THE CENTER PROBLEM AND LOCAL LIMIT CYCLES BIFURCATIONS IN POLYNOMIAL SYSTEMS

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Consider the system

\[
\frac{du}{dt} = \alpha u - \beta v + \sum_{i+j=2}^n \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = \beta u + \alpha v + \sum_{i+j=2}^n \beta_{ij} u^i v^j \tag{1}
\]

Phase space

We assume that \(\alpha = 0, \beta = 1\). Then the origin is a week (fine) focus or a center.
Poincare (return) map:

\[ P(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \ldots. \]

Limit cycles \(\longleftrightarrow\) isolated fixed points of \(P(\rho)\). The center variety:

\[ V = \{ (\alpha_{ij}, \beta_{ij}) \in E \mid \eta_3(\alpha_{ij}, \beta_{ij}) = \eta_4(\alpha_{ij}, \beta_{ij}) = \cdots = 0 \} \]

Let \(B = \langle \eta_3, \eta_5, \ldots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]\) be the ideal generated by the focus quantities \(\eta_i\). \(B\) is called the Bautin ideal of system (1). There is \(k\) such that

\[ B = \langle \eta_3, \eta_5, \ldots, \eta_{2k+1} \rangle. \]

Then

\[ P(\rho) - \rho = \eta_3(1 + \ldots)\rho^3 + \ldots \eta_{2k+1}(1 + \ldots)\rho^{2k+1}. \]

**Theorem 1 (Bautin).** If \(B = \langle \eta_3, \eta_5, \ldots, \eta_{2k+1} \rangle\) then the cyclicity of system (1) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is equal to \(k\).

The Center Problem:
Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B}$.

The Cyclicity Problem (Local Hilbert's 16th Problem):
Find a basis for the Bautin ideal $\mathcal{B}$

**Theorem 2 (Strong Hilbert Nullstellensatz)** Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ and let $I$ be an ideal of $\mathbb{C}[x_1, \ldots, x_m]$. Then $f$ vanishes on the variety of $I$ if and only if for some positive integer $\ell$ $f^\ell \in I$ ($f \in \sqrt{I}$).

Complexification: $x = u + iv$ \quad ($\bar{x} = u - iv$)

$$
\begin{align*}
\dot{x} &= i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} x^q) \\
\dot{\bar{x}} &= -i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)
\end{align*}
$$
\[
\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p) \tag{1}
\]

If \( b_{qp} = \bar{a}_{pq} \), \( y = \bar{x} \) then from (3) we obtain the "real" system.

The change of time \( d\tau = idt \) transforms (1) to the system

\[
\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p) \tag{2}
\]

where \( x, y, a_{pq}, b_{qp} \) are complex variables, \( S = \{(m, k) | m + k \geq 1\} \) is a subset of \( \{-1 \cup \mathbb{N}\} \times \mathbb{N} \), \( \mathbb{N} \) is the set of non-negative integers. Let \( l \) be the number of the elements in the set \( S \). We denote by \( E(a, b) (= \mathbb{C}^{2l}) \) the parameter space of (2), and by \( \mathbb{C}[a, b] \) \( (\mathbb{Q}[a, b]) \) the polynomial ring in the variables \( a_{pq}, b_{qp} \) over the field \( \mathbb{C} \) (over \( \mathbb{Q} \)).
What is a center for system (2)???

**Theorem 2 (Poincaré-Lyapunov).** The system

\[
\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij}u^iv^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij}u^iv^j
\]

has a center at the origin if and only if it admits a first integral of the form

\[
\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl}u^kv^l.
\]

Consider polynomial systems of the form

\[
\frac{dx}{dt} = x + F(x, y) = P(x, y), \quad \frac{dy}{dt} = -y + G(x, y) = Q(x, y),
\]

where \(F(x, y), G(x, y) \in \mathbb{C}[x, y]\) without constant and linear terms.

**Definition 1. (Dulac).** System (5) has a center at the origin if there is an analytic first integral of the form

\[
\Psi(x, y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j}x^jy^{s-j}, \quad (3)
\]
(First integral: $\frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = 0$)

For system (2) one can always find a function $\Psi$ of the form (3) such that

$$D(\Psi) := \frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots,$$

where the $g_{ii}$ are polynomials of $\mathbb{C}[a, b]$ called focus quantities. Thus system (2) with the fixed parameters $(a^*, b^*)$ has a center at the origin if and only if $g_{ii}(a^*, b^*) = 0$ for all $i = 1, 2, \ldots$, i.e. if and only if

$$(a^*, b^*) \in V(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle).$$

$V(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle) = V(\mathcal{B})$ is the the center variety.
CALCULATION OF FOCUS QUANTITIES

\[
\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^p y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p)
\]

We assume that \( S = \{\tilde{i}_1, \ldots, \tilde{i}_l\} = \{(p_1, q_1), (p_2, q_2), \ldots, (q_1, p_1)\} \) (\( \tilde{i}_s = (p_s, q_s) \)) is the ordered set of the indices of the coefficients of the first equation of system (2) and consider the map \( L : \mathbb{N}^{2l} \rightarrow \mathbb{N}^2 \) (recall that \( l \) is the number of elements in the set \( S \)), defined by

\[
L(\nu) = \begin{pmatrix} L_1(\nu) \\ L_2(\nu) \end{pmatrix} = \nu_1 \tilde{i}_1 + \nu_2 \tilde{i}_2 + \cdots + \nu_{l-1} \tilde{i}_{l-1} + \nu_l \tilde{i}_l + \nu_{l+1} \tilde{j}_l + \nu_{l+2} \tilde{j}_{l-1} + \cdots + \nu_{2l-1} \tilde{j}_2 + \nu_{2l}
\]

where \( \tilde{j}_s \) corresponds to \( \tilde{i}_s \), such that if \( \tilde{j}_s = (p_s, q_s) \), then \( \tilde{i}_s = (q_s, p_s) \).

Denote by \([\nu]\) the monomial

\[
[\nu] = a^\nu_{\tilde{i}_1} a^\nu_{\tilde{i}_2} \cdots a^\nu_{\tilde{i}_l} b^\nu_{\tilde{j}_l} b^\nu_{\tilde{j}_{l-1}} \cdots b^\nu_{\tilde{j}_1}
\]

and by \( \overline{\nu} \) the involution of the vector \( \nu \):

\[
\overline{\nu} = (\nu_{2l}, \nu_{2l-1}, \ldots, \nu_2, \nu_1).
\]
Consider the formal series

\[ V = \sum V(\nu_1, \nu_2, \cdots, \nu_{2l}) a_{\nu_1} \cdots a_{\nu_{2l}} b_{\nu_{l+1}} b_{\nu_{l+2}} \cdots b_{\nu_{2l}}, \tag{6} \]

where \( V(\nu_1, \cdots, \nu_{2l}) \) are determined by the recurrence formula:

\[
V(\nu_1, \nu_2, \cdots, \nu_{2l}) = \frac{1}{L^1(\nu) - L^2(\nu)} \left( \sum_{i=1}^{l} V(\nu_1, \cdots, \nu_{i-1}, \nu_{2l})(L^1(\nu_1, \cdots, \nu_{i-1}, \cdots, \nu_{2l} + 1) \right. \\
\left. + 1) - \sum_{i=l+1}^{2l} V(\nu_1, \cdots, \nu_{i-1}, \nu_{2l})(L^2(\nu_1, \cdots, \nu_{i-1}, \cdots, \nu_{2l}) + 1)) \tag{7} \right)
\]

if \( L^1(\nu) \neq L^2(\nu), V(\nu_1, \cdots, \nu_{2l}) = 0, \) if \( L^1(\nu) = L^2(\nu); V(0, \cdots, 0) = 1 \) and we put \( V(\nu_1, \cdots, \nu_{2l}) = 0 \) for all \( \nu = (\nu_1, \cdots, \nu_{2l}), \) such that there exists \( i : \nu_i < 0. \)

Looking for a first integral

\[ \Psi(x, y) = xy + \sum_{j+k \geq 3} v_{j-1, k-1}(a, b)x^j y^k, \]

we have the equation:

\[ D(\Psi) := \frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots, \]
Theorem 3. 1) The coefficient of \([\nu]\) in the polynomial \(v_{kn}\) is equal to \(V(\nu_1, \nu_2, \ldots, \nu_{2l})\).

2) The \(i\)-th focus quantity of the system \((2)\) is

\[
g_{ii} = \sum_{\nu: L(\nu) = \binom{i}{i}} g(\nu_1, \nu_2, \ldots, \nu_{2l}) a_{i1}^{\nu_1} a_{i2}^{\nu_2} \cdots a_{il}^{\nu_l} b_{j1}^{\nu_{l+1}} b_{j2}^{\nu_{l+2}} \cdots b_{j1}^{\nu_{2l}},
\]

where

\[
g(\nu_1, \nu_2, \ldots, \nu_{2l}) = \sum_{i=1}^{l} V(\nu_1, \ldots, \nu_{i-1}, \ldots, \nu_{2l}) (L^1(\nu_1, \ldots, \nu_{i-1}, \ldots, \nu_{2l}) + 1)
\]

\[
- \sum_{i=l+1}^{2l} V(\nu_1, \ldots, \nu_{i-1}, \ldots, \nu_{2l}) (L^2(\nu_1, \ldots, \nu_{i-1}, \ldots, \nu_{2l}) + 1)
\]

and \(V(\nu)\) are defined by \((7)\).

3) \(g(\nu) = -g(\bar{\nu})\) if \(\nu \neq \bar{\nu}\).

The equation \((7)\) is the so-called difference equation. It is often possible to pass from a given difference equation to a differential equation, and vice versa.

For general polynomial system \((2)\) we obtain the differential equation

\[
\mathcal{A}(V) = (|a| - |b|)V,
\]
where \( |a| = \sum_{(i,j) \in S} a_{ij}, \ |b| = \sum_{(j,i) \in S} b_{ij} \) and

\[
\mathcal{A}(V) = \sum_{(i,j) \in S} \frac{\partial V}{\partial a_{ij}} a_{ij}(i - j - i|a| + j|b|) + \sum_{(j,i) \in S} \frac{\partial V}{\partial b_{ij}} b_{ij}(i - j - i|a| + j|b|)
\]  

(10)
is the linear operator

\[
\mathcal{A} : \mathbb{C}[[a, b]] \longrightarrow \mathbb{C}[[a, b]]
\]

(recall that \( k[[x]] \) denotes the ring of formal power series of \( x \) over \( k \)).

Let the map

\[
\pi : \mathbb{C}[a, b][[x, y]] \longrightarrow \mathbb{C}[[a, b]]
\]

be defined by

\[
\pi \left( \sum c_{\alpha, \beta}(a, b) x^\alpha y^\beta \right) = \sum c_{\alpha, \beta}(a, b).
\]  

(11)

**Theorem 4.** The system (2) has a center at the origin for all values of the parameters \( a_{kn}, b_{nk} \) (that is for all \( (a, b) \in E(a, b) \)) if and only if there is a formal series (6) such that \( V(0, \ldots, 0) = 1 \) satisfying the equation

\[
\mathcal{A}(V) = V(|a| - |b|).
\]  

(12)

Thus, the Poincaré center problem is equivalent to the study of formal solutions of PDE (12).
THE POINCARÉ CENTER PROBLEM

System (2): \( \dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq}x^p y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp}x^q y^p) \)

\[ \Psi(x, y) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j} \]

\[ D(\Psi) := \frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = g_{11}(xy)^2 + g_{22}(xy)^3 + \cdots. \]

System (2) has a center at the origin if and only if \( g_{ii}(a^*, b^*) = 0 \) for all \( i = 1, 2, \ldots \), i.e. if and only if

\( (a^*, b^*) \in \mathbf{V}(\langle g_{11}, g_{22}, \ldots, g_{ii}, \ldots \rangle) \),

i.e., to solve the center problem means to find the variety of \( \mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle \).

The difficulty: \( g_{kk} \) are given by recurrence formula.

A way to study the problem:

- Let \( \mathcal{B}_k = \langle g_{11}, g_{22}, \ldots, g_{kk} \rangle \). Compute \( g_{11}, g_{22}, \ldots, g_{ss} \) until \( \mathbf{V}(\mathcal{B}_1) \supset \mathbf{V}(\mathcal{B}_2) \supset \cdots \mathbf{V}(\mathcal{B}_{s-1}) = \mathbf{V}(\mathcal{B}_s) \).

- Find irreducible decomposition of \( \mathbf{V}(\mathcal{B}_{ss}) \): \( \mathbf{V}(\mathcal{B}_{ss}) = V_1 \cup V_2 \cup \ldots V_m \).

- For every \( V_j \) prove existence of a Lyapunov integral.
CUBIC SYSTEM

\[ i \frac{dx}{dt} = x + P_2(x, \bar{x}) + P_3(x, \bar{x}). \]

System with homogeneous cubic nonlinearities (Malkin, 1966):

\[ i \frac{dx}{dt} = x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3. \]  
(13)

The complexification \( y = \bar{x} \) and the change of the time \( dt = id\tau \) yields the system

\[ \dot{x} = x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \]
\[ \dot{y} = -(y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3,-1}x^3). \]  
(14)

Computing the first five focus quantities of \( (14) \) we find:

\[ g_{11} = a_{11} - b_{11}; \]
\[ g_{22} = a_{20}a_{02} - b_{02}b_{20}; \]
\[ g_{33} = (3a_{20}^2a_{-13} + 8a_{20}a_{-13}b_{20} + 3a_{02}^2b_{3,-1} - 8a_{02}b_{02}b_{3,-1} - 3a_{-13}b_{20}^2 - 3b_{02}^2b_{3,-1})/8; \]
\[ g_{44} = (-9a_{20}^2a_{-13}b_{11} + a_{11}a_{-13}b_{20}^2 + 9a_{11}b_{02}^2b_{3,-1} - a_{02}^2b_{11}b_{3,-1})/16; \]
\[ g_{55} = (-9a_{20}^2a_{-13}b_{02}b_{20} + a_{20}a_{02}a_{-13}b_{20}^2 + 9a_{20}a_{02}b_{02}^2b_{3,-1} + 18a_{20}a_{-13}b_{20}b_{3,-1} + 6a_{02}^2a_{-13}b_{3,-1} - a_{02}^2b_{02}b_{20}b_{3,-1} - 18a_{02}a_{-13}b_{02}b_{3,-1}^2 - 6a_{-13}b_{20}^2b_{3,-1})/36. \]
Theorem 5. Let $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$ be the Bautin ideal of system (14). The center variety $\mathbf{V}(\mathcal{B})$ of the system (14) consists of the three irreducible components:

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\langle g_{11}, \ldots, g_{55} \rangle) = \mathbf{V}(C_1) \cup \mathbf{V}(C_2) \cup \mathbf{V}(C_3),$$

where

$C_1 = \langle a_{11} - b_{11}, 3a_{20} - b_{20}, 3b_{02} - a_{02} \rangle$,

$C_2 = \langle a_{11}, b_{11}, a_{20} + 3b_{20}, b_{02} + 3a_{02}, a_{-13}b_{3}, -1 - 4a_{02}b_{20} \rangle$,

$C_3 = \langle a_{20}^{2}a_{-13} - b_{3}, -1^{2}b_{02}, a_{20}a_{02} - b_{20}b_{02}, a_{20}a_{-13}b_{20} - a_{02}b_{3}, -1b_{02}, a_{11} - b_{11}, a_{02}b_{3}, -1 - a_{-13}b_{20}^{2} \rangle$.

Proof. Computing with $\text{minAssChar}$ or $\text{minAssGTZ}$ of $\text{Singular}$ we find that the minimal associate primes of the ideal $\langle g_{11}, g_{22}, \ldots, g_{55} \rangle$ are the ideals $C_1, C_2, C_3$. To prove that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\langle g_{11}, \ldots, g_{55} \rangle)$ it is sufficient to show that systems from $\mathbf{V}(C_1), \mathbf{V}(C_2), \mathbf{V}(C_3)$ admit first integrals.

Q.: what are the most efficient algorithms for decomposition of varieties?

$C_1$ – Hamiltonian systems. System:

$$\dot{x} = x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3, \quad \dot{y} = -(y - b_{02}y^3 - b_{11}xy^2 - b_{20}x^2y - b_{3}, -1x^3).$$

$$H = xy - \frac{b_{3}, -1x^4}{4} - a_{20}x^3y - \frac{b_{11}x^2y^2}{2} - b_{02}xy^3 - \frac{a_{-13}y^4}{4}.$$
Consider the system of differential equations

\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]  

(15)

where \( x, y \in \mathbb{C} \) and \( P \) and \( Q \) are polynomials.

The polynomial \( f(x, y) \in \mathbb{C}[x, y] \) defines an \textit{algebraic invariant curve} \( f(x, y) = 0 \) of the system (15) if and only if there exists a polynomial \( k(x, y) \in \mathbb{C}[x, y] \) such that

\[ D(f) = \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = k f. \]  

(16)

\( k \) is called \textit{cofactor} of \( f \).

Suppose that the curves defined by

\[ f_1 = 0, \ldots, f_s = 0 \]

are invariant algebraic curves of the system (15). A first integral of the system (15) of the form

\[ H = f_1^{\alpha_1} \cdots f_s^{\alpha_s} \]  

(17)

is called a \textit{Darboux integral} of (15).
If \( f_1, \ldots, f_s \) are different irreducible algebraic partial integrals such that
\[ \sum_{j=1}^s \alpha_j k_j = 0 \]
then \( H = f_1^{\alpha_1} \cdots f_s^{\alpha_s} \) is a first integral of (15).

Algebraic invariant curves:
\[
f_1 = 1 + 2b_{20} x^2 + \frac{a_{02} b_{3,-1} 2 x^4}{4b_{20}} - \frac{4b_{20}^2 y}{b_{3,-1}} - \frac{a_{02} b_{3,-1} x y}{b_{20}} - 2a_{02} b_{3,-1} x^3 y + 2a_{02} y^2 + 6a_{02} b_{20} x^2 y^2 - \frac{8a_{02} b_{20}^2 x y^3}{b_{3,-1}} + \frac{4a_{02} b_{20}^3 y^4}{b_{3,-1}^2},
\]

\[
f_2 = 8b_{20}^3 b_{3,-1} + 2a_{02} b_{3,-1}^4 + 24b_{20}^2 b_{3,-1}^2 x^2 + 6a_{02} b_{20} b_{3,-1}^4 x^2 + 12a_{02} b_{20}^2 b_{3,-1}^2 x^4 + a_{02} b_{3,-1}^6 x^6 - 48a_{02} b_{20} b_{3,-1}^3 x y - 72a_{02} b_{20}^3 b_{3,-1}^3 x^3 y - 6a_{02} b_{3,-1}^5 x^3 y - 12a_{02} b_{20}^5 b_{3,-1}^5 x^5 y + 24a_{02} b_{20}^3 b_{3,-1}^2 y^2 + 6a_{02}^2 b_{3,-1}^4 y^2 + 144a_{02} b_{20}^4 b_{3,-1}^2 x^2 y^2 + 36a_{02}^2 b_{20}^4 b_{3,-1}^2 x^2 y^2 - 60a_{02}^2 b_{20}^2 b_{3,-1}^4 x^4 y^2 - 96a_{02} b_{20}^5 b_{3,-1}^4 x y^3 - 72a_{02}^2 b_{20}^2 b_{3,-1}^3 x^3 y^3 - 160a_{02}^2 b_{20}^3 b_{3,-1}^3 x^3 y^3 + 48a_{02}^2 b_{20}^2 b_{3,-1}^2 y^4 + 240a_{02}^2 b_{20}^4 b_{3,-1}^2 x^2 y^4 - 192a_{02}^2 b_{20}^5 b_{3,-1}^2 x y^5 + 64a_{02}^2 b_{20}^6 y^6/(2b_{3,-1}^4 (4b_{20}^3 + a_{02} b_{3,-1}^3))
\]

Cofactors: \( k_1 = 4(b_{20} x^2 - a_{02} y^2) \), \( k_2 = 6(b_{20} x^2 - a_{02} y^2) \)

From the equation \( \alpha_1 k_1 + \alpha_2 k_2 = 0 \) we find \( \alpha_1 = 3, \alpha = -2 \), yielding the first integral \( \Psi = f_1^3 f_2^{-2} \).

The associated PDE:
\[
A(V) := (18)
\]
\[
\frac{\partial V}{\partial a_{02}} a_{02}(-2 + 2(-3a_{02} + b_{20} + b_{3,-1})) + \frac{\partial V}{\partial b_{20}} b_{20}(2 - 2(a_{02} - 3b_{20} + 4a_{02}b_{20}/b_{3,-1}))) + \frac{\partial V}{\partial b_{3,-1}} b_{3,-1}(4 - (8b_{20} - 12a_{02}b_{20}/b_{3,-1}) = (4a_{02} - 4b_{20} + 4a_{02}b_{20}/b_{3,-1} - b_{3,-1})V.
\]

\[\Psi = f_1^3 f_2^{-3}\] is a first integral of our system of ODE, but \(\pi(\Psi)\) is not a solution to (18), because \(\Psi\) is not of the form \(xy + h.o.t.\):

\[\Psi = 1 - \frac{3}{4} \frac{(-4 b_{20}^3 + a_{02} b_{3,-1}^2)^2}{a_{02} b_{20} b_{3,-1}^3} \frac{x y}{4 b_{20}^4 b_{3,-1} + a_{02} b_{20} b_{3,-1}^3} + \frac{3}{4} \frac{(-4 b_{20}^3 + a_{02} b_{3,-1}^2)^2}{a_{02} b_{20} b_{3,-1}^3} \left(16 b_{20}^6 - 28 a_{02} b_{20}^3 b_{3,-1}^2 + a_{02}^2 b_{3,-1}^2 \right) x^2 y^2 \]

Thus,

\[-(\pi(\Psi) - 1) \frac{4 b_{20}^4 b_{3,-1} + a_{02} b_{20} b_{3,-1}^3}{3 (-4 b_{20}^3 + a_{02} b_{3,-1}^2)^2} =

\[-(\pi(f_1)^3 \pi(f_2)^{-2} - 1) \frac{4 b_{20}^4 b_{3,-1} + a_{02} b_{20} b_{3,-1}^3}{3 (-4 b_{20}^3 + a_{02} b_{3,-1}^2)^2}\]

is a (rational) solution to (18).

Q.: Is there any algorithmic method to find solutions to equations like (18)?
$C_3$ – time-reversible systems.

**A General Algorithm for Finding Time-Reversible Systems**


Let

$$\dot{x} = P(x, \bar{x}). \tag{19}$$

be complexification of

$$\dot{u} = v + U(u, v), \quad \dot{v} = -u + V(u, v). \tag{20}$$

A straight line $L$ is an axis of symmetry of (20) if the trajectories of the system are symmetric with respect to the line $L$.

**Lemma 1.** Let $a$ denote the vector of coefficients of the polynomial $P(x, \bar{x})$ in (19), arising from the real system (20) by setting $x = u + iv$. If $a = \pm \bar{a}$ (meaning that either all the coefficients are real or all are pure imaginary), then the $u$–axis is an axis of symmetry of (20).

By the lemma the $u$–axis is an axis of symmetry for (19) if

$$P(\bar{x}, x) = -\overline{P(x, \bar{x})} \tag{21}$$
(the case \( a = -\bar{a} \)), or if
\[
P(\bar{x}, x) = \overline{P(x, \bar{x})}
\] (22)
(the case \( a = \bar{a} \)). If condition (21) is satisfied then under the change
\[
x \to \bar{x}, \quad \bar{x} \to x,
\]
(23)
\[
\dot{x} = P(x, \bar{x}) \quad \text{is transformed to its negative},
\]
\[
\dot{x} = -P(x, \bar{x}),
\]
(24)
and if condition (22) holds then (19) is unchanged. Thus condition (22) means that the system is reversible with respect to reflection across the \( u \)--axis (i.e., the transformation does not change the system) while condition (21) corresponds to time--reversibility with respect to the same transformation.

If the line of reflection is not the \( u \)--axis but a distinct line \( L \) then we can apply the rotation \( x_1 = e^{-i\varphi} x \) through an appropriate angle \( \varphi \) to make \( L \) the \( u \)--axis. (19) is time--reversible when there exists a \( \varphi \) such that
\[
e^{2i\varphi} \overline{P(x, \bar{x})} = -P(e^{2i\varphi} \bar{x}, e^{-2i\varphi} x).
\] (25)
This suggests the following natural generalization of the notion of time--reversibility to the case of two--dimensional complex systems.
Definition 2. Let $z = (x, y) \in \mathbb{C}^2$. We say that the system

$$\frac{dz}{dt} = F(z)$$  \hspace{1cm} (26)

is time–reversible if there is a linear transformation $T$, 

$$x \mapsto \alpha y, \ y \mapsto \alpha^{-1} x$$  \hspace{1cm} (27)

($\alpha \in \mathbb{C}$), such that

$$\frac{d(Tz)}{dt} = -F(Tz)$$  \hspace{1cm} (28)

For a fixed collection $(p_1, q_1), \ldots, (p_\ell, q_\ell)$ of elements of $\{-1\} \cup \mathbb{N}_+ \times \mathbb{N}_+$, and letting $\nu$ denote the element $(\nu_1, \ldots, \nu_{2\ell})$ of $\mathbb{N}_+^{2\ell}$, let $L$ be the map from $\mathbb{N}_+^{2\ell}$ to $\mathbb{N}_+^2$ (the elements of the latter written as column vectors) defined by

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell}. \hspace{1cm} (29)$$
Let $\mathcal{M}$ denote the set of all solutions $\nu = (\nu_1, \nu_2, \ldots, \nu_{2\ell})$ with non-negative components of the equation

$$L(\nu) = \binom{k}{k}$$

as $k$ runs through $\mathbb{N}_+$, and the pairs $(p_i, q_i)$ determining $L(\nu)$ come from system (2). $\mathcal{M}$ is an Abelian monoid. Let $\mathbb{C}[\mathcal{M}]$ denote the subalgebra of $\mathbb{C}[a, b]$ generated by all monomials of the form

$$[\nu] := a_{p_1q_1}^{\nu_1} a_{p_2q_2}^{\nu_2} \cdots a_{p_{\ell}q_{\ell}}^{\nu_{\ell}} b_{q_{\ell-1}p_{\ell-1}}^{\nu_{\ell+1}} b_{q_{\ell}p_{\ell}}^{\nu_{\ell+2}} \cdots b_{q_{1}p_{1}}^{\nu_{2\ell}},$$

for all $\nu \in \mathcal{M}$. For $\nu$ in $\mathcal{M}$, let $\hat{\nu}$ denote the involution of the vector $\nu$:

$$\hat{\nu} = (\nu_{2\ell}, \nu_{2\ell-1}, \ldots, \nu_1).$$

Corollary of Theorem 3: The focus quantities of system (2) belong to $\mathbb{C}[\mathcal{M}]$ and have the form

$$g_{kk} = \sum_{L(\nu) = (k, k)^T} g(\nu) ([\nu] - [\hat{\nu}]),$$

with $g(\nu) \in \mathbb{Q}, \ k = 1, 2, \ldots$.

Consider the ideal

$$I_{\text{sym}} = \langle [\nu] - [\hat{\nu}] \mid \nu \in \mathcal{M} \rangle \subset \mathbb{C}[\mathcal{M}].$$

It is clear that $\mathcal{B} \subseteq I_{\text{sym}}$, hence $\text{V}(I_{\text{sym}}) \subseteq \text{V}(\mathcal{B})$. 
Definition 3. For system (2) the variety $V(I_{sym})$ is called the Sibirsky (or symmetry) subvariety of the center variety, and the ideal $I_{sym}$ is called the Sibirsky ideal.

Every time–reversible real system with the singularity of focus or center type at the origin has a center at the origin. It is easily seen that this property is transferred to complex systems: every time–reversible system (2) has a center at the origin. Indeed, the time-reversibility condition $\alpha Q(\alpha y, x/\alpha) = -P(x, y), \alpha Q(x, y) = -P(\alpha y, x/\alpha)$ yields that system (2) is time–reversible if and only if

$$b_{qp} = \alpha^{p-q}a_{pq}, \quad a_{pq} = b_{qp}\alpha^{q-p}. \quad (32)$$

Hence in the case that (2) is time–reversible, using (32) we see that for $\nu \in M$

$$[\hat{\nu}] = \alpha^{(L^1(\nu)-L^2(\nu))}[\nu] = [\nu] \quad (33)$$

and thus from (31) we obtain $g_{kk} \equiv 0$ for all $k$, which implies that the system has a center.

By (33) every time–reversible system $(a, b) \in E(a, b)$ belongs to $V(I_{sym})$. The converse is false.

$$\dot{x} = x(1 - a_{10}x - a_{01}y), \quad \dot{y} = -y(1 - b_{10}x - b_{01}y).$$
In this case $I_{sym} = \langle a_{10}a_{01} - b_{10}b_{01} \rangle$. The system

$$\dot{x} = x(1 - a_{10}x), \quad \dot{y} = -y(1 - b_{10}x)$$

(34)

arises from $V(I_{sym})$ but (32) are not fulfilled, so (34) is not time–reversible.

**Theorem 6.** Let $\mathcal{R} \subset E(a, b)$ be the set of all time–reversible systems in the family (2). Then:

1. $\mathcal{R} \subset V(I_{sym})$;
2. $V(I_{sym}) \setminus \mathcal{R} = \{ (a, b) | \exists (p, q) \in S$ such that $a_{pq}b_{qp} = 0$ but $a_{pq} + b_{qp} \neq 0 \}$.

The theorem shows that to describe time reversible systems it is sufficient to compute $I_{sym}$.

**Algorithm for Finding Time-Reversible Systems**

**Input:** Two sequences of integers $p_1, \ldots, p_{\ell}$ ($p_i \geq -1$) and $q_1, \ldots, q_{\ell}$ ($q_i \geq 0$). (These are the coefficient labels for system (2):

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq}x^{p+1}y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{pq}x^qy^p).$$

**Output:** A finite set of generators for the Sibirsky ideal $I_{sym}$ of (2).
1. Compute a reduced Gröbner basis $G$ for the ideal

$$
\mathcal{J} = \langle a_{piq_i} - yt_1^{p_i}t_2^{q_i}, b_{qi_p} - y\ell-i+1t_1^{q_{\ell-i+1}}t_2^{p_{\ell-i+1}} \mid i = 1, \ldots, \ell \rangle
$$

$$
\subset \mathbb{C}[a, b, y_1, \ldots, y_\ell, t_1^\pm, t_2^\pm]
$$

with respect to any elimination ordering for which

$$
\{t_1, t_2\} > \{y_1, \ldots, y_\ell\} > \{a_{p_1q_1}, \ldots, b_{q_1p_1}\}.
$$

2. $I_{sym} = \langle G \cap \mathbb{C}[a, b] \rangle$.

For the cubic system:

$$
\dot{x} = x(1 - a_{20}x^2 - a_{11}xy - a_{02}y^2 - a_{-13}x^{-1}y^3)
$$

$$
\dot{y} = -y(1 - b_{3,-1}x^3y^{-1} - b_{20}x^2 - b_{11}xy - b_{02}y^2).
$$

(35)

Computing a Gröbner basis of the ideal

$$
\mathcal{J} = \langle a_{11} - t_1 t_2 y_1, b_{11} - t_1 t_2 y_1, a_{20} - t_1^2 y_2, b_{02} - t_2^2 y_2, a_{02} - t_2^2 y_3, b_{20} - t_1^2 y_3, a_{-13} - \frac{t_2^3 y_4}{t_1}, b_{3,-1} - \frac{t_1^3 y_4}{t_2}, a_{22} - t_1 t_2^2 y_5, b_{22} - t_1^2 t_2^2 y_5 \rangle
$$
with respect to lexicographic order with

\[ t_1 > t_2 > y_1 > y_2 > y_3 > y_4 > y_5 > a_{11} > b_{11} > a_{20} > b_{20} > a_{02} > b_{02} > a_{-13} > b_{3,-1} \]

we obtain a list of polynomials. According to step 2 of the algorithm above, in order to get a basis of \( I_{\text{sym}} \) we just have to pick up the polynomials that do not depend on \( t_1, t_2, y_1, y_2, y_3, y_4, y_5 \).
The cyclicity of \( (2) \):

\[
\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p)
\]

is the maximal number of limit cycles which appear from the origin after small perturbations.

**Theorem 7.** The cyclicity of

\[
i \frac{dx}{dt} = x - a_{10} x^2 - a_{01} x \bar{x} - a_{-12} \bar{x}^2.
\]

is 2 (3 if we take into account the perturbation of the linear part).


Theorem 8. The cyclicity of
\[
\frac{dx}{dt} = x - a_{20}x^3 - a_{11}x^2\bar{x} - a_{02}x\bar{x}^2 - a_{-13}\bar{x}^3.
\]
is 4 (5 if we take into account the perturbation of the linear part).

Lemma 2. The ideal of focus quantities of system (14),
\[B = \langle g_{11}, g_{22}, \ldots \rangle \subset \mathbb{Q}[a_{20}, a_{11}, \ldots, b_{02}]\]
is a radical ideal.

Proof. According to Theorem 4 \(\mathbf{V}(B) = \mathbf{V}(B_5)\). Therefore it is sufficient to show that \(B_5\) is radical. Computing the intersection of the ideals \(J_k\) we find
\[B_5 = J_1 \cap J_2 \cap J_3.\]
Hence \(B_5\) is radical because, obviously, \(J_1, J_2\) are prime (they admit rational parametrizations), and \(J_3\) is prime because the ideal produced by the Algorithm for Finding Time-Reversible Systems is always prime. \(\square\)

The proof of Theorem 8 follows from
Proposition 1. The Bautin ideal of system (14) is generated by the first five focus quantities.

Proof. Let \(B_5 := \langle g_{11}, \ldots, g_{55} \rangle\). We need to show that \(B = B_5\). It follows from the facts that \(\mathbf{V}(B) = \mathbf{V}(B_5)\) and the ideal \(B_5\) is radical. \(\square\).
Consider the cyclicity problem for the system

\[ i \dot{x} = x - a_{10}x^2 - a_{01}x\bar{x} - a_{-13}\bar{x}^3 \]  \hspace{1cm} (36)


We study along with (36) the more general system

\[ \begin{align*}
\dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-13}y^3, \\
\dot{y} &= -(y - b_{01}y^2 - b_{10}xy - b_{3,-1}x^3). 
\end{align*} \]  \hspace{1cm} (37)

**Theorem 9.** The center variety of system (37) consists of the following irreducible components:

1) \( a_{10} = a_{-13} = b_{10} = 3a_{01} - b_{01} = 0 \),
2) \( b_{01} = b_{3,-1} = a_{01} = 3b_{10} - a_{10} = 0 \),
3) \( a_{10} = a_{-13} = b_{10} = 3a_{01} + b_{01} = 0 \),
4) \( b_{01} = b_{3,-1} = a_{01} = 3b_{10} + a_{10} = 0 \),
5) \( a_{01} = a_{-13} = b_{10} = 0 \),
6) \( a_{01} = b_{3,-1} = b_{10} = 0 \),
7) \( a_{01} - 2b_{01} = b_{10} - 2a_{10} = 0 \),
8) \( a_{10}a_{01} - b_{01}b_{10} = a_{01}^4b_{3,-1} - b_{10}^4a_{-13} = a_{10}^4a_{-13} - b_{01}^4b_{3,-1} = \\
a_{10}a_{-13}b_{10}^3 - a_{01}^3b_{01}b_{3,-1} = a_{10}^2a_{-13}b_{10}^2 - a_{01}^2b_{01}^2b_{3,-1} = a_{10}^3a_{-13}b_{10} - a_{01}b_{01}^3b_{3,-1} = 0. \)
The first nine focus quantities:

\[ g_{11} = a_{10}a_{01} - b_{01}b_{10}; \]
\[ g_{22} = 0; \]
\[ g_{33} = -(2a_{10}^3a_{-13}b_{10} - a_{10}^2a_{-13}b_{10}^2 - 18a_{10}a_{-13}b_{10}^3 - 9a_{01}^4b_{3,-1} + 18a_{01}^2b_{01}b_{3,-1} + a_{01}^2b_{3,-1}^2 - 2a_{01}b_{01}b_{3,-1} + 9a_{-13}b_{10}^4)/8; \]
\[ g_{44} = -(14a_{10}b_{01}(2a_{10}a_{-13}b_{10}^3 + a_{01}^4b_{3,-1} - 2a_{01}b_{01}b_{3,-1} - a_{-13}b_{10}^4))/27; \]
\[ g_{55} = (a_{-13}b_{3,-1}(378a_{10}^4a_{-13} + 5771a_{10}^3a_{-13}b_{10} - 25462a_{10}^2a_{-13}b_{10}^2 + 11241a_{10}a_{-13}b_{10}^3 - 11241a_{01}b_{01}b_{3,-1} + 25462a_{01}^2b_{01}b_{3,-1} - 5771a_{01}b_{01}^3b_{3,-1} - 378b_{01}^4b_{3,-1}))/3240; \]
\[ g_{66} = 0; \]
\[ g_{77} = -(a_{-13}^2b_{3,-1}^2(343834a_{10}^2a_{-13}b_{10}^2 - 1184919a_{10}a_{-13}b_{10}^3 + 506501a_{-13}b_{10}^4 - 506501a_{01}^4b_{3,-1} + 1184919a_{01}b_{01}b_{3,-1} - 343834a_{01}^2b_{01}^2b_{3,-1}))/3240; \]
\[ g_{88} = 0; \]
\[ g_{99} = -a_{-13}^3b_{3,-1}^3 \left( 2a_{10}a_{-13}b_{10}^3 - a_{-13}b_{10}^4 + a_{01}^4b_{3,-1} - 2a_{01}^3b_{01}b_{3,-1} \right). \]

**Proposition 2.** The ideal \( I_5 = \langle g_{11}, g_{33}, g_{44}, g_{55} \rangle \) generated by the first five focus quantities of system (37) is not radical in \( \mathbb{C}[a_{10}, a_{01}, a_{-13}, b_{3,-1}, b_{10}, b_{01}] \).
Let us introduce new variables setting

\[ a_{10} = s_1 b_{10}, \quad b_{01} = s_2 a_{01}. \]  

(38)

By \( \tilde{g}_{kk} \) we denote the focus quantities obtained from \( g_{kk} \) after the substitution (38).

**Proposition 3.** The polynomials \( \tilde{g}_{11}, \tilde{g}_{33}, \tilde{g}_{44}, \tilde{g}_{55}, \tilde{g}_{77}, \tilde{g}_{99} \) form the basis of the ideal of focus quantities of the system (37) in the ring \( \mathbb{C}[s_1, s_2, a_{01}, a_{-13}, b_3, -1, b_{10}] \).

Proof. Denote by \([\nu]\) the monomial

\[ a_{10}^{\nu_1} a_{01}^{\nu_2} a_{-13}^{\nu_3} b_3^{\nu_4} b_{10}^{\nu_5} b_{01}^{\nu_6} \]

(where \( \nu = (\nu_1, \ldots, \nu_6) \)) and by \( \bar{\nu} \) the vector \( (\nu_6, \nu_5, \ldots, \nu_2, \nu_1) \). Focus quantities are polynomials of the ring \( \mathbb{Q}[a_{10}, b_{01}, a_{01}, b_{10}, a_{-13}, b_3, -1] \) and have the form

\[ g_{kk} = \sum_j \alpha_j ([\nu^{(j)}] - [\bar{\nu}^{(j)}]) = \sum_j \alpha_j IM[\nu^{(j)}], \]

where \( \alpha_j \in \mathbb{Q} \), \( \nu^{(j)} \) are the solutions of the equation

\[ L(\nu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nu_2 + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \nu_3 + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \nu_4 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_5 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nu_6 = \begin{pmatrix} k \\ k \end{pmatrix} \]  

(39)
and we use the notation


Denote by $M$ the monoid of all solutions of the equations (39), where $k$ runs through all $\mathbb{N}$. The Algorithm for Time-reversible systems produces the Hilbert basis of the monoid $M$: $\{(100\ 001), (110\ 000), (000\ 011), (010\ 010), (001\ 100), (040\ 100), (001\ 040), (401\ 000), (000\ 104), (101\ 030), (030\ 101), (201\ 020), (020\ 102), (301\ 010), (010\ 103)\}.

Therefore the focus quantities in the ring $\mathbb{Q}[s_1, s_2, a_{01}, a_{-13}, b_3, -1, b_{10}]$ have the form

$$\tilde{g}_{ii} = \sum_{\mu: L(\mu) = (i,i)^T} (f_\mu[\mu] - \bar{f}_\mu[\bar{\mu}]),$$

where $f_\mu \in \mathbb{Q}[s_1, s_2], \: \mu \in \tilde{M}$ and $\tilde{M}$ is the monoid of solutions of the equation

$$L(\nu) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \nu_1 + \begin{pmatrix} -1 \\ 3 \end{pmatrix} \nu_2 + \begin{pmatrix} 3 \\ -1 \end{pmatrix} \nu_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \nu_4 = \begin{pmatrix} k \\ k \end{pmatrix}$$

($k = 0, 1, 2, \ldots$). We denote by $\tilde{I}$ the ideal of focus quantities in the ring $\mathbb{C}[s_1, s_2, a_{01}, a_{-13}, b_3, -1, b_{10}], \: \text{by} \: \tilde{I}_k \: \text{the ideal generated by the first} \: k \: \text{quantities in}$
this ring, and by $-$ the involution

$$
- : \mathbb{C}[s_1, s_2][\tilde{M}] \hookrightarrow \mathbb{C}[s_1, s_2][\tilde{M}]
$$

(where $\mathbb{C}[s_1, s_2][\tilde{M}]$ is the monoid ring of the monoid $\tilde{M}$ over $\mathbb{C}[s_1, s_2]$) defined by the formula

$$
\bar{a}_{kj} = b_{jk}, \quad \bar{s}_1 = s_2.
$$

For example, if $f = s_1^u s_2^m a_{01}^5 b_{3,-1} b_{10}^1$ then $\bar{f} = s_1^m s_2^u b_{10}^5 a_{-13} a_{01}$.

Using the obvious equality

$$
IM[f(\nu + \mu)] = \frac{1}{2} IM[f \nu] RE[\mu] + \frac{1}{2} IM[\mu] RE[f \nu] \tag{40}
$$

where $f \in \mathbb{Q}[s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10}], \nu, \mu \in \tilde{M}$ we obtain

$$
\tilde{g}_{ii} \equiv h^{(i)}(s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10})[001 040] - \\
\bar{h}^{(i)}(s_1, s_2, a_{01}, a_{-13}, b_{3,-1}, b_{10})[040 010] \mod \langle \tilde{g}_{11} \rangle.
$$

It follows from the structure of the monoid $M$ that $h^{(i)}, \bar{h}^{(i)}$ are polynomials of $s_1, s_2, z, v, w, \bar{w}$, where $v = a_{01} b_{10}, z = a_{-13} b_{3,-1}, w = a_{-13} b_{10}^4, \bar{w} = b_{3,-1} a_{01}^4$. 
When $s_1 = s_2 = 1/2$ the system (37) has a center at the origin, therefore

$$\tilde{g}_{ii} \equiv ((2s_1 - 1)v_1^{(i)}w - (2s_2 - 1)v_1^{(i)}\bar{w}) + ((2s_2 - 1)v_2^{(i)}w - (2s_1 - 1)v_2^{(i)}\bar{w}) \mod \langle \tilde{g}_{11} \rangle,$$

where $v_{1,2}^{(i)} \in \mathbb{Q}[s_1, s_2, v, z, w, \bar{w}]$.

It is easy to see that we can write $\tilde{g}_{ii}$ in the form $\tilde{g}_{ii} = \tilde{g}_{ii}^{(1)} + \tilde{g}_{ii}^{(2)} + \tilde{g}_{ii}^{(3)}$, where $\tilde{g}_{ii}^{(1)}$ is a sum with rational coefficients polynomials of the form

$$f_1 = v^c((2s_1 - 1)\alpha_i w - (2s_2 - 1)\bar{\alpha}_i \bar{w}) + v^c((2s_2 - 1)\beta_i w - (2s_1 - 1)\bar{\beta}_i \bar{w}),$$

where $\alpha_i, \beta_i \in \mathbb{Q}[s_1, s_2, w, z, v], c \in \mathbb{N}, c > 0$, $\tilde{g}_{ii}^{(2)}$ is a sum of polynomials

$$f_2 = z^c((2s_1 - 1)\gamma_i w - (2s_2 - 1)\bar{\gamma}_i \bar{w})$$

where $\gamma_i, \in \mathbb{Q}[s_1, s_2, z, w], c \in \mathbb{N}, c > 0$, and $\tilde{g}_{ii}^{(3)}$ is a sum of polynomials of the form

$$f_3 = ((2s_1 - 1)\theta_i w - (2s_2 - 1)\bar{\theta}_i \bar{w})$$

where $\theta \in \mathbb{Q}[s_1, s_2, w]$ (i.e. $\tilde{g}_{ii}^{(1)}$ is the sum of all terms of $\tilde{g}_{ii}$ containing the factor $v$, $\tilde{g}_{ii}^{(2)}$ is the sum of remaining terms of $\tilde{g}_{ii}$ containing the factor $z$, and $\tilde{g}_{ii}^{(3)}$ are all the rest terms).
We will show that

\[ \tilde{f}_1 \equiv 0 \mod \tilde{I}_5, \quad \tilde{f}_2 \equiv 0 \mod \tilde{I}_9, \quad \tilde{f}_3 \equiv 0 \mod \tilde{I}_5. \quad (41) \]

First we prove that

\[ v(s_1 s_2^m (2s_1 - 1) w^k - s_1^m s_2^u (2s_2 - 1) \bar{w}^k) \in \tilde{I}_5 \quad (42) \]

and

\[ v(s_1^u s_2^m (2s_2 - 1) w^k - s_1^m s_2^u (2s_1 - 1) \bar{w}^k) \in \tilde{I}_5 \quad (43) \]

for all \( k, u, m \in \mathbb{N} \). Indeed, computing the reduced Groebner basis of \( \tilde{I}_5 \) using lex with \( s_1 > s_2 > a_{01} > b_{10} > a_{-13} > b_{3,-1} \) we see that it contains the polynomials

\[ u_1 = v(s_1 - s_2), \quad u_2 = v(2s_2 - 1)(w - \bar{w}), \]

\[ u_3 = -a_{01} z (2s_2 - 1) (w - \bar{w}), \]

\[ u_4 = -b_{10} z ((2s_1 - 1)w - (2s_2 - 1))\bar{w}, \]

\[ u_5 = a_{01} w (2s_2 - 1)(s_2 - 3)(s_2 + 3) \]

and

\[ u_6 = ((2s_1 - 1)(s_1 - 3)(s_1 + 3)w - (2s_2 - 1)(s_2 - 3)(s_2 + 3)\bar{w}). \]
It is easily checked that

\[ v((2s_1 - 1)w^k - (2s_2 - 1)\bar{w}^k) - 2u_1w^k = v(2s_2 - 1)(w^k - \bar{w}^k) \equiv 0 \mod \langle u_2 \rangle, \]

\[ v(s_1(2s_1 - 1)w^k - s_2(2s_2 - 1)\bar{w}^k) - (2s_1 + 2s_2 - 1)u_1w^k = \]

\[ vs_2(2s_2 - 1)(w^k - \bar{w}^k) \equiv 0 \mod \langle u_2 \rangle, \]

\[ v(s_2(2s_1 - 1)w^k - s_1(2s_2 - 1)\bar{w}^k) - u_1(2s_2(w^k - \bar{w}^k) + \bar{w}^k) = \]

\[ vs_2(2s_2 - 1)(w^k - \bar{w}^k) \equiv 0 \mod \langle u_2 \rangle, \]

i.e for \( \gamma = 0, 1, \ i = 1, 2 \) the polynomials

\[ (s_i^\gamma(2s_1 - 1)w^k - \bar{s}_i^\gamma(2s_2 - 1)\bar{w}^k) \]

are in the ideal \( \tilde{I}_5 \). Taking into account that

\[ (s_i^\gamma(2s_1 - 1)w^k - \bar{s}_i^\gamma(2s_2 - 1)\bar{w}^k) = \]
We now show that \( f_2 \in \tilde{I}_9 \).

Without loss of generality \( f_2 \) is of the form
\[
d_k(c) = z^c (s_1^u (2s_1 - 1) w^k - s_2^u (2s_2 - 1) \bar{w}^k)
\]
with \( k > 1 \), or of the form
\[
d_1(c) = z^c (s_1^u (2s_1 - 1) w - s_2^u (2s_2 - 1) \bar{w}).
\]

First we prove that
\[
d_k(c) \equiv 0 \mod \tilde{I}_5.
\]
It is sufficient to consider the case \( c = 1 \). We show using the induction on \( k \) that for \( k > 1 \)
\[
d_k(1) \equiv 0 \mod \tilde{I}_5
\]
and
\[
d_k^+(1) = z (w - \bar{w}) (s_1^u (2s_1 - 1) w^k + s_2^u (2s_2 - 1) \bar{w}^k) \equiv 0 \mod \tilde{I}_5.
\]
For $k = 2$ we have

$$d_2(1) + u_3b_3,-1a_{01}^3s_2^u + u_4a_{-13}b_{10}^3s_1^u = (2s_2 - 1)(w\bar{w})^2(s_1^u - s_2^u) \in \langle u_1 \rangle.$$ 

Also

$$d_2^+(1) + u_3b_3,-1a_{01}^3(s_1^uw + s_2^u\bar{w}) + u_4a_{-13}b_{10}^3s_1^u(w - \bar{w}) = 0.$$ 

Let us assume that for $2 \leq k < K$ the statement holds. Then for $k = K$ using (40) we have

$$zIM[s_1^u(2s_1-1)w^K] = \frac{1}{2}IM[s_1^u(2s_1-1)w^{K-1}]RE[w] + \frac{1}{2}RE[s_1^u(2s_1-1)w^{K-1}]IM[w].$$

Due to the induction hypothesis the both summands in the right-hand side are in $\tilde{I}_5$. Therefore (44) holds with $k = K$. The correctness of (45) follows from the formula

$$z(w - \bar{w})(s_1^u(2s_1 - 1)w^j + s_2^u(2s_2 - 1)\bar{w}^j) =$$

$$-u_3b_3,-1a_{01}^3(s_1^uw^j - s_2^u\bar{w}^j) + u_4a_{-13}b_{10}^3s_1^uw^{j-2}(w - \bar{w}).$$

Consider now the second case, namely the polynomial

$$d_1(c) = z^c(s_1^u(2s_1 - 1)w - s_2^u(2s_2 - 1)\bar{w}).$$
In fact here \( u \) can be equal only 0, 1, 2 or 3. Reducing \( d_1(3) \) modulo a Groebner basis of \( \tilde{I}_9 \) we see that all these polynomials are in \( \tilde{I}_9 \), therefore \( d_1(c) \in \tilde{I}_9 \) for \( c > 2 \). If \( c \leq 2 \) then the degree of \( d_1(c) \) is less or equal 15, but the degree of the polynomials of our interest starts from 20 (namely, the first polynomial under the consideration is \( g_{10,10} \)).

Similarly, it is possible to show that \( f_3 \in \tilde{I}_5 \). Hence \( \tilde{g}_{ii} \in \tilde{I}_9 \) for \( i > 9 \). □

Because when \( a_{01} = 0 \), \( a_{10}^4a_{-13} - a_{10}^4a_{-13} \neq 0 \) the system (36) has a focus at the origin and when \( |a_{01}| \neq 0 \) the substitution (38) is invertible we conclude that Proposition 3 yields the following statement.

**Proposition 4.** The cyclicity of the origin of the system (36) with \( a_{01} \neq 0 \) or \( a_{01} = 0 \), \( a_{10}^4a_{-13} \) is less or equal 5.

If instead of the substitution (38) we use \( a_{01} = s_1b_{01}, \quad b_{10} = s_2a_{01} \) then using similar reasoning one can prove the analog of Proposition 4. Thus, the following statement holds.

**Theorem 10.** The cyclicity of the origin of the system (36) with \( |a_{10}| + |a_{01}| \neq 0 \) is less or equal 5.