Gröbner bases and extremal combinatorics

algebraic aspects

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Term orders

Let $S = \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\prec$ be a *term order* on the monomials (total, 1 is the smallest, and from $u \prec v$ it follows that $uw \prec vw$).

Examples

1. lex: $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \prec x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ iff $i_k < j_k$ for the smallest k for which $i_k \neq j_k$.
2. deglex: smaller degree first, ties broken by lex.
3. degrevlex: smaller degree first. Same degree: $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \prec x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ iff $i_k > j_k$ for the largest k with $i_k \neq j_k$.

The *leading term* of $0 \neq f \in S$ is the $\prec$-largest monomial in the canonical expression of f. Notation: $\mathrm{lm}(f)$.

Gröbner bases, standard monomials

Let $I \trianglelefteq S$. A finite $G \subseteq I$ is a *Gröbner basis* of I if for every $0 \neq f \in I$ there is a $g \in G$ with $\mathrm{lm}(g) | \mathrm{lm}(f)$.

Theorem

G as above is a basis of I. $I \neq (0)$ has a Gröbner basis.

G can be calculated: Buchberger's algorithm...
Amazingly useful notion of reduction...

A monomial $w \in S$ is a *standard monomial* for I, if w is not $\mathrm{lm}(f)$ of any $f \in I$. $\mathrm{sm}(I)$ is the set of standard monomials for I.

Properties. $\mathrm{sm}(I)$ *is downward closed.*
$\mathrm{sm}(I)$ *is an $\mathbb{F}$-basis of S/I (a $g \in S$ has unique expression $h + f$, where $f \in I$, h is a unique linear combination of standard monomials).*

Finite sets of points

Let $\mathcal{H} \subseteq \mathbb{F}^n$ be finite.

A polynomial $f(x_1, \ldots, x_n) \in S$ defines a function $\mathcal{H} \to \mathbb{F}$. Conversely, every $\mathcal{H} \to \mathbb{F}$ is a polynomial function . The kernel of $S \to \texttt{func}(\mathcal{H}, \mathbb{F})$ is

$$I = I(\mathcal{H}) = \{f \in S : \; f(P) = 0 \text{ for } P \in \mathcal{H}\}.$$

$S/I \cong \texttt{func}(\mathcal{H}, \mathbb{F})$. In particular, $\dim_{\mathbb{F}} S/I = |\mathcal{H}|$.

Statement

The set $\mathrm{sm}(I(\mathcal{H}))$ *is an* $\mathbb{F}$*-basis of* $\texttt{func}(\mathcal{H}, \mathbb{F})$.

As a consequence, $|\mathrm{sm}(I(\mathcal{H}))| = |\mathcal{H}|$.

Standard monomials (normal set) of set families

Notation $[n] := \{1, 2, \ldots, n\}$.

A set familiy $\mathcal{F} \subseteq 2^{[n]}$ can be represented by the family $\mathcal{H} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$ of characteristic vectors of the sets $F \in \mathcal{F}$.

For $G \subseteq [n]$ write $x_G := \prod_{j \in G} x_j \quad (x_\emptyset = 1)$.

Assume that $\mathcal{H} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$.

Then $x_i^2 - x_i$ vanishes on $\mathcal{H}$: x_i^2 is a leading term for $I = I(\mathcal{H})$.

The standard monomials of I are of shape x_G for some $G \subseteq [n]$.

$$\mathrm{Sm}(\mathcal{F}) := \mathrm{Sm}(\mathcal{H}) := \{G \subseteq [n] : \ x_G \in \mathrm{sm}(I)\}.$$

This is a down-set and $|\mathrm{Sm}(\mathcal{H})| = |\mathcal{H}| = |\mathcal{F}|$.

The Hilbert function of $\mathcal{F}$

$h_{\mathcal{F}}(m) :=$ the number of deglex standard monomials of $I(\mathcal{H})$ with degree at most m.

A very important invariant of $\mathcal{F}$.

For $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ the *inclusion matrix* $I(\mathcal{F}, \mathcal{G})$ is a (0,1) matrix of size $|\mathcal{F}| \times |\mathcal{G}|$ whose rows and columns are indexed by the elements of $\mathcal{F}$ and $\mathcal{G}$, resp.. The entry at position (F, G) is 1 if $G \subseteq F$ and 0 otherwise $(F \in \mathcal{F}, G \in \mathcal{G})$.

We have

$$h_{\mathcal{F}}(m) = \mathrm{rank}_{\mathbb{F}} I(\mathcal{F}, \binom{[n]}{\leq m}).$$

Order shattering (Anstee, Sali)

$\mathcal{F} \subseteq 2^{[n]}$ *order shatters* the set $S \subseteq [n]$, iff

(a) $S = \emptyset$ and $\mathcal{F} \neq \emptyset$, or

(b) $S = \{s_1, s_2, \ldots, s_k\}$, $(s_1 < s_2 < \cdots < s_k)$, and there are subfamilies $\mathcal{F}_0, \mathcal{F}_1 \subset \mathcal{F}$ such that $s_k \notin G$ if $G \in \mathcal{F}_0$, $s_k \in H$ for $H \in \mathcal{F}_1$, both $\mathcal{F}_0$ and $\mathcal{F}_1$ order shatter $S \setminus \{s_k\}$;

moreover $T \cap F_0 = T \cap F_1$ holds where $T = \{s_k + 1, s_k + 2, \ldots, n\}$, for every $F_0 \in \mathcal{F}_0, F_1 \in \mathcal{F}_1$.

$$\mathrm{osh}(\mathcal{F}) = \{S \subseteq [n] \,:\, \mathcal{F} \text{ order shatters } S\}.$$

Lex normal set and osh

Theorem (Anstee, R, Sali)

Let $\emptyset \neq \mathcal{F} \subseteq 2^{[n]}$, $\mathbb{F}$ *a field, the ordering be* lex. *Let* $\mathcal{H} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$ *be the set of characteristic vectors of* $\mathcal{F}$. *Then* $\mathrm{osh}(\mathcal{F}) = \mathrm{Sm}(\mathcal{H})$.

Corollary For $\prec =$ *lex* the set $\mathrm{Sm}(\mathcal{H})$ does not depend on $\mathbb{F}$.

Example

$\mathcal{P} = \{G \subseteq [n] : \ |G| \text{ even } \}$. If char$\mathbb{F} = 2$ (and $x_1 \succ \ldots \succ x_n$), then $\mathrm{Sm}(\mathcal{P}) = \{G \subset [n] : \ 1 \notin G\}$.

If $\mathbb{F} = \mathbb{Q}$, $\prec$ is degree compatible, and $n = 2k+1$, then $\mathrm{Sm}(\mathcal{P}) = \{G \subset [n] : \ |G| \leq k\}$ (Delsarte).

Problem Combinatorial description for $\mathrm{Sm}(\mathcal{H})$, when $\prec$ is *deglex* or *degrevlex*.

Sketch of Proof

Put

$$\mathcal{F}_0 = \{G \in \mathcal{F} : \ n \notin G\},$$

$$\mathcal{F}_1 = \{G \setminus \{n\} : \ G \in \mathcal{F}, \ n \in G\}.$$

Then

$$\text{osh}(\mathcal{F}) = \text{osh}(\mathcal{F}_0) \cup \text{osh}(\mathcal{F}_1) \cup$$

$$\cup \{S \cup \{n\} : \ S \in \text{osh}(\mathcal{F}_0) \cap \text{osh}(\mathcal{F}_1)\}.$$

Same holds with *Sm* in the place of *osh*. Let $\mathcal{H}_i$ be the set of characteristic vectors of $\mathcal{F}_i$.

$$\text{Sm}(\mathcal{H}) = \text{Sm}(\mathcal{H}_0) \cup \text{Sm}(\mathcal{H}_1) \cup$$

$$\cup \{S \cup \{n\} : \ S \in \text{Sm}(\mathcal{H}_0) \cap \text{Sm}(\mathcal{H}_1)\}.$$

Formulae allow induction on n. □

Standard monomials for $\mathcal{F} = \binom{[n]}{d}$

Theorem (Anstee, Sali)

Put $t = \min\{d, n-d\}$. *Then* $\mathrm{osh}(\mathcal{F}) = \mathcal{M}_d$, *where* $\mathcal{M}_d$ *denotes the set of monomials* x_G *such that*

$$G = \{s_1 < s_2 < \ldots < s_j\} \subset [n]$$

for which $j \leq t$ *and* $s_i \geq 2i$ *holds for* $1 \leq i \leq j$.

$\subseteq$ is easy: a set not in $\mathcal{M}_d$ is not order shattered by $\mathcal{F}$.
$\supseteq$: counting, ballot sequences.

Theorem (Hegedűs, R)

We have $\mathrm{Sm}(\mathcal{F}) = \mathcal{M}_d$ *for* all *term orders with* $x_1 \succ \ldots \succ x_n$ *and over* all *fields* $\mathbb{F}$.

Let $0 < t \leq n/2$, and $\mathcal{H}_t$ be the set of subsets $H = \{s_1 < s_2 < \cdots < s_t\}$ of $[n]$ for which t is the smallest index j with $s_j < 2j$.

Example

$\mathcal{H}_3 = \{\{2,4,5\},\{3,4,5\}\}$.

For $J \subseteq [n]$ and $0 \leq i \leq |J|$ set

$$\sigma_{J,i} := \sum_{T \subseteq J, |T|=i} x_T \quad \in \mathbb{F}[x_1,\ldots,x_n].$$

$\sigma_{J,i}$ is an elementary symmetric polynomial.

Now let $0 < t \leq n/2$, $0 \leq d \leq n$ and $H \in \mathcal{H}_t$. Put

$$H' = H \cup \{2t, 2t+1, \ldots, n\} \subseteq [n].$$

We write

$$f_{H,d} = f_{H,d}(x_1, \ldots, x_n) := \sum_{k=0}^{t} (-1)^{t-k} \binom{d-k}{t-k} \sigma_{H',k}.$$

Example

$f_{\{2,3\},d} = \sigma_{U,2} - (d-1)\sigma_{U,1} + \binom{d}{2}$, where $U = \{2, 3, \ldots, n\}$.

We have $\mathrm{lm}(f_{H,d}) = x_H$.

Gröbner bases for $\mathcal{F}$

Theorem

Let d, n be integers, $n > 0$ and $0 \le d \le n/2$, $\mathbb{F}$ a field, and $\prec$ be a term order with $x_n \prec x_{n-1} \prec \ldots \prec x_1$. Then $\mathcal{G} \subset S$ is a Gröbner basis with respect to $\prec$ of the ideal of $\binom{[n]}{d}$:

$$\mathcal{G} = \{x_1^2 - x_1, \ldots, x_n^2 - x_n\} \cup \{x_J : \; J \in \binom{[n]}{d+1}\} \cup$$

$$\{f_{H,d} : \; H \in \mathcal{H}_t \text{ for some } 0 < t \le d\}.$$

The case $n/2 < d \le n$ is similar.
A suitable subset of $\mathcal{G}$ is a reduced Gb.
The (monic) elements of $\mathcal{G}$ are defined over $\mathbb{Z}$.

Problem

Characterize the set families $\mathcal{H}$ which have a reduced Gb 'independent' of $\mathbb{F}$ and $\prec$, as above.

An application

Let n, k, α be integers, $n, \alpha > 0$, p be a prime and $q = p^\alpha$.

$$\mathcal{F}(k, q) = \{K \subseteq [n] : \ |K| \equiv k \pmod{q}\}.$$

Theorem

Let $\prec = \mathrm{deglex}$, *and* $\ell \in \mathbb{N}$ *for which* $\ell < q$, *and* $2\ell \leq n$. *Then*

$$\mathrm{Sm}(\mathcal{F}(k, q), \mathbb{F}_p) \cap \binom{[n]}{\leq \ell} \subseteq \mathcal{M}_\ell,$$

hence

$$|\mathrm{Sm}(\mathcal{F}(k, q), \mathbb{F}_p) \cap \binom{[n]}{\leq \ell}| \leq \binom{n}{\ell}$$

The case $\alpha = 1$ is a theorem of P. Frankl.

Idea of the proof

Assume $0 \le k < q$ and recall

$$\mathcal{G} = \{x_1^2 - x_1, \ldots, x_n^2 - x_n\} \cup \{x_J : J \in \binom{[n]}{k+1}\} \cup$$

$$\{f_{H,k} : H \in \mathcal{H}_t \text{ for some } 0 < t \le d\}$$

is a Gb for $\binom{[n]}{k}$.
Here

$$f_{H,k} = \sum_{j=0}^{t} (-1)^{t-j} \binom{k-j}{t-j} \sigma_{H',j}.$$

For $t \le \ell$ and $k \equiv k' \pmod q$ we have $f_{H,k} \equiv f_{H,k'} \pmod p$.
A polynomial of degree $\le \ell$ can be reduced by the elements of $\mathcal{G}$ (of degree $\le \ell$). Gives expression with monomials from $\mathrm{sm}\binom{[n]}{\ell}$.

ℓ-wide families (Hegedűs, Friedl, R)

Let $n > 0$, k, ℓ be integers with $0 \leq \ell - 1 \leq k \leq n$, and consider the complete ℓ-*wide* family

$$\mathcal{F}^{k,\ell} = \{F \subseteq [n] : k - \ell < |F| \leq k\}.$$

Set
$D(k,\ell) = \{\{g_1 < \ldots < g_t\} \subseteq [n] : \ t \leq k$
and $g_j \geq 2j - \ell + 1$ if $1 \leq j \leq t\}$.

Theorem

(a) Let $0 \leq k < (n+\ell)/2$. *Then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(k,\ell)$$

(b) If $k \geq (n+\ell)/2$, *then*

$$\mathrm{osh}(\mathcal{F}^{k,\ell}) = D(n-k+\ell-1,\ell).$$

Let $\mathbb{F}$ be a field, $\prec$ a term order with $x_1 \succ \ldots \succ x_n$.

Theorem

Let k, n, and ℓ be as before. Then

$$\mathrm{Sm}(\mathcal{F}^{k,\ell}) = \mathrm{osh}(\mathcal{F}^{k,\ell}).$$

We found a (reduced) Gröbner basis for $\mathcal{F}^{k,\ell}$.
As in the uniform case, it is independent of $\mathbb{F}$ and $\prec$.

Let $0 \leq m \leq \min(k, n-k+\ell-1)$, and $j = \max(0, m-\ell+1)$. Then

$$h_{\mathcal{F}^{k,\ell}}(m) := |\mathrm{osh}(\mathcal{F}^{k,\ell}) \cap \binom{[n]}{\leq m}| = \sum_{i=j}^{m} \binom{n}{i}.$$

R. M. Wilson's rank formula

Consider $M := I(\binom{[n]}{t}, \binom{[n]}{d})$, where $d \leq t \leq n - d$.

Modulo p rank formula

Let p be a prime. Then with $\mathbb{F} = \mathbb{F}_p$

$$\operatorname{rank}_{\mathbb{F}} M = \sum_{\substack{0 \leq i \leq d \\ p \nmid \binom{t-i}{d-i}}} \binom{n}{i} - \binom{n}{i-1}.$$

With the aid of *osh*, *sm* we have a simple proof and a generalization (Friedl, R).

A generalized rank formula (1)

Let $0 \leq d_1 < d_2 \cdots < d_r \leq t \leq n - d_r$ be integers, p be a prime and

$$M = I\left(\binom{[n]}{t}, \binom{[n]}{d_1} \cup \binom{[n]}{d_2} \cup \ldots \cup \binom{[n]}{d_r}\right).$$

Then

$$\mathrm{rank}_{\mathbb{F}} I = \sum_{\substack{0 \leq i \leq d_r \\ p \not| n_i}} \binom{n}{i} - \binom{n}{i-1},$$

where $n_i = \gcd\left(\binom{t-i}{d_1-i}, \binom{t-i}{d_2-i}, \ldots, \binom{t-i}{d_r-i}\right)$.

A generalized rank formula (2)

Let $M = M_{d_1,t} + M_{d_2,t} + \cdots + M_{d_r,t}$ be the group of all functions $V\binom{[n]}{t} \to \mathbb{Z}$ spanned by the monomials x_G, where $|G| \in \{d_1, \ldots, d_r\}$.

Theorem

There exsits a $\mathbb{Z}$-basis

$$B^* = \{z_G : G \in \mathrm{osh}\binom{[n]}{d_r}\} \subseteq M$$

of M for which $z_G = n_G \cdot x_G$ (as functions on $V\binom{[n]}{t}$) and $n_G = \gcd\left(\binom{t-|G|}{d_1-|G|}, \ldots, \binom{t-|G|}{d_r-|G|}\right)$.

Wilson's formula and a dual Specht filtration

Assume $0 \le d \le t \le n-d$. P^j denotes the space of polynomials spanned by the multilinear monomials of degree j over $\mathbb{F}$.
We give a map $r^{t,d}$ from P^d to P^t. For $H \subseteq [n]$, $|H| = d$ put

$$r^{t,d}(x_H) := \sum_{H \subseteq G,\ |G|=t} x_G.$$

$I_{\mathbb{F}}(\binom{[n]}{t}, \binom{[n]}{d})$ is the matrix of $r^{t,d}$.

Theorem (R)

The module $Q := \mathrm{Im}(r^{t,d})$ admits a filtration with $\mathbb{F}S_n$ modules

$$Q =: Q_d \ge Q_{d-1} \ge \ldots \ge Q_0 \ge Q_{-1} = (0),$$

where $Q_i = Q_{i-1}$ if $\binom{t-i}{d-i}$ is 0 in $\mathbb{F}$, and $Q_i/Q_{i-1} \cong S_i$ otherwise. Here S_i is a dual Specht module whose dimension over $\mathbb{F}$ is $\binom{n}{i} - \binom{n}{i-1}$.

Partitions and tableaux

A sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of integers is a *partition* of n, if $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$.

Notation: $\lambda \vdash n$.

Ferrers diagram, tableau...

Example

$n = 7$, $\lambda = (4, 2, 1)$

2	5	3	6
4	1		
7			

and

1	2	4	5
3	7		
6			

A tableau t is *standard* if the rows and the columns of t are increasing sequences.

Lattice permutations and tableaux

A *lattice permutation* is a sequence of nonnegative integers $m = i_1 i_2 \ldots i_n$ such that, for any prefix $m_k = i_1 \ldots i_k$ and any nonnegative integer l, the number of l's in m_k is at least as large as the number of $(l+1)$'s in that prefix.

Example $m = 0010021$.

Lattice permutations correspond to standard tableaux. Given a standard tableau t, form $m = i_1 i_2 \ldots i_n$, where $i_k = i - 1$ if k appears in row i of t.

A lattice permutation m has type λ iff the t is a standard λ-tableau.

The tableau of m is

1	2	4	5
3	7		
6			

A set of points associated to λ

Let $\alpha_0, \dots, \alpha_{k-1}$ be k different elements of $\mathbb{F}$, $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ and V_λ be the set of all vectors $v = (v_1, \dots, v_n) \in \mathbb{F}^n$ such that

$$|\{j \in [n] : \ v_j = \alpha_i\}| = \lambda_{i+1}$$

for $0 \leq i \leq k-1$.

Example

If $\alpha_0 = 0$, $\alpha_1 = 1$, and $\lambda = (n-d, d)$, then V_λ is the set of characteristic vectors of $\binom{[n]}{d}$.

Problem

Describe the standard monomials and Gröbner bases of V_λ.

Standard monomials of V_λ

Let $\lambda \vdash n$. We define $St(\lambda)$ to be the set of monomials

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \mathbb{F}[x_1, \ldots, x_n]$$

where $i_1 i_2 \ldots i_n$ is a lattice permutation of type λ.
Let B_λ be the set of monomials u for which there exists a monomial $v \in St(\lambda)$ such that u divides v.

Example

For $n = 7$, and $\lambda = (4, 2, 1)$ we have $x_3 {x_6}^2 x_7 \in St(\lambda)$ and $x_6 \in B_\lambda$.

Theorem

Let $\mathbb{F}$ be an arbitrary field. Then

$$\mathrm{sm}(\prec_{lex}, V_\lambda) = \mathrm{sm}(\prec_{deg}, V_\lambda) = B_\lambda.$$

The deglex part is due to A. Garsia nad C. Procesi, the lex to Hegedűs and R., uses combinatorics, representation theory (Specht modules).

T. Harima's theorem for set families

Theorem

Let $\mathbb{F}$ be a field, $\emptyset \neq \mathcal{F} \subset 2^{[n]}$ and $\mathcal{G} := 2^{[n]} \setminus \mathcal{F}$. Then for $m = 0, 1, \ldots, n$ we have

$$h_{\mathcal{G}}(m) := \sum_{j=0}^{m} \binom{n}{i} + h_{\mathcal{F}}(n-1-m) - |\mathcal{F}|.$$

Harima's (more general) proof uses multiplicity theory for Gorenstein rings.
Calculations with polynomial functions give a simple, direct proof.

Admits generalization to interesting coefficient rings, such as $\mathbb{Z}_k$.

The lex game (Felszeghy, R)

$\mathbb{F}$ a field, $V \subseteq \mathbb{F}^n$ finite and nonempty, $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$.
The game $LS(V; \mathbf{w})$ has two players Lea and Stan.
Stan thinks a point $\mathbf{y} = (y_1, \ldots, y_n) \in V$. Lea is to find out a coordinate of $\mathbf{y}$ under the following rules.

First Lea guesses w_n times for y_n. If she finds out, then wins.
Otherwise Stan reveals y_n. In the next round Lea tries to guess y_{n-1} with w_{n-1} questions.
...
We stop when either Lea correctly declares one of the y_i (and then wins the game), or Stan reveals y_1. In that case Stan wins.

An example

Let $n = 5$, $\alpha, \beta \in \mathbb{F}$ different elements and let V be the set of all α-β sequences in $\mathbb{F}^5$ in which the number of the α coordinates is 1, 2 or 3. Then Lea can win with the question vector $\boldsymbol{w} = (11100)$.

Indeed, if Stan gives only β for the last 2 coordinates, then Lea always guesses α.
If Stan gives an α, then Lea keeps on guessing β.

Lea loses for $\boldsymbol{w} = (01110)$.

Stans starts with $y_5 = \beta$, and keeps on saying *no*.

Theorem

Let $V \subseteq \mathbb{F}^n$ be a finite set and $\mathbf{w} \in \mathbb{N}^n$. Lea wins $LS(V; \mathbf{w})$ if and only if $\mathbf{x}^{\mathbf{w}} \in \mathrm{Lm}\,(I(V))$.

We have also the following equivalent statement.

Theorem

Stan wins $LS(V; \mathbf{w})$ if and only if $\mathbf{x}^{\mathbf{w}} \in \mathrm{Sm}\,(I(V))$.

Applications
A fast combinatorial algorithm for $\mathrm{Sm}\,(I(V))$.
Explicit calculation of $\mathrm{Sm}(\mathcal{F})$ for some interesting $\mathcal{F}$.

Proof of $(\Rightarrow)$

If Lea wins $LS(V;\mathbf{w})$, then $\mathbf{x}^{\mathbf{w}} \in \mathrm{Lm}\,(I(V))$.

Proof.

$$s(\mathbf{x}) = s(x_1,\dots,x_n) := \left(\prod_{i=1}^{w_n}(x_n - f_{n,i})\right)\cdot\left(\prod_{i=1}^{w_{n-1}}(x_{n-1} - f_{n-1,i}(x_n))\right)\cdot \left(\prod_{i=1}^{w_{n-2}}(x_{n-2} - f_{n-2,i}(x_{n-1},x_n))\right)\cdots \cdots\left(\prod_{i=1}^{w_1}(x_1 - f_{1,i}(x_2,\dots,x_n))\right),$$

where $f_{j,i}$ $(i = 1,\dots,w_j)$ are the guesses of Lea for y_j.

Recursive structure of standard monomials

Notation
$V_\beta := \{(v_1, \ldots, v_{n-1}) \ : \ (v_1, \ldots, v_{n-1}, \beta) \in V\}$

Theorem

For $n > 1$ we have $x_1^{w_1} \ldots x_n^{w_n} \in \mathrm{Sm}\,(I(V)) \iff$ there are at least $w_n + 1$ values β for which $x_1^{w_1} \ldots x_{n-1}^{w_{n-1}} \in \mathrm{Sm}\,(I(V_\beta))$.

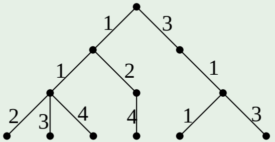
Proof. It suffices to see that Stan wins $LS(V; (w_1, \ldots, w_n)) \iff$ there are at least $w_n + 1$ values β such that Stan wins $LS(V_\beta; (w_1, \ldots, w_{n-1}))$. $\square$

An algorithm to compute standard monomials

Trie: labelled, rooted tree. The word of a node v is the sequence of labels on path form root to v.

Example

The reverse trie of
$V = \{(2,1,1),(3,1,1),(4,1,1),(4,2,1),(1,1,3),(3,1,3)\}$:



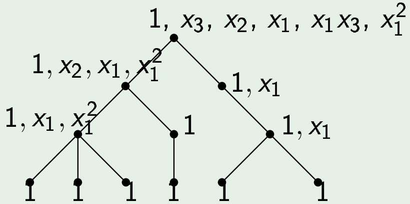
Claim. The subtree at child β of the root is the reverse trie of V_β.

Plan. To every node attach the normal set of its subtree. At level $n-i$. we see monomials in $x_1,\ldots,x_i$ (essentially Cerlienco and Mureddu).

Simple implementation

Proceed level by level from the bottom (from level n).
At level n: write 1 to every vertex.
Level $n-i$: write $x_1^{w_1} \dots x_{i-1}^{w_{i-1}} x_i^{w_i}$ to v, if $x_1^{w_1} \dots x_{i-1}^{w_{i-1}}$ occurs in at least $w_i + 1$ children of v.

Example



About the fast algorithm

Using the reverse trie of V, we build the trie for the (exponent vectors of) standard monomials.

This latter phase can be done in linear time.

Cost is dominated by the building of the reverse trie of V.

Total cost is $O(n|V|r)$, where r is the maximal outdegree in the trie of V.

This can be $O(n|V|)$, if $\mathbb{F}$ is small, eg., for $\mathbb{F} = \mathbb{F}_2$.

Cost is $O(n|V|\log r)$, if we have a good ordering on $\mathbb{F}$.

Applications of the game: permutations

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ be pairwise different elements and

$$V = \left\{ \left(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \ldots, \alpha_{\pi(n)} \right) \; : \; \pi \in S_n \right\}.$$

Then

$$\mathrm{Sm}\,(I(V)) = \{x_1^{w_1} \ldots x_n^{w_n} \; : \; w_i < i \; (\forall\, i)\}$$

If $w_i \geq i$, then Lea wins; here we are left with only i possibilities for y_i.

We obtain the *Hall monomials*.

Applications: full families of sets with given sizes

Let $D \subseteq \mathbb{N}$ be arbitrary. Put

$$\mathcal{F}_D := \{Z \subseteq [n] \ : \ |Z| \in D\},$$
$$V_D := \{\boldsymbol{y} \subseteq \{0,1\}^n \ : \ \text{the Hamming weight of } \boldsymbol{y} \ \in D\}.$$

$D^{(0)} := D \cup (D-1)$ és $D^{(1)} := D \cap (D-1)$.

For $\boldsymbol{w} \in \{0,1\}^n$ write $D^{(\boldsymbol{w})} := \left(\ldots\left(\left(D^{(w_1)}\right)^{(w_2)}\right)^{\cdots}\right)^{(w_n)}$.

Example

Let $n = 2$, $\boldsymbol{w} = (0,1)$, $D = \{1,3,4\}$. Then

$$D^{(\boldsymbol{w})} = \{0,1,2,3,4\}^{(1)} = \{0,1,2,3\}.$$

Theorem

$x^{\boldsymbol{w}} \in \mathrm{Sm}\,(I(V_D)) \iff \boldsymbol{w} \in \{0,1\}^n$ *and* $0 \in D^{(\boldsymbol{w})}$.

Let d, r, l be integers, $0 \le d < r$ and $1 \le l < r$. Put

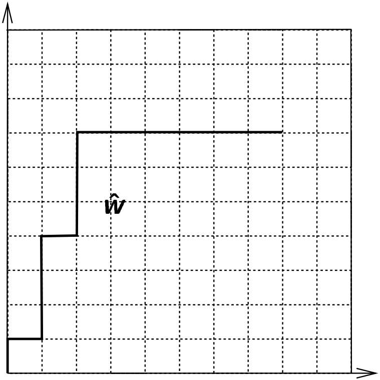
$$D = \{a \in \mathbb{Z} : \; d \le a \bmod r \le d + l - 1\}.$$

For which vectors $\mathbf{w} \in \{0,1\}^n$ *do we have* $0 \in D^{(\mathbf{w})}$*?*

We attach to $\mathbf{w} \in \{0,1\}^n$ a lattice path $\hat{\mathbf{w}}$ in the plane. It starts from the origin, proceeds in unit steps. The ith step is *up* if $w_i = 0$, and it is to the *right*, if $w_i = 1$.

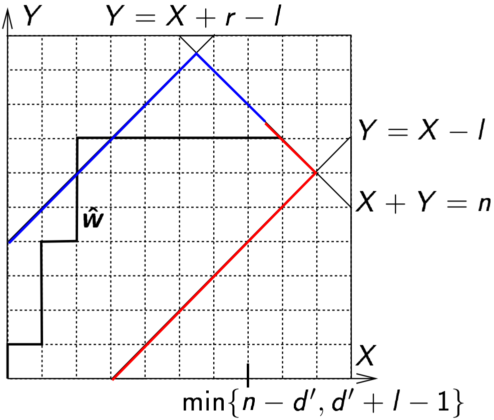
An example

The lattice path $\hat{\boldsymbol{w}}$ attached to the exponent vector $\boldsymbol{w} = (2, 6, 10, 11, 12, 13, 14, 15)$.



Theorem $0 \in D^{(\mathbf{w})} \iff \hat{\mathbf{w}}$ *meets blue line before red one.*

Ex. $n = 15,\ r = 7,\ l = 3,\ d = 1,\ (\mathbf{w}) = (2, 6, 10, 11, 12, 13, 14, 15)$.



d' is the integer for which $d' \equiv d \pmod{r}$ and $\frac{n-r-l}{2} < d' \leq \frac{n+r-l}{2}$.

A theorem of A. Bernasconi and L. Egidi

Let $\mathbb{F} = \mathbb{Q}$, $V \subseteq [n]$ and $\mathcal{F} := V_D$.
Suppose further that $0 \leq m \leq n$, and

$$D = \{l_1, \ldots, l_s\} \cup \{m_1, \ldots, m_t\},$$

where $l_j \leq m$ and $m < m_1 < m_2 < \cdots < m_t$. Assume also that

$$\{0, 1, \ldots, m\} \setminus \{l_1, \ldots, l_s\} = \{n_1, n_2, \ldots, n_{m+1-s}\},$$

with $n_1 > n_2 > \cdots > n_{m+1-s}$ and $u = \min\{t, m+1-s\}$.

Then we have

$$h_{\mathcal{F}}(m) = \sum_{j=1}^{s} \binom{n}{l_j} + \sum_{j=1}^{u} \min\{\binom{n}{m_j}, \binom{n}{n_j}\}.$$

Problem

Determine (deglex) Gröbner bases and standard monomials of $\mathcal{F}$.

A (finite) 0-1 sequence is a *ballot sequence* if in each prefix the number of zeros is not smaller than the number of ones. A (finite) 0-1 sequence is a *k-ballot sequence* if by putting k zeros in front of the original sequence we get a ballot sequence.

Example

11010011 is a 2-ballot sequence but not 1-ballot.

A (finite) increasing sequence of positive integers is k-ballot if its characteristic sequence is a k-ballot sequence. Similarly, a squarfree monomial is k-ballot if the characteristic sequence of its variables in increasing order is a k-ballot sequence.

Example

$x_1 x_3 x_5$ is 1-ballot but not (0-)ballot.

Remark. If a monomial is k-ballot then it is also l-ballot for $l \geq k$.

Standard monomials of some V_D (Pintér, R)

Suppose that $D = \{c_1, \ldots, c_k\} \subseteq [n]$ and for each i at most one of i and $n - i$ is in D. We seek the deglex standard monomials of $V = V_D$ over $\mathbb{Q}$.

Put $d_i = \min\{c_i, n - c_i\}$ and assume that $d_1 < \cdots < d_k$.

Theorem

The standard monomials of V of degree at most $d_1 + k - 1$ are the $(k-1)$-ballots. The standard monomials for V of degree at least $d_{j-1} + k - j + 2$ and at most $d_j + k - j$ are the $(k-j)$-ballots for $j = 2, \ldots, k$.

Example

Let $n = 6$, $D = \{1, 4\}$. The standard monomials for V are: 1; $x_1, \ldots, x_6$; $x_1x_3, \ldots, x_1x_6$, $x_2x_6, \ldots, x_5x_6$, the 1-ballots of degree at most 2.