Implementation of Involutive Bases in Maple and C++

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Outline

- Janet’s algorithm
- Involutive bases
- Maple / C++ packages
- Examples

Multiple-closed sets of monomials

$$\mathbb{M} := \mathrm{Mon}(x_1, \ldots, x_n) := \{x^i \mid i \in (\mathbb{Z}_{\geq 0})^n\}$$

$S \subseteq \mathbb{M}$ is $\mathbb{M}$-*multiple-closed* if

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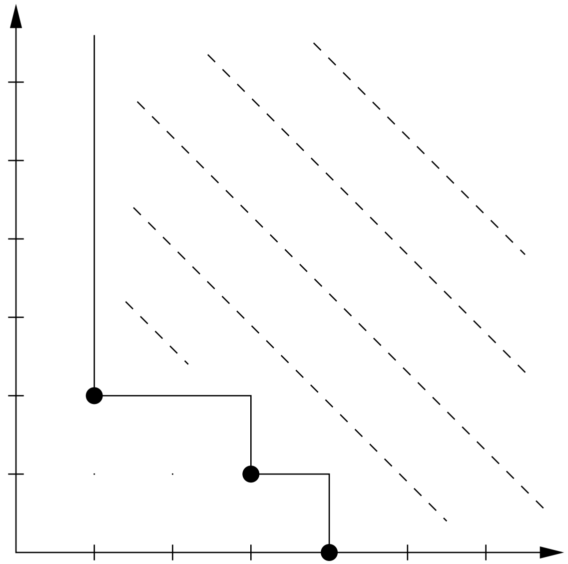

$\mathbb{M}$-multiple-closed set
generated by $x_1 x_2^2$, $x_1^3 x_2$, x_1^4

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$\Rightarrow$ Termination of Janet's and the involutive algorithms

Finally achieve $S = \mathrm{lm}(I)$ as in Buchberger's algorithm

Decomposition into disjoint cones

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if $\exists\, v \in C$ such that $C = \mathrm{Mon}(\mu)\, v = \{m\, v \mid m \in \mathrm{Mon}(\mu)\}$.

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$\{\, (C_1, \mu_1),\, \ldots,\, (C_l, \mu_l)\,\} \subset \mathcal{P}(\mathbb{M}) \times \mathcal{P}(\{x_1, \ldots, x_n\})$
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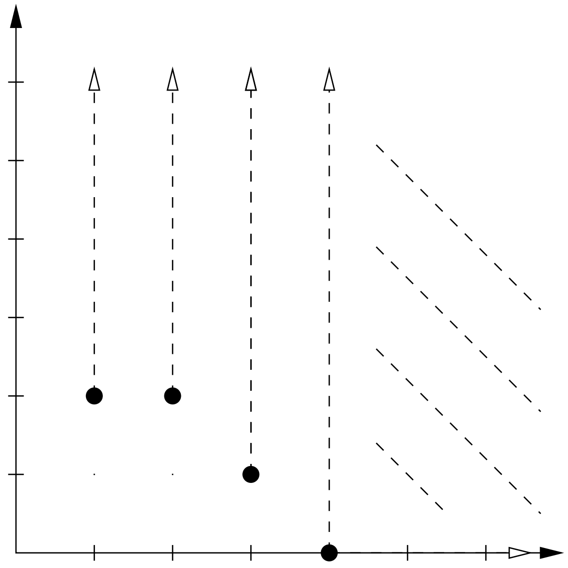

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involutive divisions (Gerdt, Blinkov et. al.)

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Janet division:

Let $M \subset \mathbb{M} = \mathrm{Mon}(x_1, \ldots, x_n)$ be finite.

For a cone with vertex $m = x_1^{a_1} \cdots x_n^{a_n} \in M$

x_i is a *multiplicative variable*, i.e. $x_i \in \mu$, iff

$$a_i = \max\{\, b_i \mid x^b \in M;\; b_j = a_j \;\forall\, j < i \,\}.$$

Janet reduction

NF(p, T, $\prec$) $\qquad p \in k[x_1, \ldots, x_n], \qquad T = \{\, (d_1, \mu_1), \ldots, (d_l, \mu_l) \,\}$

$r \leftarrow 0$

while $p \neq 0$ do

if $\exists\, (d, \mu) \in T : \mathrm{lm}(p) \in \mathrm{Mon}(\mu)\, d$ then

$$p \leftarrow p - \frac{\mathrm{lc}(p)}{\mathrm{lc}(d)}\, \frac{\mathrm{lm}(p)}{\mathrm{lm}(d)}\, d$$

else

$$r \leftarrow r + \mathrm{lc}(p)\ \mathrm{lm}(p)$$

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fi

od

return r

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return r $\qquad$ Disj. cones $\Rightarrow$ course of Alg. is uniquely determined

Janet's algorithm

JanetBasis(F, $\prec$) $\qquad$ $F \subset k[x_1, \ldots, x_n]$ finite

$G \leftarrow F$

do

$\quad G \leftarrow$ auto-reduce G

$\quad J \leftarrow \{ (d_1, \mu_1), \ldots, (d_l, \mu_l) \}$ s.t. $\{ (\mathrm{lm}(d_1), \mu_1), \ldots, (\mathrm{lm}(d_l), \mu_l) \}$

$\qquad$ decomp. into disj. cones of $[\mathrm{lm}(G)]$

$\quad P \leftarrow \{ \mathrm{NF}(x \cdot p, J) \mid (p, \mu) \in J, \, x \notin \mu \}$

$\quad G \leftarrow \{ p \mid (p, \mu) \in J \} \cup P$

while $\;P \neq \{0\}$

return J

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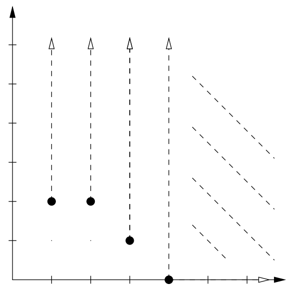

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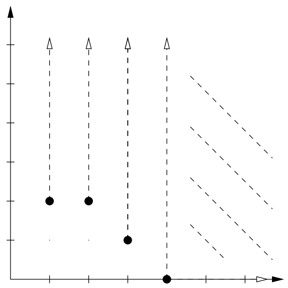

(b) Let T be a decomp. of $\mathrm{Mon}(R) - \mathrm{lm}(\langle F \rangle)$ into disjoint cones.

A k-basis of $R/\langle F \rangle$ is given by the cosets represented by $\bigcup_{(m,\mu)\in T} \mathrm{Mon}(\mu)\, m$.

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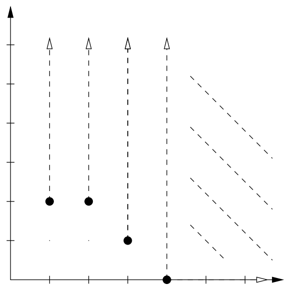

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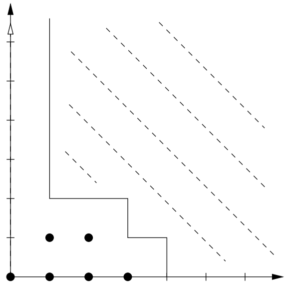

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Define $\pi : R^{|J|} \to R : e_i \mapsto g_i$*.*

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$$x_j \, e_i - \sum_l \alpha_{i,j,l} \, e_l, \qquad x_j \notin \mu_i, \quad i = 1, \ldots, r,$$

form a Janet basis of $\ker \pi$ for a suitable monomial ordering, where $x_j \, g_i = \sum_l \alpha_{i,j,l} \, g_l$.

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$\rightsquigarrow$ direct construction of a free resolution of $R/\langle F \rangle$.

Combinatorial tool

Let $R := k[x_1, \ldots, x_n]$.

Generalized Hilbert series of $S \subseteq \mathrm{Mon}(R)$ is

$$H_S(x_1, \ldots, x_n) := \sum_{m \in S} m$$

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Most important case: $I \trianglelefteq R, \qquad S = \mathrm{Mon}(R) - \mathrm{lm}(I)$

$H_S(x_1, \ldots, x_n)$ enumerates the k-basis of R/I

formed by the residue classes of the $m \in S$

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usual Hilbert series of $R/\langle \mathrm{lm}(I) \rangle$ (with std. grading) is
$H_S(\lambda, \ldots, \lambda)$

Parametric derivatives

Problem: Find all formal power series solutions of

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} - y\,\dfrac{\partial^2 u}{\partial t^2} = 0 \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = 0 \end{array} \right.$$

Janet basis . . .

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Janet basis . . .   see worksheet later on . . .

Generalized Hilbert series:

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Generalized Hilbert series:

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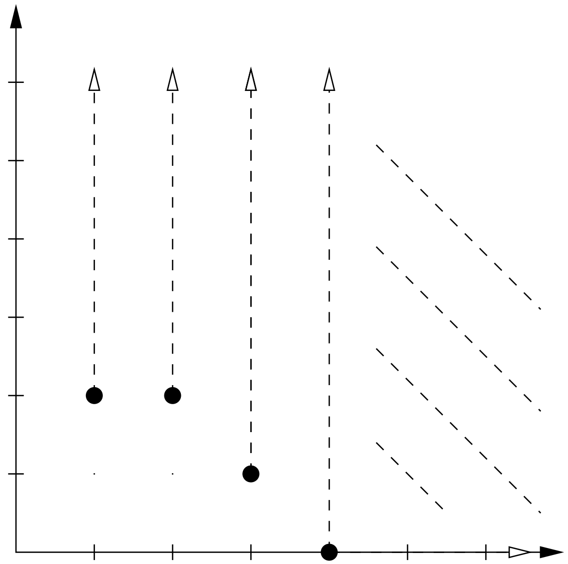
Formal power series solutions:

$$\underbrace{c + c_t\,t + c_y\,y + c_x\,x + c_{t^2}\,t^2}_{\text{free}} + \ldots + \underbrace{c_{y^2}\,y^2 + c_{x^2}\,x^2 + c_{yt^2}\,yt^2}_{\text{det.}} + \ldots$$

Involutive Basis Algorithm (Gerdt)

choose $f \in F$ with the lowest $\operatorname{lm}(f)$ w.r.t. $\prec$
$G \leftarrow \{f\}$; $Q \leftarrow F \setminus G$
do
    $h \leftarrow 0$
    while $Q \neq \emptyset$ and $h = 0$ do
        choose $p \in Q$ with the lowest $\operatorname{lm}(p)$ w.r.t. $\prec$
        $Q \leftarrow Q \setminus \{p\}$; $h \leftarrow \operatorname{NF}(p, G, \prec)$
    if $h \neq 0$ then
        for all $\{g \in G \mid \operatorname{lm}(g) = x^i \operatorname{lm}(h),\ |i| > 0\}$ do
            $Q \leftarrow Q \cup \{g\}$; $G \leftarrow G \setminus \{g\}$
        $G \leftarrow G \cup \{h\}$
        $Q \leftarrow Q \cup \{\, x \cdot g \mid g \in G,\ x$ non-mult. for $g \,\}$
while $Q \neq \emptyset$
return G

Janet-like Gröbner Bases



Idea: do not store all the *prolongations*

V. P. Gerdt, Y. A. Blinkov, *Janet-like Monomial Division*. *Janet-like Gröbner Bases*. CASC 2005, LNCS 3781, Springer, 2005

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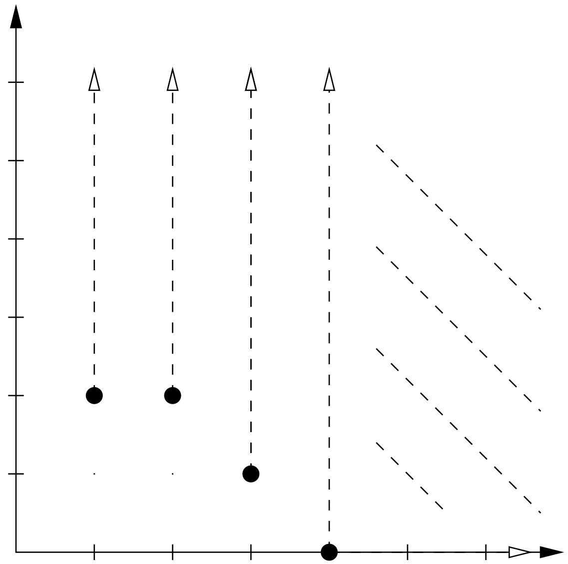

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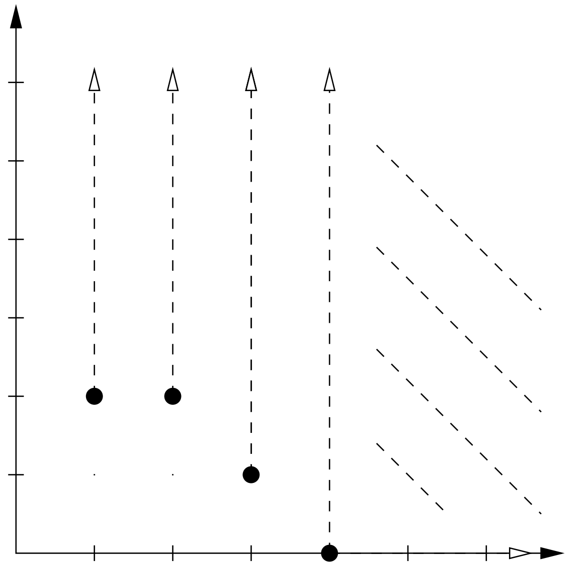

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Note: In general, the minimal Gröbner basis is still a proper subset of the Janet-like Gröbner basis.

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Maple packages. . .

. . . at Lehrstuhl B für Mathematik, RWTH Aachen

implementing the involutive basis technique:

`Involutive` / `Janet`

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`LDA` (**L**inear **D**ifference **A**lgebra)

in cooperation with V. P. Gerdt & Y. A. Blinkov

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- an abstract Maple package for homological algebra

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`jets`

- jet calculus, formal differential geometry

Authors: M. Barakat, G. Hartjen, A. Lorenz

Involutive

- Janet (-like Gröbner) bases for submodules of free modules over a commutative polynomial ring

web: `http://wwwb.math.rwth-aachen.de/Janet`

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  TOP / POT
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- Applications:
  commutative algebra
  solving systems of algebraic equations

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Main procedures of Involutive

`InvolutiveBasis`

compute Janet(-like Gröbner) basis

`PolInvReduce`

involutive reduction modulo Janet basis

`FactorModuleBasis`

vector space basis of residue class module

`Syzygies`

syzygies

`PolHilbertSeries`, `PolHilbertPolynomial`, etc.

combinatorial devices

`PolMinPoly`, `PolRepres`, etc.

computing in residue class rings

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- Elementary divisor algorithm (Jacobson normal form)
- parametric derivatives
- formal power series solutions, polynomial solutions

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Main procedures of Janet

`JanetBasis`

compute Janet(-like Gröbner) basis

`InvReduce`

involutive reduction modulo Janet basis

`ParamDeriv`

parametric derivatives

`CompCond`

compatibility conditions (syzygies)

`HilbertSeries`, `HilbertPolynomial`, etc.

combinatorial devices

`SolSeries`, `PolySol`

formal power series / polynomial solutions

ginv

- C++ module for Python

web: `http://invo.jinr.ru`

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- comp. of Gröbner bases using involutive algorithms

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- contributions by V. Brendt, S. Jambor, D. Robertz

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ginv

```
import ginv
st = ginv.SystemType("Polynomial")
im = ginv.MonomInterface("DegRevLex", st, ['x', 'y'])
ic = ginv.CoeffInterface("GmpZ", st)
ip = ginv.PolyInterface("PolyList", st, im, ic)
iw = ginv.WrapInterface("CritPartially", ip)
iD = ginv.DivisionInterface("Janet", iw)
eqs = ["x^2+y^2", ...]
basis = ginv.basisBuild("TQ", iD, eqs)
```

JanetOre

- Janet (-like Gröbner) bases for left submodules of free modules over Ore algebras

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- term orderings:
  degrevlex, plex
  TOP / POT
  block / elimination orderings

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Main procedures of JanetOre

`JBasis`

compute Janet(-like Gröbner) basis

`JInvReduce`

involutive reduction modulo Janet basis

`JFactorModuleBasis`

vector space basis of residue class module

`JSyzygies`

syzygies

`JHilbertSeries`, `JHilbertPolynomial`, etc.

combinatorial devices

`JMinPoly`, `JRepres`, etc.

computing in residue class rings

LDA

Linear **D**ifference **A**lgebra

- Janet (-like Gröbner) bases for linear shift equations

Gerdt – R., *A Maple Package for Computing Gröbner Bases for Linear Recurrence Relations*, Nuclear Instruments and Methods in Physics Research, Section A, 559 (1), 2006

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- Janet (-like Gröbner) bases for linear shift equations
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- can naturally be extended to quasilinear systems

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LDA

Linear **D**ifference **A**lgebra

- Janet (-like Gröbner) bases for linear shift equations
- equations with non-constant coefficients
- can naturally be extended to quasilinear systems
- Applications:
  difference schemes for PDEs
  Feynman integrals (Tarasov, Smirnov et. al.)

Gerdt – R., *A Maple Package for Computing Gröbner Bases for Linear Recurrence Relations*, Nuclear Instruments and Methods in Physics Research, Section A, 559 (1), 2006

Main procedures of LDA

`JanetBasis`

compute Janet(-like Gröbner) basis

`InvReduce`

involutive reduction modulo Janet basis

`CompCond`

compatibility conditions (syzygies)

`ResidueClassBasis`, `ShiftRepres`, etc.

residue class basis of factor module, etc.

`HilbertSeries`, `HilbertPolynomial`, etc.

combinatorial devices

`Pol2Shift / Shift2Pol`

conversion between shift operators and eq's

Applications

`Involutive`:

- Invariant theory of finite groups
  Plesken – R., *Exp. Math.*, 2005
- Construction of matrix representations of groups
  Plesken – R., *J. Algebra*, to appear

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- Invariant theory of finite groups
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- Construction of matrix representations of groups
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`LDA`:

- Constr. of finite difference schemes for PDEs
  Gerdt, Blinkov, Mozzhilkin, *SIGMA*, to appear
- Reduction of Feynman integrals . . .
  (ongoing work)

homalg

- package is independent of ring

Barakat – R., *homalg: An abstract package for homological algebra*, in preparation

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- given a functor “on objects”, its part “on morphisms” is provided by the package
- composition and left / right derivation of functors is automatic
- easy to define new functors

Barakat – R., *homalg: An abstract package for homological algebra*, in preparation

Main procedures of homalg

`Hom`, `Ext`, `Tensorproduct`, `Tor`

predefined (co-) functors (on objects)

`HomMap`, `ExtMap`, `TensorproductMap`, `TorMap`

predefined (co-) functors (on morphisms)

`IsHom`, `Kernel`, `Image`, `Cokernel`

homomorphisms

`ResolveShortExactSeq`, `ConnectingHom`

snake lemma, etc.

`ComposeFunctors`

compose functors

`LeftDerivedFunctor`, `RightDerivedCofunctor`

derive functors

OreModules

- constructive module theory over Ore algebras
- syzygies, free resolutions, extension groups. . .

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for multidimensional linear systems:

- decide controllability and parametrizability,

web: `wwwb.math.rwth-aachen.de/OreModules`

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- constructive module theory over Ore algebras
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for multidimensional linear systems:

- decide controllability and parametrizability,
- construct (minimal) parametrizations,
- compute Bezout identities,
- decide flatness (also π-freeness), etc.

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Main procedures of OreModules

`DefineOreAlgebra`

define an Ore algebra for the current session

`SyzygyModule`, `FreeResolution`, etc.

constructive module theory over Ore algebras

`Exti`

compute a presentation of $\mathrm{ext}^i_D(M, D)$

`Parametrization`

construct a parametrization of a linear system

`LeftInverse`, `RightInverse`

compute left / right inverses of matrices

`Brunovsky`, `ControllabilityMatrix`, etc.

tools for linear control theory

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