f \\ B_1\,u = \ldots = B_n\,u = 0 \end{array}$$

$[a, b]$ . . . . Finite interval in $\mathbb{R}$

$C^\infty[a, b]$ . . . . Smooth functions $[a, b] \to \mathbb{C}$

T . . . . Linear differential operator of order n

$B_1, \ldots, B_n$ . . . . Two-point boundary operators

For simplicity, we assume constant coefficients in T:

$$\begin{array}{llll} T & : & C^\infty[a, b] & \to C^\infty[a, b] \\ & & u & \mapsto c_0\,u + c_1\,u' + c_2\,u'' + \ldots + c_{n-1}\,u^{(n-1)} + u^{(n)} \end{array}$$

Two-point boundary operators:

$$\begin{array}{llll} B_i & : & C^\infty[a, b] & \to \mathbb{C} \\ & & u & \mapsto p_{i,0}\,u^{(n-1)}(a) + \ldots + p_{i,n-1}\,u'(a) + p_{i,n}\,u(a) \quad + \end{array}$$

Traditional Solution by the Green's Function

We assume regularity:

$$\underset{f}{\forall}\ \underset{u}{\exists!}\ \ (\#)$$

Then every linear two-point BVP admits to write the solution as:

$$u(x) = \int_a^b g(x, \xi)\ \ f(\xi)\ \ d\xi$$

g	....	Green's function $[a, b]^2 \to \mathbb{C}$
f	....	Given forcing function $[a, b] \to \mathbb{C}$
u	....	Desired solution function $[a, b] \to \mathbb{C}$

Traditional solution method (see e.g. Kamke): Matrix inversion based on a fundamental system for T.

Importance of Green's function: Response to "point sources".

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An Example: Beam Deflection

Beam deflected under a loading σ:

$$-u''''(x) + \underbrace{\sigma(x)\,/\,E\,I}_{f(x)}\,u(x) = 0 \qquad \longrightarrow \qquad u''''(x) = f(x)$$

Fixed at both ends, but allowed to bend:

$$\begin{array}{ll} u''''(x) = f(x) & \\ \hline u(0) = 0 & u''(0) = 0 \\ u(1) = 0 & u''(1) = 0 \end{array}$$

Green's function:

$$g(x, \xi) = \begin{cases} \frac{1}{3} x \xi - \frac{1}{6} \xi^3 - \frac{1}{2} x^2 \xi + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{if} \quad 0 \le \xi \le x \le 1 \\ \frac{1}{3} x \xi - \frac{1}{2} x \xi^2 - \frac{1}{6} x^3 + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{if} \quad 0 \le x \le \xi \le 1 \end{cases}$$

The Operator Perspective of BVPs

Everything seems to be on the functional level:

- Problem statement: For every **function** f, find the **function** u such that …
- Solution method: Linear algebra over **function** fields.
- Final solution: The Green’s **function**.

$$u \underset{\longleftarrow}{\overset{T}{\longmapsto}}_{G} f$$

But *actually*, everything happens on the operator level:

- What we really want is the Green’s **operator** G mapping forcing functions to solutions (via g or not).
- In some sense, it is the inverse of the given differential **operator** T.
- We will present a new method that works directly on (an algebraic model of) the **operators**.

Integro-Differential Algebras

A complex algebra $\mathcal{F}$ with basis $\mathcal{F}^{\#}$, plus 5 linear operations fulfilling these axioms:

Boundary operators as abbreviations: $f^{\leftarrow} \equiv f - \int^{*} f'$ *and* $f^{\rightarrow} \equiv \int_{*} f' - f$.

Differentiation	$\ldots'$	$: \mathcal{F} \to \mathcal{F}$	$(fg)' = f'g + fg'$
Integral	$\int^* \ldots$	$: \mathcal{F} \to \mathcal{F}$	$\int^* f' = f - f^{\leftarrow},\ (\int^* f)' = f$
Cointegral	$\int_* \ldots$	$: \mathcal{F} \to \mathcal{F}$	$\int_* f' = f^{\rightarrow} - f,\ (\int_* f)' = -f$
Left Boundary Value	$\ldots^{\leftarrow}$	$: \mathcal{F} \to \mathbb{C}$	$(fg)^{\leftarrow} = f^{\leftarrow} g^{\leftarrow}$
Right Boundary Value	$\ldots^{\rightarrow}$	$: \mathcal{F} \to \mathbb{C}$	$(fg)^{\rightarrow} = f^{\rightarrow} g^{\rightarrow}$

Standard example: $\mathcal{F} = C^\infty[0, 1]$ with:

$$f' = \frac{\partial f}{\partial x}$$
$$\int^* f = \int_0^x f(\xi)\, d\xi$$
$$\int_* f = \int_x^1 f(\xi)\, d\xi$$
$$f^{\leftarrow} = f(0)$$
$$f^{\rightarrow} = f(1)$$

Common subalgebra: Exponential polynomials

$$\mathcal{E}xp^{\#} = \{x^k\, e^{\lambda x} \mid k \in \mathbb{N} \wedge \lambda \in \mathbb{C}\}$$

Other example: $\mathcal{F} = C^\infty(\mathbb{R}^2)$ with:

$$f' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$
$$\int^* f = \int_0^x f(t, t - x + y)\, dt$$
$$\int_* f = \int_x^1 f(t, t - x + y)\, dt$$
$$f^{\leftarrow} = f(0, -x + y)$$
$$f^{\rightarrow} = f(1, 1 - x + y)$$

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Integro-Differential Polynomials

Analogous to differential algebra → differential polynomials:

Idea: *Integral / differential / boundary / multiplication operators*

~ "formal linear expressions" of $\mathcal{F}$ containing an indeterminate u.

Introduce abbreviations for basic operators:

Differential operator	u'	$\rightharpoonup$	D
Integral Operator	$\int^{*} u$	$\rightharpoonup$	A
Cointegral Operator	$\int_{*} u$	$\rightharpoonup$	B
Left Boundary Operator	$u^{\leftarrow}$	$\rightharpoonup$	L
Right Boundary Operator	$u^{\rightarrow}$	$\rightharpoonup$	R
Multiplication Operator	f	$\rightharpoonup$	$\lceil f \rceil$

Hence we construct the noncommutative algebra of integro-differential polynomials:

$$\mathcal{An}(\mathcal{F}) = \mathbb{C}\langle\{D, A, B, L, R\} \cup \{\lceil f \rceil \mid f \in \mathcal{F}^{\#}\}\rangle / \mathcal{Grn}$$

Here $\mathcal{Grn}$ is the ideal generated by a set of polynomial identities ("Green's system").

Somehow mirroring integro-differential axioms.

The Green's System 1, 2

```
System["1. Equalities for Isolating Differential Operators", any[f],
  D A = 1                          "DA"
  D B = -1                         "DB"
  D ⌈f⌉ = ⌈f⌉ D + ⌈f′⌉             "DM"
  D L = 0                          "DL"
  D R = 0                          "DR"
]
```

```
System["2. Equalities for Isolating Boundary Operators", any[f],

 L A = 0                                          "LA"
 R A = A + B                                      "RA"
 L B = A + B                                      "LB"
 R B = 0                                          "RB"
 ------------------------------------------------------
 L ⌈f⌉ = f← L                                     "LM"
 R ⌈f⌉ = f→ R                                     "RM"
 ------------------------------------------------------
 L L = L                                          "LL"
 L R = R                                          "LR"
 R L = L                                          "RL"
 R R = R                                          "RR"

]
```

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The Green's System 0, 3

```
System["Equalities for Algebraic Simplication", any[f, g],
 ⌈f⌉ ⌈g⌉ = ⌈f g⌉                                  "MM"
]
```

```
System["3. Equalities for Contracting Integration Operators", any[f],

 A ⌈f⌉ A = ⌈∫* f⌉ A - A ⌈∫* f⌉                    "AMA"
 A ⌈f⌉ B = ⌈∫* f⌉ B + A ⌈∫* f⌉                    "AMB"
 B ⌈f⌉ A = ⌈∫_* f⌉ A + B ⌈∫_* f⌉                  "BMA"
 B ⌈f⌉ B = ⌈∫_* f⌉ B - B ⌈∫_* f⌉                  "BMB"
 ------------------------------------------------------
 A A = ⌈∫* 1⌉ A - A ⌈∫* 1⌉                        "AA"
 A B = ⌈∫* 1⌉ B + A ⌈∫* 1⌉                        "AB"
 B A = ⌈∫_* 1⌉ A + B ⌈∫_* 1⌉                      "BA"
 B B = ⌈∫_* 1⌉ B - B ⌈∫_* 1⌉                      "BB"

]
```

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The Green's System 4

```
System["4. Equalities for Absorbing Integration Operators", any[f],
```

$A\lceil f\rceil D = -f^{\leftarrow} L + \lceil f\rceil - A\lceil f'\rceil$	"AMD"
$B\lceil f\rceil D = f^{\rightarrow} R - \lceil f\rceil - B\lceil f'\rceil$	"BMD"
$A D = -L + 1$	"AD"
$B D = R - 1$	"BD"
$A\lceil f\rceil L = \lceil \int^{*} f\rceil L$	"AML"
$B\lceil f\rceil L = \lceil \int_{*} f\rceil L$	"BML"
$A\lceil f\rceil R = \lceil \int^{*} f\rceil R$	"AMR"
$B\lceil f\rceil R = \lceil \int_{*} f\rceil R$	"BMR"
$A L = \lceil \int^{*} 1\rceil L$	"AL"
$B L = \lceil \int_{*} 1\rceil L$	"BL"
$A R = \lceil \int^{*} 1\rceil R$	"AR"
$B R = \lceil \int_{*} 1\rceil R$	"BR"

```
]
```

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New Formulation of BVPs

Remember: Given $f \in C^{\infty}[a, b]$, find $u \in C^{\infty}[a, b]$ such that $T u = f$ and $B_1 u = \ldots = B_n u = 0$.

$$\begin{array}{llll} G & : & C^{\infty}[a, b] & \longrightarrow C^{\infty}[a, b] \\ & & f & \longmapsto u \end{array}$$

Hence, for every $f \in C^{\infty}[a, b]$ we must have:

$$T G f = f$$
$$B_1 G f = \ldots = B_n G f = 0$$

Or purely on the level of operators:

$$TG = 1$$
$$B_1\, G = \ldots = B_n\, G = 0$$

So we search an operator G that is a right inverse of T and annihilated by $\langle B_1, \ldots, B_n \rangle$.

Characteristic data of BVP: Differential operator T and boundary (vector) operator $\langle B_1, \ldots, B_n \rangle$.

Beam Deflection Again

Functional formulation from before:

$$\begin{array}{l} u''''(x) = f(x) \\ \hline u(0) = 0 \quad u''(0) = 0 \\ u(1) = 0 \quad u''(1) = 0 \end{array} \qquad g(x, \xi) = \begin{cases} \frac{1}{3} x \xi - \frac{1}{6} \xi^3 - \frac{1}{2} x^2 \xi + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{if} \quad 0 \le \xi \le x \le 1 \\ \frac{1}{3} x \xi - \frac{1}{2} x \xi^2 - \frac{1}{6} x^3 + \frac{1}{6} x \xi^3 + \frac{1}{6} x^3 \xi & \text{if} \quad 0 \le x \le \xi \le 1 \end{cases}$$

New operator formulation:

$$\begin{array}{l} D^4\, G = 1 \\ \hline LG = 0 \quad LD^2 G = 0 \\ RG = 0 \quad RD^2 G = 0 \end{array} \qquad \begin{aligned} G = &-\tfrac{1}{6} A\lceil x^3 \rceil - \tfrac{1}{6} \lceil x^3 \rceil B + \tfrac{1}{3} \lceil x \rceil A \lceil x \rceil + \tfrac{1}{6} \lceil x \rceil A \lceil x^3 \rceil + \tfrac{1}{3} \lceil x \rceil B \lceil x \rceil \\ &- \tfrac{1}{2} \lceil x \rceil B \lceil x^2 \rceil + \tfrac{1}{6} \lceil x \rceil B \lceil x^3 \rceil - \tfrac{1}{2} \lceil x^2 \rceil A \lceil x \rceil + \tfrac{1}{6} \lceil x^3 \rceil A \lceil x \rceil + \tfrac{1}{6} \lceil x^3 \rceil B \lceil x \rceil \end{aligned}$$

Characteristic data: $T = D^4$ and $\langle B_1, B_2, B_3, B_4 \rangle = \langle LG, RG, LD^2 G, RD^2 G \rangle$.

Computation within the *Theorema* system:

```
Compute[Green[D^4, ⟨L, R, L D^2, R D^2⟩], by → GreenEvaluator]
```

$$-\frac{1}{6} A \lceil x^3 \rceil - \frac{1}{6} \lceil x^3 \rceil B + \frac{1}{3} \lceil x \rceil A \lceil x \rceil + \frac{1}{6} \lceil x \rceil A \lceil x^3 \rceil + \frac{1}{3} \lceil x \rceil B \lceil x \rceil - \frac{1}{2} \lceil x \rceil B \lceil x^2 \rceil + \frac{1}{6} \lceil x \rceil B \lceil x^3 \rceil - \frac{1}{2} \lceil x^2 \rceil A \lceil x \rceil + \frac{1}{6} \lceil x^3 \rceil A \lceil x \rceil + \frac{1}{6} \lceil x^3 \rceil B \lceil x \rceil$$

Solving BVPs—A Primitive Approach

Heuristic strategy:

For every concrete problem characterized by T and $\langle B_1, \ldots, B_n \rangle$ *throw together*

the equations $TG = 1$, $B_1 G = \ldots = B_n G = 0$

and an appropriate segment of the above "Green's system" identities.

Then solve the resulting system for G by using noncommutative Gröbner bases.

Problematic points in the above strategy:

- In the noncommutative case, (finite) Gröbner bases need not exist.
- If it exists, how do we know that G always gets isolated as above?
- Computing a noncommutative Gröbner basis for each example is expensive!

Observe:

- The $n+1$ equations $TG = 1$, $B_1 G = \ldots = B_n G = 0$ are the only ones changing in each computation.
- Replacing them by a single equation, $\mathcal{E}(G)$, might yield an expression for G by a uniform method.
- Then we could use the Green's system only for normalizing this expression.

$TG = 1,\ B_1 G = \ldots = B_n G = 0 \quad \Longleftrightarrow \quad \mathcal{E}(G) \quad ??$

Solving BVPs—A Refined Approach

Sketch of Solution Algorithm

$TG = 1$
$B_1 G = \ldots = B_n G = 0$

$\mathcal{E}(G): \qquad GT = 1 - P$

- Compute the solution space N of the homogeneous equation $Tu = 0$.
- Determine a projector P onto N such that $M = (1 - P)\, C^\infty[a, b]$ fulfills the boundary conditions.

```
P = Compute[Proj[D^4, 〈L, R, L D^2, R D^2〉], by → GreenEvaluator]
```

$$L - \lceil x \rceil L + \lceil x \rceil R - \frac{1}{3} \lceil x \rceil L D^2 - \frac{1}{6} \lceil x \rceil R D^2 + \frac{1}{2} \lceil x^2 \rceil L D^2 - \frac{1}{6} \lceil x^3 \rceil L D^2 + \frac{1}{6} \lceil x^3 \rceil R D^2$$

- Find the right inverse $T^{\blacklozenge}$ of T as specified above.

```
T◆ = Compute[(D^4)^◆, by → GreenEvaluator]
```

$$-\frac{1}{6} A \lceil x^3 \rceil + \frac{1}{6} \lceil x^3 \rceil A + \frac{1}{2} \lceil x \rceil A \lceil x^2 \rceil - \frac{1}{2} \lceil x^2 \rceil A \lceil x \rceil$$

- Build up $\boxed{G = (1 - P)\, T^{\blacklozenge}}$ as the crude Green's operator.
- Reduce G with respect to the Green's system for obtaining a standard representation.

The Role of Noncommutative Gröbner Bases

Crucial Properties of the Green's System:

- *Noetherianity*: Every reduction terminates → finitary basis.
- *Confluence*: It provides a unique normal form for each polynomial → Gröbner basis!
- *Adequacy*: Enough reductions for algebraizing relevant analytic knowledge → correctness claim.
- *Standardization*: Normal forms correspond exactly to the Green's functions → extraction algorithm.

But there are 233 such S-polynomials! → Proof automated in *Theorema* (approximately 2000 lines).

Proof is relative to the axioms of integro-differential algebras.

Logfile

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The Problem Monoid—Ideas

Additive Structure on Boundary Conditions:

$$\mathbf{B}: \mathcal{F} \to \mathbb{C}^m, \quad f \mapsto \tilde{\mathbf{B}} f \qquad \tilde{\mathbf{B}}: \mathcal{F} \to \mathbb{C}^n, \quad f \mapsto \tilde{\mathbf{B}} f \qquad \mathbf{B} \oplus \tilde{\mathbf{B}}: \mathcal{F} \to \mathbb{C}^{m+n}, \quad f \mapsto \langle \mathbf{B} f, \tilde{\mathbf{B}} f \rangle$$

Encoding of Boundary Value Problems:

$$(T, B_1 \oplus \ldots \oplus B_n) \quad := \quad \boxed{\begin{array}{l} T u = f \\ (B_1 \oplus \ldots \oplus B_n)\, u = 0 \end{array}}$$

Identifying Boundary Value Problems

$$(T, B) \sim (\tilde{T}, \tilde{B}) \quad :\Leftrightarrow \quad T = \tilde{T} \wedge \mathrm{Ker}(B) = \mathrm{Ker}(\tilde{B})$$

Noncommutative Multiplicative Structure on Boundary Value Problems:

$\mathcal{B} = \{(T, B) \mid \text{regular}\} / \sim$
$[T, B] \cdot [\tilde{T}, \tilde{B}] = [T\tilde{T}, B\tilde{T} \oplus \tilde{B}]$

Example of Multiplication / Factorization

Decomposing Boundary Value Problems:

$$\begin{bmatrix} u' = f \\ \int_0^1 u(\xi)\, d\xi = 0 \end{bmatrix} \cdot \begin{bmatrix} u' = f \\ u(0) = 0 \end{bmatrix} = \begin{bmatrix} u'' = f \\ u(0) = u(1) = 0 \end{bmatrix}$$

In the Problem Monoid:

$$[D, F] \cdot [D, L] = [D^2, L \oplus R]$$

Abbreviation:

$$F \equiv A + B = \int_0^x + \int_x^1 = \int_0^1$$
$$F = RA = LB$$

The Problem Monoid—Exact Definition

Stieltjes Boundary Conditions:

$$\beta u = \sum_{i=0}^{n-1} (a_i\, u^{(i)}(0) + b_i\, u^{(i)}(1)) + \int_0^1 \varphi(\xi)\, u(\xi)\, d\xi$$

$$\beta = \sum_{i=0}^{n-1} (a_i\, L D^i + b_i\, R D^i) + F\lceil\varphi\rceil$$

↖ … for any $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in \mathbb{C}$ and $\varphi \in \mathcal{E}xp$:

Formulation in the Language of Green's Algebra:

$$\mathcal{S}tj = \{L, R\} \cdot \mathcal{A}n(\mathcal{F})$$
$$\mathcal{S}tj_n = \{B_1 \oplus \ldots \oplus B_n \mid B_1, \ldots, B_n \in \mathcal{S}tj\}$$

Recognizing Regularity of BVPs:

$$[T, B] \text{ regular } :\Leftrightarrow \underset{f \in C^\infty[0,1]}{\forall} \; \underset{u \in C^\infty[0,1]}{\exists!} \; (Tu = f \wedge Bu = 0)$$

$$[T, B] \text{ regular } \Leftrightarrow \operatorname{Ker}(T) \dotplus \operatorname{Ker}(B) = \mathcal{F} \Leftrightarrow \begin{pmatrix} B_1 \varphi_1 & \cdots & B_1 \varphi_n \\ \vdots & \ddots & \vdots \\ B_n \varphi_1 & \cdots & B_n \varphi_n \end{pmatrix} \text{ regular}$$

… where $B = B_1 \oplus \ldots \oplus B_n$ and $\{\varphi_1, \ldots, \varphi_n\}$ is any basis for $\operatorname{Ker}(T)$ ↗

Problem Monoid Revisited

$$\mathcal{B} = \{[T, B] \mid T \in \mathbb{C}[\partial]_n \wedge B \in \mathcal{S}tj_n \wedge \operatorname{Ker}(T) \dotplus \operatorname{Ker}(B) = \mathcal{F}\} / \sim$$

Problem Factorization versus Operator Factorization

Easy to check:

$$\underset{G}{\underset{\uparrow}{[T, B]}} \cdot \underset{\tilde{G}}{\underset{\uparrow}{[\tilde{T}, \tilde{B}]}} = \underset{\tilde{G} \cdot G}{\underset{\uparrow}{[T\tilde{T}, B\tilde{T} \oplus \tilde{B}]}}$$

Notation:

$$[T, B]^{-1} := G \qquad [\tilde{T}, \tilde{B}]^{-1} := \tilde{G} \qquad \longrightarrow \qquad ([T, B] \cdot [\tilde{T}, \tilde{B}])^{-1} = [\tilde{T}, \tilde{B}]^{-1} \cdot [T, B]^{-1}$$

Example from before:

$$\underset{C}{\underbrace{\underset{(A - AF)}{\underset{\uparrow}{[D, F]}}}} \cdot \underset{A}{\underset{\uparrow}{[D, L]}} = \underset{G_2}{\underbrace{\underset{XAX + XBX - AX - XB}{\underset{\uparrow}{[D^2, L \oplus R]}}}} = \underset{C}{\underbrace{\underset{A - AF}{\underset{\uparrow}{[D, F]}}}} \cdot \underset{-B}{\underset{\uparrow}{[D, R]}}$$

The Factorization Lemma

Every factorization of the differential operator can be lifted to the level of problems:

> *For every regular problem* $[T, B]$ *and for every factorization*
>
> $$T = T_1\, T_2$$
>
> *there are boundary conditions* B_1, B_2 *with* $Ker(B) \subseteq Ker(B_1)$ *such that*
>
> $$[T, B] = [T_1, B_1] \cdot [T_2, B_2]$$
>
> *and both* $[T_1, B_1]$ *and* $[T_2, B_2]$ *are regular.*

Iteration yields:

$$[T, B] = [D - \lambda_1, \beta_1] \cdot \ldots \cdot [D - \lambda_n, \beta_n]$$
$$[T, B]^{-1} = [D - \lambda_n, \beta_n]^{-1} \cdot \ldots \cdot [D - \lambda_1, \beta_1]^{-1}$$

First-Order Green’s Operators

$$[D - \lambda, \beta]^{-1} = (1 - P_{\lambda,\beta}) \lceil e^{\lambda x} \rceil A \lceil e^{-\lambda x} \rceil \qquad \text{with} \qquad P_{\lambda,\beta} \equiv \beta(e^{\lambda x})^{-1} \lceil e^{\lambda x} \rceil \beta$$

Solving Inhomogeneous Initial Value Problems à la Mikusiński

Recall Duhamel Convolution:

$$u * \tilde{u}\,(x) = \int_0^x u(x - t)\, \tilde{u}(t)\, dt$$

Note:

- Makes $\mathfrak{L} \equiv C(0, \infty)$ into a commutative(!) ring.
- Tichmarsh’s Theorem: No zero divisors!
- Field of fractions $\mathfrak{M}$ introduced by Mikusiński in 1959.
- Integral operator $l \equiv 1 \in \mathfrak{L}$, so $l * u\,(x) = \int_0^x u(t)\, dt = Au\,(x)$.
- Differential operator $s \equiv l^{-1} \in \mathfrak{M}$

How to Solve Inhomogeneous Initial Value Problems (Constant Coefficients):

Fundamental Formula : $\boxed{s * u = u' + u(0)\,\delta_0}$

Iteration : $s * s * u = u'' + u'(0)\,\delta_0 + u(0)\,\delta_0'$

Dirac Distribution : $\delta_0 \equiv s * 1 = f \,//\, f \longrightarrow \delta_0 * u = u,\ l * \delta_0 = 1$

Example:

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = a,\ u'(0) = b \end{array}} \qquad \begin{array}{l} s * s * u = f + a\,\delta_0 + b\,\delta_0' \\ \longrightarrow u = (l * l) * f + a\,(l * 1) + b = x * f + a x + b \end{array}$$

Solution:

$$u(x) = a + b x + \int_0^x t\, f(t)\; dt$$

How about Boundary Value Problems?

We need both A and B for representing Green's operators for boundary value problems.

Since AB ≠ BA, Mikusiński cannot do this!

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Returning to Green's Operators—on the Functional Level à la Mikusiński

Recall Green's Functions:

$$G f(x) = \int_0^1 g(x, \xi)\, f(\xi)\, d\xi$$

$$G = \mathrm{Fred}(g)$$

Introduce Multiplication on Collection $\mathfrak{G}$ of all Green's Functions of Regular Problems in $\mathfrak{B}$:

$$g * \tilde{g}(x, y) = \int_0^1 g(x, t)\, \tilde{g}(t, y)\, dt$$

Note:

- Introduced by Volterra in 1913.

- Makes $\mathfrak{K} \equiv L^2(I \times I) \supseteq \mathfrak{G}$ into a noncommutative ring.
- Key property is $\mathrm{Fred}(g * \tilde{g}) = \mathrm{Fred}(g) \circ \mathrm{Fred}(\tilde{g})$.
- We will often identify g with $\mathrm{Fred}(g)$ and drop $*$.

Factorization on Three Levels:

$[D^2, L \oplus R]$	$=$	$[D, F]$	$\cdot$	
$XAX + XBX - AX - XB$	$=$	A	$\circ$	
$-h(\xi - x)\, x - h(x - \xi)\, x + h(\xi - x)\, x\xi + h(x - \xi)\, x\xi$	$=$	$h(\xi - x)$	$*$	$(-h(x -$ …

Anti-Isomorphism : $\mathfrak{G} \simeq \mathfrak{B}^{\mathrm{op}}$

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Localization in Noncommutative Rings

Necessary and Sufficient Condition for Constructing a «Ring of Fractions»:

For localizing R at $S \subseteq R$ into RS^{-1} we require:

<u>*Multiplicativity:*</u> $$\underset{s,\tilde{s} \in S}{\forall}\; s\tilde{s} \in S$$

<u>*Ore Condition:*</u> $$\underset{r \in R}{\forall}\; \underset{s \in S}{\forall}\; \underset{\tilde{r} \in R}{\exists}\; \underset{\tilde{s} \in S}{\exists}\; r\tilde{s} = s\tilde{r}$$

<u>*Reversibility:*</u> $$\underset{r \in R}{\forall}\left(\underset{s \in S}{\exists}\; sr = 0 \Rightarrow \underset{\tilde{s} \in S}{\exists}\; r\tilde{s} = 0\right)$$

Even if R has no zero divisors, it may not have a ring of fractions, i.e. a quotient (skew) field!

Failing Attempts at Localizing in $\mathfrak{K}$:

- First Attempt: Take $R = \mathfrak{K}$ and $S = \mathfrak{K}^{\diamond}$. Too many denominators!
- Second Attempt: Take $R = \mathfrak{K}$ and $S = \langle A, B \rangle$. Too few denominators!
- Third Attempt: Take $R = \mathfrak{K}$ and $S = \mathfrak{G}$. More balanced, but still fails!

The Ore condition turns out to be very tough!

Winning Idea:

Let R be any ring and $S \subseteq R$ a multiplicative subset fulfilling the Ore condition just within S.

Then the ring S^+ generated by S in R fulfills the Ore condition when localized at S.

Successful choice: $R = \mathfrak{G}^+$ and $S = \mathfrak{G}$:

- Hence it suffices to prove the Ore condition within $\mathfrak{G}$.
- Since $\mathfrak{G} \simeq \mathfrak{B}^{op}$, we may as well prove it within $\mathfrak{B}$.

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The Ore Condition in the Problem Monoid

Regularization Lemma:

For every $T \in \mathbb{C}[\partial]_m$ and every $B = \mathcal{S}tj_m$

there is a regular $[\tilde{T}, \tilde{B}]$ with $T \mid \tilde{T}$ and $Ker(B) \supseteq Ker(\tilde{B})$.

Division Lemma:

For regular problems $[T, B]$ and $[T_1, B_1]$ with $T_1 \mid T$ and $Ker(B_1) \supseteq Ker(B)$

there is exactly one regular problem $[T_2, B_2]$ with $[T_1, B_1] \cdot [T_2, B_2] = [T, B]$.

Ore Condition:

Given regular problems $[T_1, B_1]$ and $[T_2, B_2]$, there are regular problems $[\tilde{T}_1, \tilde{B}_1]$ and $[\tilde{T}_2, \tilde{B}_2]$

such that $[T_1, B_1] \cdot [\tilde{T}_1, \tilde{B}_1] = [T_2, B_2] \cdot [\tilde{T}_2, \tilde{B}_2]$.

Proof of the Ore Condition:

- Regularization Lemma $\longrightarrow [T, B]$ with $T_1\ T_2 \mid T$ and $\mathrm{Ker}(B_1 \oplus B_2) \supseteq \mathrm{Ker}(B)$.
- Division Lemma $\longrightarrow [\tilde{T}_1, \tilde{B}_1]$ and $[\tilde{T}_2, \tilde{B}_2]$.

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Lots of Fundamental Formulae

The Fundamental Formulae à la Mikusiński:

$$AD = 1 - L \qquad -BD = 1 - R$$
$$Au' = u - u(0) \qquad -Bu' = u - u(1)$$
$$u' = A^{-1} u - u(0)\, A^{-1}\, 1 \qquad u' = -B^{-1} u + u(1)\, B^{-1}\, 1$$
$$\boxed{A^{-1} u = u' + u(0)\, \delta_0} \qquad \boxed{B^{-1} u = -u' + u(1)\, \delta_1}$$
$$\delta_0 \equiv A^{-1}\, 1 \nearrow \qquad \delta_1 \equiv B^{-1}\, 1 \nearrow$$

Example of a Different Fundamental Formula:

$$CD = 1 - F$$
$$Cu' = u - \int_0^1 u(\xi)\, d\xi$$
$$u' = C^{-1} u - \left(\int_0^1 u(\xi)\, d\xi\right) C^{-1}\, 1$$
$$\boxed{C^{-1} u = u' + \left(\int_0^1 u(\xi)\, d\xi\right) \varepsilon}$$
$$\varepsilon \equiv C^{-1}\, 1 \nearrow$$

Recall: $C = [D, F]^{-1} = A - AF$

Solving Inhomogeneous Boundary Value Problems à la Mikusiński

Recall the Factorization:

$$\underbrace{[D, F]^{-1}}_{\uparrow\ (A - AF)} \cdot \underbrace{[D, L]^{-1}}_{\uparrow\ A} = \underbrace{[D^2, L \oplus R]^{-1}}_{\uparrow\ XAX + XBX - AX - XB}$$
$$\underbrace{(A - AF)}_{C} \cdot A = \underbrace{XAX + XBX - AX - XB}_{G_2}$$

A Custom-Tailored Fundamental Formula:

$$\boxed{G_2^{-1} u = u'' + u(0)\, \delta_0' + u(1)\, \varepsilon}$$
$$\delta_0' \equiv A^{-2}\, 1 \nearrow$$

Example:

$$\boxed{\begin{array}{l} u'' = f \\ u(0) = a,\ u(1) = b \end{array}} \qquad G_2^{-1} u = f + a\, \delta_0' + b\, \varepsilon$$
$$\longrightarrow u \overset{*}{=} G_2\, f + a\, A\, (\delta_0 - 1) + b\, A1 = G_2\, f + a\, (1 - x) + b\, x$$

Solution:

$$u(x) = a\,(1 - x) + b\,x + \int_0^1 g_2(x, \xi)\, f(\xi)\, dt$$

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Conclusion

Computer algebra tools (noncommutative polynomials) for handling

- Problem **and** solution of
- Differential equation **and** boundary conditions.

Two new recent achievements:

- **Factorization** of any regular BVP into irreducibles.
- The **Mikusiński calculus** extended to cover boundary conditions.

Setting still very **simple**: variable coefficients, partial differential equations, systems, …?

Hope of extending some ideas there!