THE CENTER PROBLEM AND LOCAL LIMIT CYCLES BIFURCATIONS IN POLYNOMIAL SYSTEMS

Valery Romanovski
Center for Applied Mathematics and Theoretical Physics,
University of Maribor, Krekova 2, SI-2000 Maribor
Slovenia

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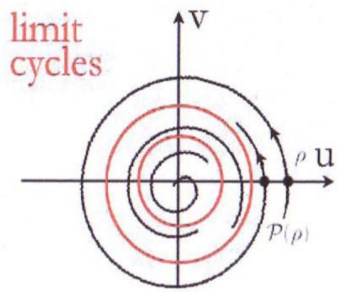
Consider the system

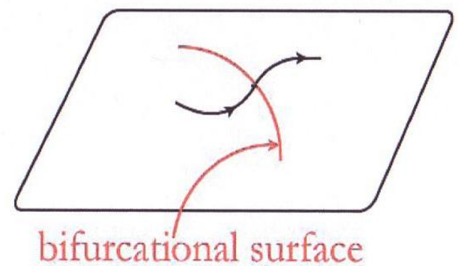
$$\frac{du}{dt} = \alpha u - \beta v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = \beta u + \alpha v + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \quad (1)$$

Phase space

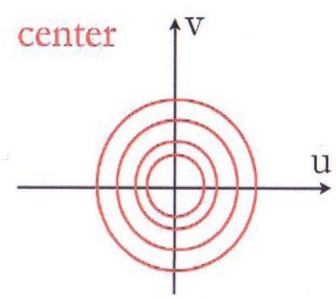


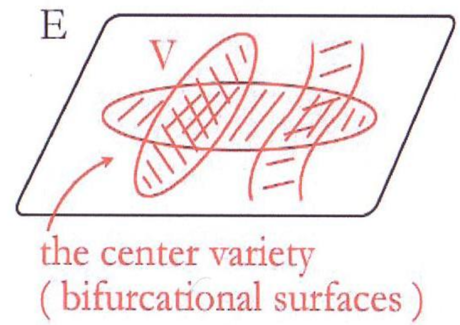
Parameter space

E=($\alpha, \beta, \alpha_{ij}, \beta_{ij}$)



We assume that $\alpha = 0,\ \beta = 1$. Then the origin is a week (fine) focus or a center.





Poincare (return) map:

$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Limit cycles $\longleftrightarrow$ isolated fixed points of $\mathcal{P}(\rho)$. The center variety:

$$\mathbf{V} = \{(\alpha_{ij}, \beta_{ij}) \in \mathcal{E} \mid \eta_3(\alpha_{ij}, \beta_{ij}) = \eta_4(\alpha_{ij}, \beta_{ij}) = \cdots = 0\}$$

Let $\mathcal{B} = \langle \eta_3, \eta_4, \dots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$ be the ideal generated by the focus quantities η_i. $\mathcal{B}$ is called the *Bautin ideal* of system (1). There is k such that

$$\mathcal{B} = \langle \eta_3, \eta_5, \dots, \eta_{2k+1} \rangle.$$

Then

$$\mathcal{P}(\rho) - \rho = \eta_3(1 + \dots)\rho^3 + \dots \eta_{2k+1}(1 + \dots)\rho^{2k+1}.$$

Theorem 1 (Bautin). *If $\mathcal{B} = \langle \eta_3, \eta_5, \dots, \eta_{2k+1} \rangle$ then the cyclicity of system (1) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is equal to k.*

Proof. Bautin N.N. Mat. Sb. (1952) v.30, 181-196 (Russian); Trans. Amer. Math. Soc. (1954) v.100
Roussarie R. Bifurcations of planar vector fields and Hilbert's 16th problem (1998), Birkhauser.

The Center Problem:

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B}$.

The Cyclicity Problem (Local Hilbert's 16th Problem):

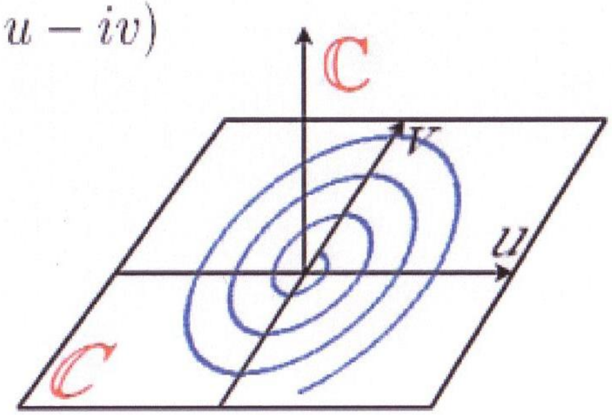
Find a basis for the Bautin ideal $\mathcal{B}$

Theorem 2 (Strong Hilbert Nullstellensatz) *Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ and let I be an ideal of $\mathbb{C}[x_1, \ldots, x_m]$. Then f vanishes on the variety of I if and only if for some positive integer ℓ $f^\ell \in I$ ($f \in \sqrt{I}$).*

Complexification: $x = u + iv \quad (\bar{x} = u - iv)$

$$\dot{x} = \; i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)$$

$$\dot{\bar{x}} = - \, i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)$$



$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p) \tag{1}$$

If $b_{qp} = \bar{a}_{pq}$, $y = \bar{x}$ then from (3) we obtain the "real" system.

The change of time $d\tau = idt$ transforms (1) to the system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p) \tag{2}$$

where x, y, a_{pq}, b_{qp} are complex variables, $S = \{(m, k) | m + k \geq 1\}$ is a subset of $\{-1 \cup \mathbb{N}\} \times \mathbb{N}$, $\mathbb{N}$ is the set of non-negative integers. Let l be the number of the elements in the set S. We denote by $E(a, b) (= \mathbb{C}^{2l})$ the parameter space of (2), and by $\mathbb{C}[a, b]$ ($\mathbb{Q}[a, b]$) the polynomial ring in the variables a_{pq}, b_{qp} over the field $\mathbb{C}$ (over $\mathbb{Q}$).

What is a center for system (2)???

Theorem 2 (Poincaré-Lyapunov). *The system*

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j$$

has a center at the origin if and only if it admits a first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl} u^k v^l.$$

Consider polynomial systems of the form

$$\frac{dx}{dt} = x + F(x,y) = P(x,y), \quad \frac{dy}{dt} = -y + G(x,y) = Q(x,y),$$

where $F(x,y), G(x,y) \in \mathbb{C}[x,y]$ without constant and linear terms.

Definition 1. *(Dulac). System (5) has a center at the origin if there is an analytic first integral of the form*

$$\Psi(x,y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}, \tag{3}$$

(First integral: $\frac{\partial \Psi}{\partial x} P(x,y) + \frac{\partial \Psi}{\partial y} Q(x,y) = 0.$)

For system (2) one can always find a function Ψ of the form (3) such that

$$D(\Psi) := \frac{\partial \Psi}{\partial x} P(x,y) + \frac{\partial \Psi}{\partial y} Q(x,y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots ,$$

where the g_{ii} are polynomials of $\mathbb{C}[a,b]$ called *focus quantities*. Thus system (2) with the fixed parameters (a^*, b^*) has a center at the origin if and only if $g_{ii}(a^*, b^*) = 0$ for all $i = 1, 2, \ldots$, i.e. if and only if

$$(a^*, b^*) \in \mathbf{V}(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle).$$

$\mathbf{V}(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle) = \mathbf{V}(\mathcal{B})$ is the *the center variety.*

CALCULATION OF FOCUS QUANTITIES

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p)$$

We assume that $S = \{\bar{\imath}_1, \ldots, \bar{\imath}_l\} = \{(p_1, q_1), (p_2, q_2), \ldots, (q_1, p_1)\}$ ($\bar{\imath}_s = (p_s, q_s)$) is the ordered set of the indices of the coefficients of the first equation of system (2) and consider the map $L : \mathbb{N}^{2l} \to \mathbb{N}^2$ (recall that l is the number of elements in the set S), defined by

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \nu_1 \bar{\imath}_1 + \nu_2 \bar{\imath}_2 + \cdots + \nu_{l-1} \bar{\imath}_{l-1} + \nu_l \bar{\imath}_l + \nu_{l+1} \bar{\jmath}_l + \nu_{l+2} \bar{\jmath}_{l-1} + \cdots + \nu_{2l-1} \bar{\jmath}_2 + \nu_{2l} \tag{4}$$

where $\bar{\jmath}_s$ corresponds to $\bar{\imath}_s$, such that if $\bar{\jmath}_s = \binom{p_s}{q_s}$, then $\bar{\imath}_s = \binom{q_s}{p_s}$.

Denote by $[\nu]$ the monomial

$$[\nu] = a_{\bar{\imath}_1}^{\nu_1} a_{\bar{\imath}_2}^{\nu_2} \ldots a_{\bar{\imath}_l}^{\nu_l} b_{\bar{\jmath}_l}^{\nu_{l+1}} b_{\bar{\jmath}_{l-1}}^{\nu_{l+2}} \ldots b_{\bar{\jmath}_1}^{\nu_{2l}}$$

and by $\overline{\nu}$ the involution of the vector ν:

$$\overline{\nu} = (\nu_{2l}, \nu_{2l-1}, \ldots, \nu_2, \nu_1). \tag{5}$$

Consider the formal series

$$V = \sum V_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a_{\bar{\imath}_1}^{\nu_1} a_{\bar{\imath}_2}^{\nu_2} \ldots a_{\bar{\imath}_l}^{\nu_l} b_{\bar{\jmath}_l}^{\nu_{l+1}} b_{\bar{\jmath}_{l-1}}^{\nu_{l+2}} \ldots b_{\bar{\jmath}_1}^{\nu_{2l}}, \tag{6}$$

where $V_{(\nu_1,\ldots,\nu_{2l})}$ are determined by the recurrence formula:

$$V_{(\nu_1,\nu_2,\ldots,\nu_{2l})} = \frac{1}{L^1(\nu) - L^2(\nu)} (\sum_{i=1}^{l} V_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^1(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})$$

$$+1) - \sum_{i=l+1}^{2l} V_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^2(\nu_1,\cdots,\nu_i-1,\ldots,\nu_{2l})+1)) \tag{7}$$

if $L^1(\nu) \neq L^2(\nu)$, $V_{(\nu_1,\ldots,\nu_{2l})} = 0$, if $L^1(\nu) = L^2(\nu)$; $V_{(0,\ldots,0)} = 1$ and we put $V_{(\nu_1,\ldots,\nu_{2l})} = 0$ for all $\nu = (\nu_1,\ldots,\nu_{2l})$, such that there exists $i : \nu_i < 0$.

Looking for a first integral

$$\Psi(x,y) = xy + \sum_{j+k\geq 3} v_{j-1,k-1}(a,b) x^j y^k ,$$

we have the equation:

$$D(\Psi) := \frac{\partial \Psi}{\partial x} P(x,y) + \frac{\partial \Psi}{\partial y} Q(x,y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots ,$$

Theorem 3. *1) The coefficient of* $[\nu]$ *in the polynomial* v_{kn} *is equal to* $V_{(\nu_1,\nu_2,\ldots,\nu_{2l})}$.
2) The i-th focus quantity of the system (2) is

$$g_{ii} = \sum_{\nu : L(\nu)=\binom{i}{i}} g_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a_{\bar{\imath}_1}^{\nu_1} a_{\bar{\imath}_2}^{\nu_2} \ldots a_{\bar{\imath}_l}^{\nu_l} b_{\bar{\jmath}_l}^{\nu_{l+1}} b_{\bar{\jmath}_{l-1}}^{\nu_{l+2}} \ldots b_{\bar{\jmath}_1}^{\nu_{2l}}, \tag{8}$$

where

$$g_{(\nu_1,\nu_2,\ldots,\nu_{2l})} = \sum_{i=1}^{l} V_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^1(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+1) \tag{9}$$

$$-\sum_{i=l+1}^{2l} V_{(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})}(L^2(\nu_1,\ldots,\nu_i-1,\ldots,\nu_{2l})+1)$$

and $V_{(\nu)}$ *are defined by (7).*
3) $g_{(\nu)} = -g_{(\bar{\nu})}$ *if* $\nu \neq \bar{\nu}$.

The equation (7) is the so-called difference equation. It is often possible to pass from a given difference equation to a differential equation, and vice versa.

For general polynomial system (2) we obtain the differential equation

$$\mathcal{A}(V) = (|a| - |b|)V,$$

where $|a| = \sum_{(i,j)\in S} a_{ij}$, $|b| = \sum_{(j,i)\in S} b_{ij}$ and

$$\mathcal{A}(V) = \sum_{(i,j)\in S} \frac{\partial V}{\partial a_{ij}} a_{ij}(i - j - i|a| + j|b|) + \sum_{(j,i)\in S} \frac{\partial V}{\partial b_{ij}} b_{ij}(i - j - i|a| + j|b|) \quad (10)$$

is the linear operator

$$\mathcal{A} : \mathbb{C}[[a, b]] \longrightarrow \mathbb{C}[[a, b]]$$

(recall that $k[[x]]$ denotes the ring of formal power series of x over k).

Let the map

$$\pi : \mathbb{C}[a, b][[x, y]] \longrightarrow \mathbb{C}[[a, b]]$$

be defined by

$$\pi\left(\sum c_{\alpha,\beta}(a, b)x^{\alpha}y^{\beta}\right) = \sum c_{\alpha,\beta}(a, b). \quad (11)$$

Theorem 4. *The system (2) has a center at the origin for all values of the parameters a_{kn}, b_{nk} (that is for all $(a, b) \in E(a, b)$) if and only if there is a formal series (6) such that $V_{(0,\dots,0)} = 1$ satisfying the equation*

$$\mathcal{A}(V) = V(|a| - |b|). \quad (12)$$

Thus, the Poincaré center problem is equivalent to the study of formal solutions of PDE (12).

THE POINCARÉ CENTER PROBLEM

System (2): $\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q)$, $\dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p)$

$$\Psi(x,y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

$$D(\Psi) := \frac{\partial \Psi}{\partial x} P(x,y) + \frac{\partial \Psi}{\partial y} Q(x,y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots .$$

System (2) has a center at the origin if and only if $g_{ii}(a^*, b^*) = 0$ for all $i = 1, 2, \ldots,$ i.e. if and only if

$$(a^*, b^*) \in \mathbf{V}(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle),$$

i.e., to solve the center problem means to find the variety of $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$.
The difficulty: g_{kk} are given by recurrence formula.
A way to study the problem:

- Let $\mathcal{B}_k = \langle g_{11}, g_{22}, \ldots, g_{kk} \rangle$. Compute $g_{11}, g_{22}, \ldots, g_{ss}$ until $\mathbf{V}(\mathcal{B}_1) \supset \mathbf{V}(\mathcal{B}_2) \supset \ldots \mathbf{V}(\mathcal{B}_{s-1}) = \mathbf{V}(\mathcal{B}_s)$.
- Find irreducible decomposition of $\mathbf{V}(\mathcal{B}_{ss})$: $\mathbf{V}(\mathcal{B}_{ss}) = V_1 \cup V_2 \cup \ldots V_m$.
- For every V_j prove existence of a Lyapunov integral.

CUBIC SYSTEM

$$i\frac{dx}{dt} = x + P_2(x, \bar{x}) + P_3(x, \bar{x}).$$

System with homogeneous cubic nonlinearities (Malkin, 1966):

$$i\frac{dx}{dt} = x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3. \tag{13}$$

The complexification $y = \bar{x}$ and the change of the time $dt = id\tau$ yields the system

$$\begin{aligned} \dot{x} &= x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \\ \dot{y} &= -(y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3,-1}x^3). \end{aligned} \tag{14}$$

Computing the first five focus quantities of (14) we find:

$$\begin{aligned} g_{11} &= a_{11} - b_{11}; \\ g_{22} &= a_{20}a_{02} - b_{02}b_{20}; \\ g_{33} &= (3a_{20}^2a_{-13} + 8a_{20}a_{-13}b_{20} + 3a_{02}^2b_{3,-1} - 8a_{02}b_{02}b_{3,-1} - 3a_{-13}b_{20}^2 - 3b_{02}^2b_{3,-1})/ \\ g_{44} &= (-9a_{20}^2a_{-13}b_{11} + a_{11}a_{-13}b_{20}^2 + 9a_{11}b_{02}^2b_{3,-1} - a_{02}^2b_{11}b_{3,-1})/16; \\ g_{55} &= (-9a_{20}^2a_{-13}b_{02}b_{20} + a_{20}a_{02}a_{-13}b_{20}^2 + 9a_{20}a_{02}b_{02}^2b_{3,-1} + 18a_{20}a_{-13}^2b_{20}b_{3,-1} + \\ &\quad 6a_{02}^2a_{-13}b_{3,-1}^2 - a_{02}^2b_{02}b_{20}b_{3,-1} - 18a_{02}a_{-13}b_{02}b_{3,-1}^2 - 6a_{-13}^2b_{20}^2b_{3,-1})/36. \end{aligned}$$

Theorem 5. *Let $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$ be the Bautin ideal of system (14). The center variety $\mathbf{V}(\mathcal{B})$ of the system (14) consists of the three irreducible components:*

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\langle g_{11}, \ldots, g_{55} \rangle) = \mathbf{V}(C_1) \cup \mathbf{V}(C_2) \cup \mathbf{V}(C_3),$$

where
$C_1 = \langle a_{11} - b_{11}, 3a_{20} - b_{20}, 3b_{02} - a_{02} \rangle,$
$C_2 = \langle a_{11}, b_{11}, a_{20} + 3b_{20}, b_{02} + 3a_{02}, a_{-13}b_{3,-1} - 4a_{02}b_{20} \rangle$
$C_3 = \langle a_{20}^2 a_{-13} - b_{3,-1}b_{02}^2, a_{20}a_{02} - b_{20}b_{02},\ a_{20}a_{-13}b_{20} - a_{02}b_{3,-1}b_{02},$
$a_{11} - b_{11}, a_{02}^2 b_{3,-1} - a_{-13}b_{20}^2 \rangle.$

Proof. Computing with $minAssChar$ or $minAssGTZ$ of $Singular$ we find that the minimal associate primes of the ideal $\langle g_{11}, g_{22}, \ldots, g_{55} \rangle$ are the ideals C_1, C_2, C_3. To prove that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\langle g_{11}, \ldots, g_{55} \rangle)$ it is sufficient to show that systems from $\mathbf{V}(C_1), \mathbf{V}(C_2), \mathbf{V}(C_3)$ admit first integrals.

Q.: what are the most efficient algorithms for decomposition of varieties?
C_1 – Hamiltonian systems. System:

$$\dot{x} = x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \quad \dot{y} = -(y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3,-1}x^3).$$

$$H = xy - \frac{b_{3,-1}\,x^4}{4} - a_{20}\,x^3\,y - \frac{b_{11}\,x^2\,y^2}{2} - b_{02}\,x\,y^3 - \frac{a_{-13}\,y^4}{4}$$

C_2 – Darboux integrable systems.

Consider the system of differential equations

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{15}$$

where $x, y \in \mathbb{C}$ and P and Q are polynomials.

The polynomial $f(x, y) \in \mathbb{C}[x, y]$ defines an *algebraic invariant curve* $f(x, y) = 0$ of the system (15) if and only if there exists a polynomial $k(x, y) \in \mathbb{C}[x, y]$ such that

$$D(f) = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = kf. \tag{16}$$

k is called *cofactor* of f.
Suppose that the curves defined by

$$f_1 = 0, \ldots, f_s = 0$$

are invariant algebraic curves of the system (15). A first integral of the system (15) of the form

$$H = f_1^{\alpha_1} \cdots f_s^{\alpha_s} \tag{17}$$

is called a *Darboux integral* of (15).

If $f_1, \ldots, f_s$ are different irreducible algebraic partial integrals such that $\sum_{j=1}^{s} \alpha_j k_j = 0$ then $H = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is a first integral of (15).

Algebraic invariant curves:

$$f_1 = 1 + 2b_{20}x^2 + \frac{a_{02}{b_{3,-1}}^2 x^4}{4b_{20}} - \frac{4{b_{20}}^2 xy}{b_{3,-1}} - \frac{a_{02}b_{3,-1}xy}{b_{20}} - 2a_{02}b_{3,-1}x^3y + 2a_{02}y^2 + 6a_{02}b_{20}x^2y^2 - \frac{8a_{02}{b_{20}}^2 xy^3}{b_{3,-1}} + \frac{4a_{02}{b_{20}}^3 y^4}{{b_{3,-1}}^2},$$

$$f_2 = 8b_{20}^3 b_{3,-1}^2 + 2a_{02}b_{3,-1}^4 + 24b_{20}^4 {b_{3,-1}}^2 x^2 + 6a_{02}b_{20}b_{3,-1}^4 x^2 + 12a_{02}b_{20}^2 b_{3,-1}^4 x^4 + {a_{02}}^2 b_{3,-1}^6 x^6 - 48a_{02}b_{20}^2 {b_{3,-1}}^3 xy - 72a_{02}b_{20}^3 b_{3,-1}^3 x^3 y - 6a_{02}^2 b_{3,-1}^5 x^3 y - 12a_{02}^2 b_{20} b_{3,-1}^5 x^5 y + 24a_{02}{b_{20}}^3 {b_{3,-1}}^2 y^2 + 6{a_{02}}^2 {b_{3,-1}}^4 y^2 + 144a_{02}{b_{20}}^4 {b_{3,-1}}^2 x^2y^2 + 36a_{02}^2 b_{20} b_{3,-1}^4 x^2 y^2 + 60{a_{02}}^2 {b_{20}}^2 b_{3,-1}^4 x^4 y^2 - 96a_{02}b_{20}^5 b_{3,-1} xy^3 - 72a_{02}^2 b_{20}^2 b_{3,-1}^3 xy^3 - 160a_{02}^2 b_{20}^3 b_{3,-1}^3 x^3y^3 + 48a_{02}^2 b_{20}^3 b_{3,-1}^2 y^4 + 240a_{02}^2 b_{20}^4 b_{3,-1}^2 x^2 y^4 - 192a_{02}^2 b_{20}^5 b_{3,-1} xy^5 + 64a_{02}^2 b_{20}^6 y^6 / (2b_{3,-1}^2 (4b_{20}^3 + a_{02}b_{3,-1}^2))$$

Cofactors: $k_1 = 4(b_{20}x^2 - a_{02}y^2)$, $k_2 = 6(b_{20}x^2 - a_{02}y^2)$

From the equation $\alpha_1 k_1 + \alpha_2 k_2 = 0$ we find $\alpha_1 = 3, \alpha = -2$, yielding the first integral $\Psi = f_1^3\, f_2^{-2}$.

The associated PDE:

$$\mathcal{A}(V) := \tag{18}$$

$$\frac{\partial V}{\partial a_{02}} a_{02}(-2+2(-3a_{02}+b_{20}+b_{3,-1}))+\frac{\partial V}{\partial b_{20}} b_{20}(2-2(a_{02}-3b_{20}+4a_{02}b_{20}/b_{3,-1})))+$$
$$\frac{\partial V}{\partial b_{3,-1}} b_{3,-1}(4-(8b_{20}-12a_{02}b_{20}/b_{3,-1})=(4a_{02}-4b_{20}+4a_{02}b_{20}/b_{3,-1}-b_{3,-1})V.$$

$\Psi = f_1^3 f_2^{-3}$ is a first integral of our system of ODE, but $\pi(\Psi)$ is not a solution to (18), because Ψ is not of the form $xy + h.o.t.$:

$$\Psi = 1 - \frac{3\left(-4\,{b_{20}}^3+a_{02}\,{b_{3,-1}}^2\right)^2 x\,y}{4\,{b_{20}}^4\,b_{3,-1}+a_{02}\,b_{20}\,{b_{3,-1}}^3} +$$
$$\frac{3\left(-4\,{b_{20}}^3+a_{02}\,{b_{3,-1}}^2\right)^2\left(16\,{b_{20}}^6-28\,a_{02}\,{b_{20}}^3\,{b_{3,-1}}^2+{a_{02}}^2\,{b_{3,-1}}^4\right)x^2\,y^2}{\left(4\,{b_{20}}^4\,b_{3,-1}+a_{02}\,b_{20}\,{b_{3,-1}}^3\right)^2} + \ldots$$

Thus,

$$-(\pi(\Psi)-1)\frac{4\,{b_{20}}^4\,b_{3,-1}+a_{02}\,b_{20}\,{b_{3,-1}}^3}{3\left(-4\,{b_{20}}^3+a_{02}\,{b_{3,-1}}^2\right)^2} =$$
$$-(\pi(f_1)^3\pi(f_2)^{-2}-1)\frac{4\,{b_{20}}^4\,b_{3,-1}+a_{02}\,b_{20}\,{b_{3,-1}}^3}{3\left(-4\,{b_{20}}^3+a_{02}\,{b_{3,-1}}^2\right)^2}$$

is a (rational) solution to (18).

Q.: Is there any algorithmic method to find solutions to equations like (18)?

C_3 – time-reversible systems.

A General Algorithm for Finding Time-Reversible Systems

Jarrah , Laubenbacher and R. (2003):

Let

$$\dot{x} = P(x, \bar{x}). \tag{19}$$

be complexification of

$$\dot{u} = v + U(u, v), \qquad \dot{v} = -u + V(u, v). \tag{20}$$

A straight line L is an *axis of symmetry* of (20) if the trajectories of the system are symmetric with respect to the line L.

Lemma 1. *Let a denote the vector of coefficients of the polynomial $P(x, \bar{x})$ in (19), arising from the real system (20) by setting $x = u + iv$. If $a = \pm\bar{a}$ (meaning that either all the coefficients are real or all are pure imaginary), then the u–axis is an axis of symmetry of (20).*

By the lemma the u–axis is an axis of symmetry for (19) if

$$P(\bar{x}, x) = -\overline{P(x, \bar{x})} \tag{21}$$

(the case $a = -\bar{a}$), or if

$$P(\bar{x}, x) = \overline{P(x, \bar{x})} \tag{22}$$

(the case $a = \bar{a}$). If condition (21) is satisfied then under the change

$$x \to \bar{x}, \qquad \bar{x} \to x, \tag{23}$$

$\dot{x} = P(x, \bar{x})$ is transformed to its negative,

$$\dot{x} = -P(x, \bar{x}), \tag{24}$$

and if condition (22) holds then (19) is unchanged. Thus condition (22) means that the system is reversible with respect to reflection across the u–axis (i.e., the transformation does not change the system) while condition (21) corresponds to time–reversibility with respect to the same transformation.

If the line of reflection is not the u–axis but a distinct line L then we can apply the rotation $x_1 = e^{-i\varphi}x$ through an appropriate angle φ to make L the u–axis. (19) is time–reversible when there exists a φ such that

$$e^{2i\varphi}\overline{P(x, \bar{x})} = -P(e^{2i\varphi}\bar{x}, e^{-2i\varphi}x). \tag{25}$$

This suggests the following natural generalization of the notion of time–reversibility to the case of two–dimensional complex systems.

Definition 2. *Let* $\mathbf{z} = (x, y) \in \mathbb{C}^2$. *We say that the system*

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \tag{26}$$

is time–reversible *if there is a linear transformation* T,

$$x \mapsto \alpha y, \;\; y \mapsto \alpha^{-1} x \tag{27}$$

($\alpha \in \mathbb{C}$), such that

$$\frac{d(T\mathbf{z})}{dt} = -F(T\mathbf{z}) \tag{28}$$

For a fixed collection $(p_1, q_1), \ldots, (p_\ell, q_\ell)$ of elements of $(\{-1\} \cup \mathbb{N}_+) \times \mathbb{N}_+$, and letting ν denote the element $(\nu_1, \ldots, \nu_{2\ell})$ of $\mathbb{N}_+^{2\ell}$, let L be the map from $\mathbb{N}_+^{2\ell}$ to $\mathbb{N}_+^2$ (the elements of the latter written as column vectors) defined by

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell}. \tag{29}$$

Let $\mathcal{M}$ denote the set of all solutions $\nu = (\nu_1, \nu_2, \dots, \nu_{2l})$ with non–negative components of the equation

$$L(\nu) = \binom{k}{k} \tag{30}$$

as k runs through $\mathbb{N}_+$, and the pairs (p_i, q_i) determining $L(\nu)$ come from system (2). $\mathcal{M}$ is an Abelian monoid. Let $\mathbb{C}[\mathcal{M}]$ denote the subalgebra of $\mathbb{C}[a, b]$ generated by all monomials of the form

$$[\nu] := a_{p_1 q_1}^{\nu_1} a_{p_2 q_2}^{\nu_2} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} b_{q_{\ell-1} p_{\ell-1}}^{\nu_{\ell+2}} \cdots b_{q_1 p_1}^{\nu_{2\ell}},$$

for all $\nu \in \mathcal{M}$. For ν in $\mathcal{M}$, let $\hat{\nu}$ denote the involution of the vector ν:

$$\hat{\nu} = (\nu_{2\ell}, \nu_{2\ell-1}, \dots, \nu_1).$$

Corollary of Theorem 3: The focus quantities of system (2) belong to $\mathbb{C}[\mathcal{M}]$ and have the form

$$g_{kk} = \sum_{L(\nu)=(k,k)^T} g_{(\nu)}([\nu] - [\hat{\nu}]), \tag{31}$$

with $g_{(\nu)} \in \mathbb{Q}, \; k = 1, 2, \dots$.

Consider the ideal

$$I_{sym} = \langle [\nu] - [\hat{\nu}] \mid \nu \; \in \mathcal{M} \rangle \subset \mathbb{C}[\mathcal{M}].$$

It is clear that $\mathcal{B} \subseteq I_{sym}$, hence $\mathbf{V}(I_{sym}) \subseteq \mathbf{V}(\mathcal{B})$.

Definition 3. *For system* (2) *the variety* $\mathbf{V}(I_{sym})$ *is called the* Sibirsky (or symmetry) subvariety *of the center variety, and the ideal* I_{sym} *is called the* Sibirsky ideal.

Every time–reversible real system with the singularity of focus or center type at the origin has a center at the origin. It is easily seen that this property is transferred to complex systems: *every time–reversible system* (2) *has a center at the origin*. Indeed, the time-reversibility condition $\alpha Q(\alpha y, x/\alpha) = -P(x,y),\ \alpha Q(x,y) = -P(\alpha y, x/\alpha)$ yields that system (2) is time–reversible if and only if

$$b_{qp} = \alpha^{p-q} a_{pq}, \qquad a_{pq} = b_{qp}\alpha^{q-p}. \tag{32}$$

Hence in the case that (2) is time–reversible, using (32) we see that for $\nu \in \mathcal{M}$

$$[\hat{\nu}] = \alpha^{(L^1(\nu)-L^2(\nu))}[\nu] = [\nu] \tag{33}$$

and thus from (31) we obtain $g_{kk} \equiv 0$ for all k, which implies that the system has a center.

By (33) every time–reversible system $(a,b) \in E(a,b)$ belongs to $\mathbf{V}(I_{sym})$. The converse is false.

$$\dot{x} = x(1 - a_{10}x - a_{01}y), \qquad \dot{y} = -y(1 - b_{10}x - b_{01}y).$$

In this case $I_{sym} = \langle a_{10}a_{01} - b_{10}b_{01} \rangle$. The system

$$\dot{x} = x(1 - a_{10}x), \qquad \dot{y} = -y(1 - b_{10}x) \tag{34}$$

arises from $\mathbf{V}(I_{sym})$ but (32) are not fulfilled, so (34) is not time–reversible.

Theorem 6. *Let $\mathcal{R} \subset E(a,b)$ be the set of all time–reversible systems in the family* (2). *Then:*
1. $\mathcal{R} \subset \mathbf{V}(I_{sym})$;
2. $\mathbf{V}(I_{sym}) \setminus \mathcal{R} = \{(a,b) \mid \exists (p,q) \in S$ such that $a_{pq}b_{qp} = 0$ but $a_{pq} + b_{qp} \neq 0\}$.

The theorem shows that to describe time reversible systems it is sufficient to compute I_{sym}.

Algorithm for Finding Time-Reversible Systems

Input: Two sequences of integers $p_1, \ldots, p_\ell$ $(p_i \geq -1)$ and $q_1, \ldots, q_\ell$ $(q_i \geq 0)$.
(These are the coefficient labels for system (2):
$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq}x^{p+1}y^q),\ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp}x^q y^p)$.)

Output: A finite set of generators for the Sibirsky ideal I_{sym} of (2).

1. Compute a reduced Gröbner basis G for the ideal

$$\mathcal{J} = \langle a_{p_i q_i} - y_i t_1^{p_i} t_2^{q_i}, b_{q_i p_i} - y_{\ell-i+1} t_1^{q_{\ell-i+1}} t_2^{p_{\ell-i+1}} \mid i = 1, \ldots, \ell \rangle$$
$$\subset \mathbb{C}[a, b, y_1, \ldots, y_\ell, t_1^{\pm}, t_2^{\pm}]$$

with respect to any elimination ordering for which

$$\{t_1, t_2\} > \{y_1, \ldots, y_d\} > \{a_{p_1 q_1}, \ldots, b_{q_1 p_1}\}.$$

2. $I_{sym} = \langle G \cap \mathbb{C}[a, b] \rangle$.

For the cubic system:

$$\begin{aligned} \dot{x} &= x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3) \\ \dot{y} &= -y(1 - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2)\,. \end{aligned} \tag{35}$$

Computing a Gröbner basis of the ideal

$$\mathcal{J} = \langle a_{11} - t_1\, t_2\, y_1, b_{11} - t_1\, t_2\, y_1, a_{20} - t_1^2\, y_2, b_{02} - t_2^2\, y_2, a_{02} - t_2^2\, y_3, b_{20} - t_1^2\, y_3,$$
$$a_{-13} - \frac{t_2^3\, y_4}{t_1}, b_{3,-1} - \frac{t_1^3\, y_4}{t_2}, a_{22} - t_1^2\, t_2^2\, y_5, b_{22} - t_1^2\, t_2^2\, y_5\rangle$$

with respect to lexicographic order with

$$t_1 > t_2 > y_1 > y_2 > y_3 > y_4 > y_5$$
$$> a_{11} > b_{11} > a_{20} > b_{20} > a_{02} > b_{02} > a_{-13} > b_{3,-1}$$

we obtain a list of polynomials. According to step 2 of the algorithm above, in order to get a basis of I_{sym} we just have to pick up the polynomials that do not depend on $t_1, t_2, y_1, y_2, y_3, y_4, y_5$.

```
am13 b20^2 - a02^2 b3m1, - a11 + b11, a20 am13 b20 - a02 b02 b3m1,
- a02 a20 + b02 b20, a20^2 am13 - b02^2 b3m1, b02 b3m1^2 y3^2 - a20 b20^2 y4^2,
- am13 b3m1 y3^2 + a02 b20 y4^2, - am13^2 b20 y3^2 + a02^3 y4^2, a02 b3m1^2 y3^2 - b20^3 y4^2,
a20 am13^2 y3^2 - a02^2 b02 y4^2, b02 b3m1 y2 - a20 b11 y4, am13 b20 y2 - a02 b11 y4,
a02 b3m1 y2 - b11 b20 y4, a20 am13 y2 - b02 b11 y4, - b11 b3m1 y3^2 + b20^2 y2 y4,
a20 am13 b11 y3^2 - a02 b02 b20 y2 y4, am13 b11 y3^2 - a02^2 y2 y4, - b02 b20^2 y2^2 + a20 b11^2 y3^2,
am13 b3m1 y2^2 - b11^2 y4^2, - a02 b20 y2^2 + b11^2 y3^2, - b20 y1 + a20 y3,
- a02 y1 + b02 y3, b02 b20 y2^2 - b11^2 y1 y3, - b11 b3m1 y1 y3 + a20 b20 y2 y4,
b02 b3m1^2 y1 y3 - a20^2 b20 y4^2, am13 b11 y1 y3 - a02 b02 y2 y4, am13 b3m1 y1 y3 - b02 b20 y4^2,
- a20 am13^2 y1 y3 + a02 b02^2 y4^2, b11^2 y1^2 - a20 b02 y2^2, - b11 b3m1 y1^2 + a20^2 y2 y4,
b02 b3m1^2 y1^2 - a20^3 y4^2, am13 b11 y1^2 - b02^2 y2 y4, - am13 b3m1 y1^2 + a20 b02 y4^2,
- a20 am13^2 y1^2 + b02^3 y4^2, - a20 am13^2 y1 + b02^2 t2^2 y4^2, - am13 b3m1 y3 + b20 t2^2 y4^2,
a20 am13^2 y3 - a02 b02 t2^2 y4^2, - am13^2 b20 y3 + a02^2 t2^2 y4^2, - am13 b3m1 y1 + a20 t2^2 y4^2,
am13^2 b3m1 y1 y2 - b02 b11 t2^2 y4^3, am13^2 b3m1 y2 y3 - a02 b11 t2^2 y4^3, a02 - t2^2 y3,
am13 b11 y1 - b02 t2^2 y2 y4, am13 b11 y3 - a02 t2^2 y2 y4, - b02 b11 b3m1 y3 + a20 b20 t2^2 y2 y4,
b02 b11 b3m1 y1 - a20^2 t2^2 y2 y4, - b20 t2^2 y2^2 + b11^2 y3, b11^2 y1 - a20 t2^2 y2^2, b02 - t2^2 y1,
a20 am13^2 - b02 t2^4 y4^2, am13^2 b20 - a02 t2^4 y4^2, - am13^2 b3m1 y2 + b11 t2^4 y4^3,
am13 b11 - t2^4 y2 y4, - b3m1 t2^4 y2^3 + b11^3 y4, - am13^3 b3m1 + t2^8 y4^4, b02/t2 - t2 y1,
```

CYCLICITY

The cyclicity of (2):

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p)$$

is the maximal number of limit cycles which appear from the origin after small perturbations.

Theorem 7. *The cyclicity of*

$$i\frac{dx}{dt} = x - a_{10}x^2 - a_{01}x\bar{x} - a_{-12}\bar{x}^2.$$

is 2 (3 if we take into account the perturbation of the linear part).

Bautin, N. *Mat. Sb.* **30** (1952) 181–196.
Żołądek, H. *J. Differential Equations* **109** (1994) 223–273.
Yakovenko, S. A geometric proof of Bautin theorem. Concerning the Hilbert Sixteenth Problem. Advances in Mathematical Sciences, Vol. 23; *Amer. Math. Soc. Transl.* **165** (1995) 203–219.

Theorem 8. *The cyclicity of*

$$i\frac{dx}{dt} = x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3.$$

is 4 (5 if we take into account the perturbation of the linear part).
Lemma 2. *The ideal of focus quantities of system (14),*
$\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle \subset \mathbb{Q}[a_{20}, a_{11}, \ldots, b_{02}]$ *is a radical ideal.*

Proof. According to Theorem 4 $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_5)$. Therefore it is sufficient to show that $\mathcal{B}_5$ is radical. Computing the intersection of the ideals J_k we find

$$\mathcal{B}_5 = J_1 \cap J_2 \cap J_3.$$

Hence $\mathcal{B}_5$ is radical because, obviously, J_1, J_2 are prime (they admit rational parametrizations), and J_3 is prime because the ideal produced by the Algorithm for Finding Time-Reversible Systems is always prime. □

The proof of Theorem 8 follows from
Proposition 1. *The Bautin ideal of system (14) is generated by the first five focus quantities.*

Proof. Let $\mathcal{B}_5 := \langle g_{11}, \ldots, g_{55} \rangle$. We need to show that $\mathcal{B} = \mathcal{B}_5$. It follows from the facts that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_5)$ and the ideal $\mathcal{B}_5$ is radical. □.

Consider the cyclicity problem for the system

$$i\dot{x} = x - a_{10}x^2 - a_{01}x\bar{x} - a_{-13}\bar{x}^3 \tag{36}$$

(Jarrah, Laubenbacher and R., to appear).

We study along with (36) the more general system

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-13}y^3, \\ \dot{y} &= -(y - b_{01}y^2 - b_{10}xy - b_{3,-1}x^3). \end{aligned} \tag{37}$$

Theorem 9. *The center variety of system (37) consists of the following irreducible components:*
1) $a_{10} = a_{-13} = b_{10} = 3a_{01} - b_{01} = 0$,
2) $b_{01} = b_{3,-1} = a_{01} = 3b_{10} - a_{10} = 0$,
3) $a_{10} = a_{-13} = b_{10} = 3a_{01} + b_{01} = 0$,
4) $b_{01} = b_{3,-1} = a_{01} = 3b_{10} + a_{10} = 0$,
5) $a_{01} = a_{-13} = b_{10} = 0$,
6) $a_{01} = b_{3,-1} = b_{10} = 0$,
7) $a_{01} - 2b_{01} = b_{10} - 2a_{10} = 0$,
8) $a_{10}a_{01} - b_{01}b_{10} = a_{01}^4 b_{3,-1} - b_{10}^4 a_{-13} = a_{10}^4 a_{-13} - b_{01}^4 b_{3,-1} = a_{10}a_{-13}b_{10}^3 - a_{01}^3 b_{01} b_{3,-1} = a_{10}^2 a_{-13} b_{10}^2 - a_{01}^2 b_{01}^2 b_{3,-1} = a_{10}^3 a_{-13} b_{10} - a_{01} b_{01}^3 b_{3,-1} = 0$.

The first nine focus quantities:

$$
\begin{aligned}
g_{11} &= a_{10}a_{01} - b_{01}b_{10};\\
g_{22} &= 0;\\
g_{33} &= -(2a_{10}^3a_{-13}b_{10} - a_{10}^2a_{-13}b_{10}^2 - 18a_{10}a_{-13}b_{10}^3 - 9a_{01}^4b_{3,-1} +\\
&\quad 18a_{01}^3b_{01}b_{3,-1} + a_{01}^2b_{01}^2b_{3,-1} - 2a_{01}b_{01}^3b_{3,-1} + 9a_{-13}b_{10}^4)/8;\\
g_{44} &= -(14a_{10}b_{01}(2a_{10}a_{-13}b_{10}^3 + a_{01}^4b_{3,-1} - 2a_{01}^3b_{01}b_{3,-1} - a_{-13}b_{10}^4))/27;\\
g_{55} &= (a_{-13}b_{3,-1}(378a_{10}^4a_{-13} + 5771a_{10}^3a_{-13}b_{10} - 25462a_{10}^2a_{-13}b_{10}^2\\
&\quad +11241a_{10}a_{-13}b_{10}^3 - 11241a_{01}^3b_{01}b_{3,-1} + 25462a_{01}^2b_{01}^2b_{3,-1} -\\
&\quad 5771a_{01}b_{01}^3b_{3,-1} - 378b_{01}^4b_{3,-1}))/3240;\\
g_{66} &= 0;\\
g_{77} &= -(a_{-13}^2b_{3,-1}^2(343834a_{10}^2a_{-13}b_{10}^2 - 1184919a_{10}a_{-13}b_{10}^3 + 506501a_{-13}b_{10}^4 -\\
&\quad 506501{a_{01}}^4b_{3,-1} + 1184919a_{01}^3b_{01}b_{3,-1} - 343834a_{01}^2b_{01}^2b_{3,-1}));\\
g_{88} &= 0;\\
g_{99} &= -a_{-13}^3{b_{3,-1}}^3\left(2a_{10}a_{-13}{b_{10}}^3 - a_{-13}{b_{10}}^4 + {a_{01}}^4b_{3,-1} - 2{a_{01}}^3b_{01}b_{3,-1}\right).
\end{aligned}
$$

Proposition 2. *The ideal $I_5 = \langle g_{11}, g_{33}, g_{44}, g_{55}\rangle$ generated by the first five focus quantities of system (37) is not radical in $\mathbb{C}[a_{10}, a_{01}, a_{-13}, b_{3,-1}, b_{10}, b_{01}]$.*

Let us introduce new variables setting

$$a_{10} = s_1 b_{10}, \quad b_{01} = s_2 a_{01}. \tag{38}$$

By $\tilde{g}_{kk}$ we denote the focus quantities obtained from g_{kk} after the substitution (38).

Proposition 3. *The polynomials $\tilde{g}_{11}, \tilde{g}_{33}, \tilde{g}_{44}, \tilde{g}_{55}, \tilde{g}_{77}, \tilde{g}_{99}$ form the basis of the ideal of focus quantities of the system (37) in the ring $\mathbb{C}[s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10}]$.*

Proof. Denote by $[\nu]$ the monomial

$$a_{10}^{\nu_1} a_{01}^{\nu_2} a_{-13}^{\nu_3} b_{3,-1}^{\nu_4} b_{10}^{\nu_5} b_{01}^{\nu_6}$$

(where $\nu = (\nu_1, .., \nu_6)$) and by $\bar{\nu}$ the vector $(\nu_6, \nu_5, \dots, \nu_2, \nu_1)$. Focus quantities are polynomials of the ring $\mathbb{Q}[a_{10}, b_{01}, a_{01}, b_{10}, a_{-13}, b_{3,-1}]$ and have the form

$$g_{kk} = \sum_j \alpha_j([\nu^{(j)}] - [\bar{\nu}^{(j)}]) = \sum_j \alpha_j IM[\nu^{(j)}],$$

where $\alpha_j \in \mathbb{Q}$, $\nu^{(j)}$ are the solutions of the equation

$$L(\nu) = \binom{1}{0}\nu_1 + \binom{0}{1}\nu_2 + \binom{-1}{3}\nu_3 + \binom{3}{-1}\nu_4 + \binom{1}{0}\nu_5 + \binom{0}{1}\nu_6 = \binom{k}{k} \tag{39}$$

and we use the notation

$$IM[\nu] = [\nu] - [\bar{\nu}], \ RE[\nu] = [\nu] + [\bar{\nu}].$$

Denote by M the monoid of all solutions of the equations (39), where k runs through all $\mathbb{N}$. The Algorithm for Time-reversible systems produces the Hilbert basis of the monoid M: $\{(100\ 001), (110\ 000),\ (000\ 011),\ (010\ 010),\ (001\ 100),\ (040\ 100),$ $(001\ 040), (401\ 000), (000\ 104),\ (101\ 030),\ (030\ 101),\ (201\ 020),\ (020\ 102),$ $(301\ 010), (010\ 103)\}$.

Therefore the focus quantities in the ring $\mathbb{Q}[s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10}]$ have the form

$$\tilde{g}_{ii} = \sum_{\mu : L(\mu)=(i,i)^T} (f_\mu[\mu] - \bar{f}_\mu[\bar{\mu}]),$$

where $f_\mu \in \mathbb{Q}[s_1, s_2]$, $\mu \in \tilde{M}$ and $\tilde{M}$ is the monoid of solutions of the equation

$$L(\nu) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nu_1 + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \nu_2 + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \nu_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_4 = \begin{pmatrix} k \\ k \end{pmatrix}$$

$(k = 0, 1, 2, \dots)$. We denote by $\tilde{I}$ the ideal of focus quantities in the ring $\mathbb{C}[s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10}]$, by $\tilde{I}_k$ the ideal generated by the first k quantities in this ring, and by $^-$ the involution

$$^- : \mathbb{C}[s_1, s_2][\tilde{M}] \mapsto \mathbb{C}[s_1, s_2][\tilde{M}]$$

(where $\mathbb{C}[s_1, s_2][\tilde{M}]$ is the monoid ring of the monoid $\tilde{M}$ over $\mathbb{C}[s_1, s_2]$) defined by the formula

$$\bar{a}_{kj} = b_{jk}, \quad \bar{s}_1 = s_2.$$

For example, if $f = s_1^u s_2^m a_{01}^5 b_{3,-1} b_{10}$ then $\bar{f} = s_1^m s_2^u b_{10}^5 a_{-13} a_{01}$.

Using the obvious equality

$$IM[f(\nu + \mu)] = \frac{1}{2} IM[f\nu] RE[\mu] + \frac{1}{2} IM[\mu] RE[f\nu] \tag{40}$$

where $f \in \mathbb{Q}[s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10}], \nu, \mu \in \tilde{M}$ we obtain

$$\tilde{g}_{ii} \equiv h^{(i)}(s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10})[001\ 040] - \bar{h}^{(i)}(s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10})[040\ 010] \mod \langle \tilde{g}_{11} \rangle.$$

It follows from the structure of the monoid M that $h^{(i)}, \bar{h}^{(i)}$ are polynomials of $s_1, s_2, z, v, w, \bar{w}$, where $v = a_{01} b_{10}, z = a_{-13} b_{3,-1}, w = a_{-13} b_{10}^4, \bar{w} = b_{3,-1} a_{01}^4$.

When $s_1 = s_2 = 1/2$ the system (37) has a center at the origin, therefore

$$\tilde{g}_{ii} \equiv ((2s_1 - 1)v_1^{(i)}w - (2s_2 - 1)\bar{v}_1^{(i)}\bar{w}) + ((2s_2 - 1)v_2^{(i)}w - (2s_1 - 1)\bar{v}_2^{(i)}\bar{w}) \mod \langle \tilde{g}_{11} \rangle,$$

where $v_{1,2}^{(i)} \in \mathbb{Q}[s_1, s_2, v, z, w, \bar{w}]$.

It is easy to see that we can write $\tilde{g}_{ii}$ in the form $\tilde{g}_{ii} = \tilde{g}_{ii}^{(1)} + \tilde{g}_{ii}^{(2)} + \tilde{g}_{ii}^{(3)}$, where $\tilde{g}_{ii}^{(1)}$ is a sum with rational coefficients polynomials of the form

$$f_1 = v^c((2s_1 - 1)\alpha_i w - (2s_2 - 1)\bar{\alpha}_i \bar{w}) + v^c((2s_2 - 1)\beta_i w - (2s_1 - 1)\bar{\beta}_i \bar{w}),$$

where $\alpha_i, \beta_i \in \mathbb{Q}[s_1, s_2, w, z, v]$, $c \in \mathbb{N}, c > 0$, $\tilde{g}_{ii}^{(2)}$ is a sum of polynomials

$$f_2 = z^c((2s_1 - 1)\gamma_i w - (2s_2 - 1)\bar{\gamma}_i \bar{w})$$

where $\gamma_i, \in \mathbb{Q}[s_1, s_2, z, w]$, $c \in \mathbb{N}, c > 0$, and $\tilde{g}_{ii}^{(3)}$ is a sum of polynomials of the form

$$f_3 = ((2s_1 - 1)\theta_i w - (2s_2 - 1)\bar{\theta}_i \bar{w})$$

where $\theta \in \mathbb{Q}[s_1, s_2, w]$ (i.e. $\tilde{g}_{ii}^{(1)}$ is the sum of all terms of $\tilde{g}_{ii}$ containing the factor v, $\tilde{g}_{ii}^{(2)}$ is the sum of remaining terms of $\tilde{g}_{ii}$ containing the factor z, and $\tilde{g}_{ii}^{(3)}$ are all the rest terms).

We will show that

$$\tilde{f}_1 \equiv 0 \bmod \tilde{I}_5,\ \ \tilde{f}_2 \equiv 0 \bmod \tilde{I}_9,\ \ \tilde{f}_3 \equiv 0 \bmod \tilde{I}_5. \tag{41}$$

First we prove that

$$v(s_1^u s_2^m (2s_1 - 1)w^k - s_1^m s_2^u (2s_2 - 1)\bar{w}^k) \in \tilde{I}_5 \tag{42}$$

and

$$v(s_1^u s_2^m (2s_2 - 1)w^k - s_1^m s_2^u (2s_1 - 1)\bar{w}^k) \in \tilde{I}_5 \tag{43}$$

for all $k, u, m \in \mathbb{N}$. Indeed, computing the reduced Groebner basis of $\tilde{I}_5$ using lex with $s_1 > s_2 > a_{01} > b_{10} > a_{-13} > b_{3,-1}$ we see that it contains the polynomials

$$u_1 = v(s_1 - s_2),\ \ u_2 = v(2s_2 - 1)(w - \bar{w}),$$

$$u_3 = -a_{01} z (2s_2 - 1)\left(w - \bar{w}\right),$$
$$u_4 = -b_{10} z\left((2s_1 - 1)w - (2s_2 - 1))\bar{w}\right),$$
$$u_5 = a_{01} w (2s_2 - 1)(s_2 - 3)(s_2 + 3)$$

and

$$u_6 = ((2s_1 - 1)(s_1 - 3)(s_1 + 3)w - (2s_2 - 1)(s_2 - 3)(s_2 + 3)\bar{w}).$$

It is easily checked that

$$v((2s_1-1)w^k-(2s_2-1)\bar{w}^k)-2u_1w^k=v(2s_2-1)(w^k-\bar{w}^k)\equiv 0 \bmod \langle u_2\rangle,$$

$$v(s_1(2s_1-1)w^k-s_2(2s_2-1)\bar{w}^k)-(2s_1+2s_2-1)u_1w^k= \\ vs_2(2s_2-1)(w^k-\bar{w}^k)\equiv 0 \bmod \langle u_2\rangle,$$

$$v(s_2(2s_1-1)w^k-s_1(2s_2-1)\bar{w}^k)-u_1(2s_2(w^k-\bar{w}^k)+\bar{w}^k)= \\ vs_2(2s_2-1)(w^k-\bar{w}^k)\equiv 0 \bmod \langle u_2\rangle,$$

i.e for $\gamma=0,1,\ i=1,2$ the polynomials

$$(s_i^\gamma(2s_1-1)w^k-\bar{s}_i^\gamma(2s_2-1)\bar{w}^k)$$

are in the ideal $\tilde{I}_5$. Taking into account that

$$(s_i^\gamma(2s_1-1)w^k-\bar{s}_i^\gamma(2s_2-1)\bar{w}^k)=$$

$(s_i^{\gamma-1}(2s_1-1)w^k-\bar{s}_i^{\gamma-1}(2s_2-1)\bar{w}^k)(s_i+\bar{s}_i)-s_i\bar{s}_i(s_i^{\gamma-2}(2s_1-1)w^k-\bar{s}_i^{\gamma-2}(2s_2-1)\bar{w}^k)$
and using the induction on γ we conclude that (42) holds. Similarly one can verify (43). Therefore $f_1 \in \tilde{I}_5$.

We now show that $f_2 \in \tilde{I}_9$.

Without loss of generality f_2 is of the form

$$d_k(c) = z^c(s_1^u(2s_1 - 1)w^k - s_2^u(2s_2 - 1)\bar{w}^k)$$

with $k > 1$, or of the form

$$d_1(c) = z^c(s_1^u(2s_1 - 1)w - s_2^u(2s_2 - 1)\bar{w}).$$

First we prove that

$$d_k(c) \equiv 0 \bmod \tilde{I}_5. \tag{44}$$

It is sufficient to consider the case $c = 1$. We show using the induction on k that for $k > 1$

$$d_k(1) \equiv 0 \bmod \tilde{I}_5$$

and

$$d_k^+(1) = z(w - \bar{w})(s_1^u(2s_1 - 1)w^k + s_2^u(2s_2 - 1)\bar{w}^k) \equiv 0 \bmod \tilde{I}_5. \tag{45}$$

For $k = 2$ we have

$$d_2(1) + u_3 b_{3,-1} a_{01}^3 s_2^u + u_4 a_{-13} b_{10}^3 s_1^u = (2s_2 - 1))(w\bar{w})^2 (s_1^u - s_2^u) \in \langle u_1 \rangle.$$

Also

$$d_2^+(1) + u_3 b_{3,-1} a_{01}^3 (s_1^u w + s_2^u \bar{w}) + u_4 a_{-13} b_{10}^3 s_1^u (w - \bar{w}) = 0.$$

Let us assume that for $2 \le k < K$ the statement holds. Then for $k = K$ using (40) we have

$$zIM[s_1^u(2s_1-1)w^K] = \frac{1}{2} IM[s_1^u(2s_1-1)w^{K-1}] RE[w] + \frac{1}{2} RE[s_1^u(2s_1-1)w^{K-1}] IM[w].$$

Due to the induction hypothesis the both summands in the right-hand side are in $\tilde{I}_5$. Therefore (44) holds with $k = K$. The correctness of (45) follows from the formula

$$z(w - \bar{w})(s_1^u(2s_1 - 1)w^j + s_2^u(2s_2 - 1)\bar{w}^j) = \\ - u_3 b_{3,-1} a_{01}^3 (s_1^u w^{j-1} + s_2^u \bar{w}^{j-1}) + u_4 a_{-13} b_{10}^3 s_1^u w^{j-2} (w - \bar{w}).$$

Consider now the second case, namely the polynomial

$$d_1(c) = z^c (s_1^u(2s_1 - 1)w - s_2^u(2s_2 - 1)\bar{w}).$$

In fact here u can be equal only 0,1,2 or 3. Reducing $d_1(3)$ modulo a Groebner basis of $\tilde{I}_9$ we see that all these polynomials are in $\tilde{I}_9$, therefore $d_1(c) \in \tilde{I}_9$ for $c > 2$. If $c \leq 2$ then the degree of $d_1(c)$ is less or equal 15, but the degree of the polynomials of our interest starts from 20 (namely, the first polynomial under the consideration is $g_{10,10}$).

Similarly, it is possible to show that $f_3 \in \tilde{I}_5$. Hence $\tilde{g}_{ii} \in \tilde{I}_9$ for $i > 9$. $\square$

Because when $a_{01} = 0$, $a_{10}^4 a_{-13} - \overline{a_{10}^4 a_{-13}} \neq 0$ the system (36) has a focus at the origin and when $|a_{01}|| \neq 0$ the substitution (38) is invertible we conclude that Proposition 3 yields the following statement.

Proposition 4. *The cyclicity of the origin of the system (36) with $a_{01} \neq 0$ or $a_{01} = 0$, $a_{10}^4 a_{-13}$ is less or equal 5.*

If instead of the substitution (38) we use $a_{01} = s_1 b_{01}, \quad b_{10} = s_2 a_{01}$ then using similar reasoning one can prove the analog of Proposition 4. Thus, the following statement holds.

Theorem 10. *The cyclicity of the origin of the system (36) with $|a_{10}| + |a_{01}| \neq 0$ is less or equal 5.*