SINGULAR and Applications

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Technische Universität Kaiserslautern

A Computer Algebra System for Polynomial Computations

with special emphasize on the needs of algebraic geometry, commutative algebra, and singularity theory

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The computer is not the philosopher's stone but the philosopher's whetstone

Hugo Battus, Rekenen op taal 1983

Birth of SINGULAR

- 1984

Fields

- rational numbers $\mathbb{Q}$ (charakteristic 0)
- finite fields $\mathbb{Z}/p\mathbb{Z}(p <= 2147483629)$
- finite fields $\mathbb{F}_{p^n}(p^n < 2^{15})$

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- algebraic extensions of $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$
  $K[t]$/MinPoly

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  $K[t]$/MinPoly
- floating point real and complex numbers

Rings

- polynomial rings $K[x_1, \ldots, x_n]$
- localizations $K[x_1, \ldots, x_n]_M$
  M maximal ideal
- factor rings $K[x_1, \ldots, x_n]/J$ oder $K[x_1, \ldots, x_n]_M/J$

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- non-commutative G–algebras
  $K\langle x_1, \ldots, x_n \mid x_j x_i = C_{ij} x_i x_j + D_{ij} \rangle$
  $C_{ij} \in K \;,\; LM(D_{ij}) < x_i x_j$
- factor algebras of G–algebras by two-sided ideals

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- factor algebras of G–algebras by two-sided ideals
- tensor products of the rings above

Algorithms in the Kernel (C/C_{++})

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- Standard basis algorithms (Buchberger, SlimGB, factorizing Buchberger, FGLM, Hilbert–driven Buchberger, ...)

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- Characteritstic sets (Wu)

Examples for libraries

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Examples for libraries

- primdec.lib
- absfact.lib

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dynamic modules

History

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Greuel/Pfister: Exist singularities (not quasi-homogeneous and complete intersection) with exact Poincaré-complex?

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Neuendorf/Pfister: Implementation of the Gröbner basis algorithm in basic at ZX-Spectrum

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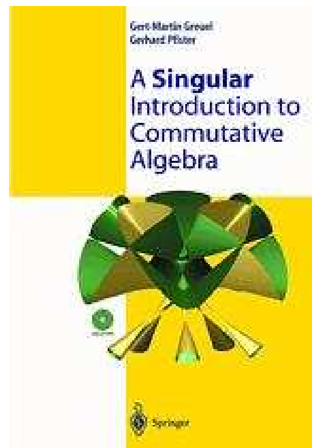

2002

Book: A SINGULAR Introduction to Commutative Algebra
(G.-M. Greuel and G. Pfister, with contributions by O. Bachmann, C. Lossen and H. Schönemann).

History

2004

Jenks Price

for:

Excellence in Software Engineering

awarded at ISSAC in Santander

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- Supported by: Deutsche Forschungsgemeinschaft, Stiftung Rheinland-Pfalz für Innovation, Volkswagen Stiftung
- SINGULAR is free software (Gnu Public Licence)

Team

T. Wichmann, C. Lossen, G.-M. Greuel, H. Schönemann, W. Pohl, G. Pfister, V. Levandovskyy, E. Westenberger, A. Frühbis-Krüger, Oscar, K. Krüger

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Kaiserslautern
Saarbrücken
Cottbus
Berlin
Mainz
Dortmund
Valladolid
La Laguna
Buenos Aires

Saito's result

Theorem (K. Saito 1971): Let $(X,0)$ be the germ of an isolated hypersurface singularity. The following conditions are equivalent:

- $(X,0)$ is quasi-homogeneous.
- $\mu(X,0) = \tau(X,0)$.
- The Poincaré complex of $(X,0)$ is exact.

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We wanted to generalize this theorem to the case of isolated complete intersection singularities.

Poincaré complex

Let $(X_{l,k}, 0)$ be the germ of the unimodal space curve singularity $FT_{k,l}$ of the classifi cation of Terry Wall defi ned by the equations

$$xy + z^{l-1} = 0$$
$$xz + yz^2 + y^{k-1} = 0$$

$$4 \leq l \leq k, 5 \leq k.$$

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The Poincaré complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_{X_{l,k},0} \longrightarrow \Omega^1_{X_{l,k},0} \longrightarrow \Omega^2_{X_{l,k},0} \longrightarrow \Omega^3_{X_{l,k},0} \longrightarrow 0$$

is exact.

But $(X_{l,k}, 0)$ is not quasi-homogeneous:

$$\mu(X, 0) = \tau(X, 0) + 1 = k + l + 2.$$

μ and τ

Let $(X, 0)$ be a germ of a space curve singularity defined by $f = g = 0$, with $f, g \in \mathbb{C}\{x, y, z\}$

- $\mu(X, 0) = dim_{\mathbb{C}}(\Omega^1_{X,0}/d\mathcal{O}_{(X,0)})$

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- $\tau(X, 0) = dim_{\mathbb{C}}(\mathbb{C}\{x, y, z\}/ < f, g, M_1, M_2, M_3 >)$

here the M_i are the 2-minors of the Jacobian matrix of f, g.

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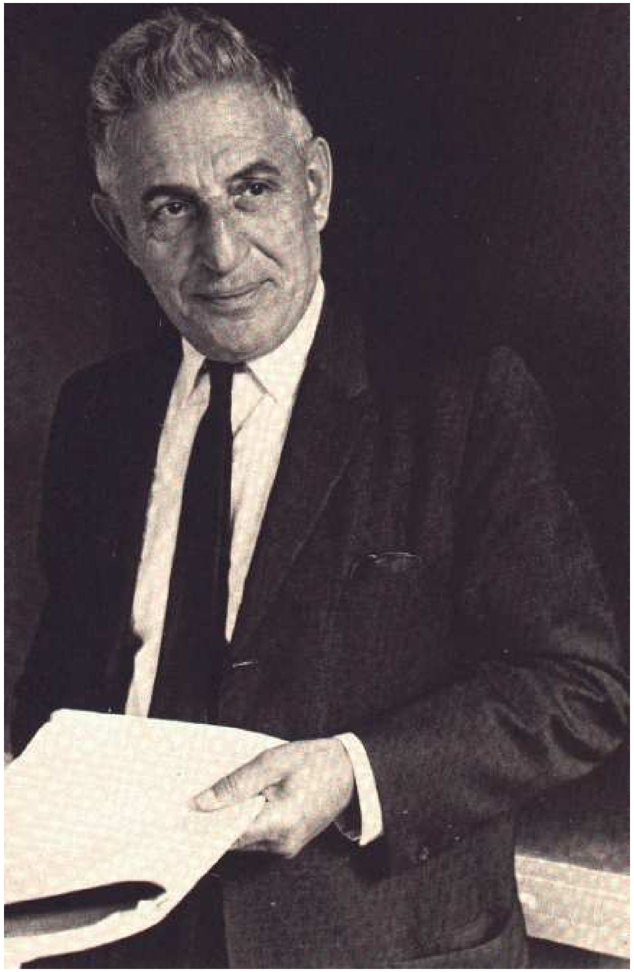
  here the M_i are the 2-minors of the Jacobian matrix of f,g.
- Reiffen: The Poincaré complex is exact if and only if

$$< f,g > \Omega^3_{\mathbb{C}^3,0} \subset d(< f,g > \Omega^2_{\mathbb{C}^3,0})$$

and

$$\mu(X,0) = dim_{\mathbb{C}}(\Omega^2_{X,0}) - dim_{\mathbb{C}}(\Omega^3_{X,0})$$

Zariski's conjecture



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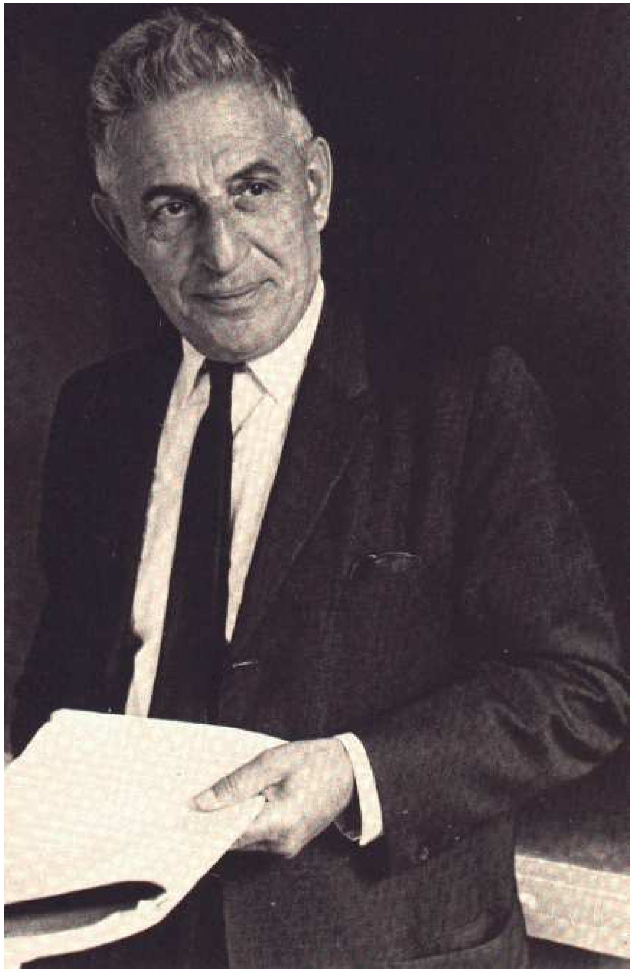
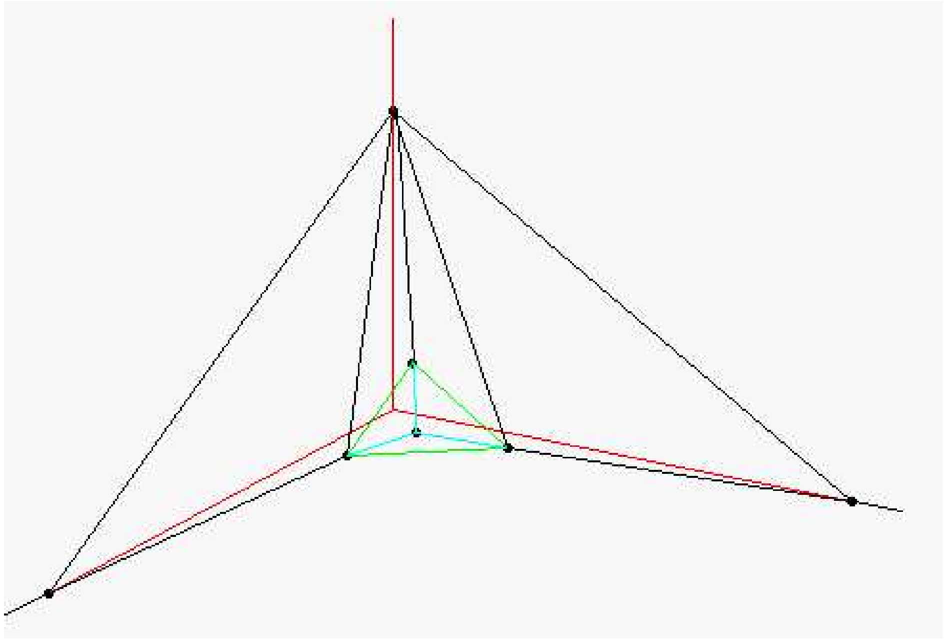


Conjecture (Zariski 1971) :
A μ–constant deformation of an isolated hypersurface singularity is a deformation with constant multiplicity.

Zariski's conjecture

$$F_t = x^a + y^b + z^{3c} + x^{c+2}y^{c-1} + x^{c-1}y^{c-1}z^3 + x^{c-2}y^c(y^2 + tx)^2$$



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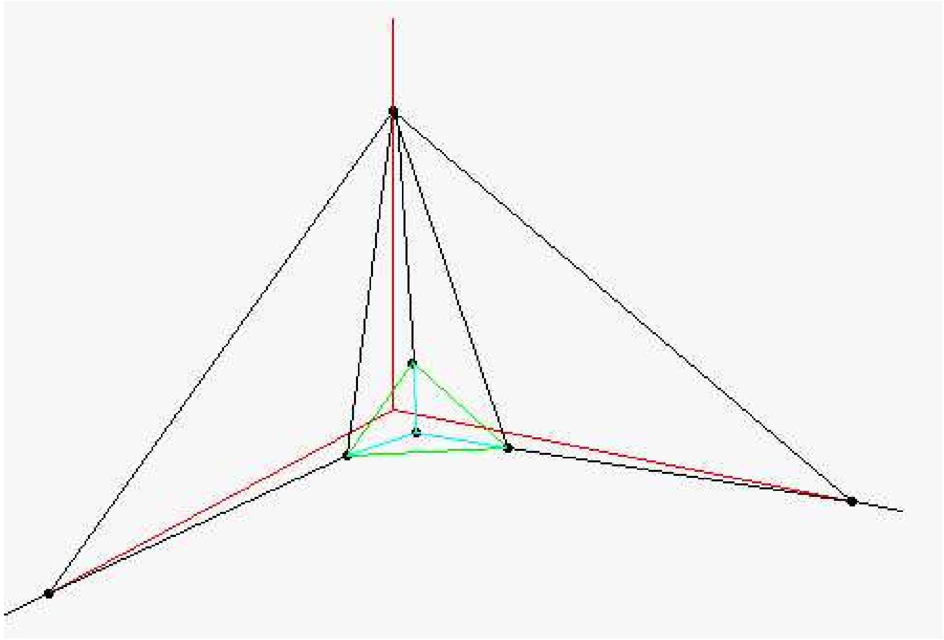

$(a, b, c) = (40, 30, 8)$

$\mu(F_0) = 10661$

$\mu(F_t) = 10655$

Applications

- **mathematical**
  - experimental tool
  - proving theorems

Applications

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- **mathematical**
  - experimental tool
  - proving theorems
- **non–mathematical**
  - engineering (glas melting, robotics, chemical models, analog and digital microelectronics)
  - equilibrian problems in economics
  - theoretical physics

Computeralgebra and finite Groups

Problem: Characterize the class of fi nite solvable groups G by 2–variable identities.

Computeralgebra and finite Groups

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Problem: Characterize the class of finite solvable groups G by 2–variable identities.

Example:

- G is **abelian** $\Leftrightarrow xy = yx \; \forall \; x, y \in G$
- (Zorn, 1930) A finite group G is **nilpotent** $\Leftrightarrow \exists \; n \geq 1$, such that $v_n(x, y) = 1 \; \forall \; x, y \in G$
  (**Engel Identity**)

  $v_1 := [x, y] = xyx^{-1}y^{-1}$ (commutator)
  $v_{n+1} := [v_n, y]$

nilpotent groups

Let G be a finite group

$$G^{(1)} := [G,G] = \langle aba^{-1}b^{-1} \mid a,b \in G \rangle \, .$$

Let $G^{(i)} := [G^{(i-1)}, G]$, then G is called nilpotent, if $G^{(m)} = \{e\}$ for a suitable m.

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- abelian groups are nilpotent.
- if the order of the group is a power of a prime it is nilpotent.
- G ist nilpotent $\Leftrightarrow$ it is the direct product of its Sylow groups.
- S_3 is not nilpotent.

solvable groups

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- nilpotente groups are solvable.
- S_3, S_4 are solvable.
- groups of odd order are solvable.
- S_5, A_5 are not solvable.

Main result

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Theorem (T. Bandman, G.-M. Greuel, F. Grunewald, B. Kunyavsky, G. Pfi ster, E. Plotkin)

$$U_1 = U_1(x,y) := x^2 y^{-1} x,$$

$$U_{n+1} = U_{n+1}(x,y) = [x U_n x^{-1}, y U_n y^{-1}].$$

A fi nite group G is **solvable** $\Leftrightarrow \exists\, n$, such that $U_n(x,y) = 1 \;\forall\, x, y \in G$.

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- $U_1(x, y) = 1 \Leftrightarrow y = x^{-1}$
- $U_1(x, y) = U_2(x, y)$ $\Leftrightarrow x^{-1} y x^{-1} y^{-1} x^2 = y x^{-2} y^{-1} x y^{-1}$
- Let $x, y \in G$ such that $y \neq x^{-1}$ and $U_1(x, y) = U_2(x, y) \Rightarrow U_n(x, y) \neq 1 \;\forall\, n \in \mathbb{N}$.

Proof

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G solvable $\Rightarrow$ Identity is true (by defi nition).

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Let G minimally not solvable. Then G is one of the following groups:

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If is enough to prove (for G in Thompson's list): $\exists\ x, y \in G$, such that $y \neq x^{-1}$ and $U_1(x, y) = U_2(x, y)$.

Motivation of the choice of the word

Let w be a word in X, Y, X^{-1}, Y^{-1} and

$$U_1 = w$$
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A Computer–search through the 10,000 shortest words in X, X^{-1}, Y, Y^{-1} found the following four words such that the equation $U_1 = U_2$ has a non-trivial solution in PSL$(2, p)$ for all $p < 1000$:

$$w_1 = X^{-2}Y^{-1}X$$

$$w_2 = X^{-1}YXY^{-1}X$$

$$w_3 = Y^{-2}X^{-1}$$

$$w_4 = XY^{-2}X^{-1}YX^{-1}$$

PSL

$$\mathsf{PSL}(2,K) = \mathsf{SL}(2,K) / \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \;\middle|\; a^2 = 1 \right\}$$

PSL

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especially

$$\mathsf{PSL}(2,\mathbb{F}_5) = \left\{\left[\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)\right],\ a_{11}a_{22} - a_{21}a_{12} = 1\right\}$$

$$\left[\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)\right] = \left\{\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right),\ \begin{pmatrix} 4a_{11} & 4a_{12} \\ 4a_{21} & 4a_{22} \end{pmatrix}\right\}\ .$$

PSL

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$$[\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)] = \left\{ \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right),\ \begin{pmatrix} 4a_{11} & 4a_{12} \\ 4a_{21} & 4a_{22} \end{pmatrix} \right\}.$$

It holds:

$$\mathsf{PSL}(2,\mathbb{F}_5) \cong \mathsf{PSL}(2,\mathbb{F}_4) \cong A_5$$

Translation to algebraic Geometry

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Let us consider $G = \textbf{PSL}(2, \mathbb{F}_p),\ p \geq 5$

Translation to algebraic Geometry

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Consider the matrices

$$x = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & b \\ c & 1+bc \end{pmatrix}$$

$x^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & t \end{smallmatrix}\right)$ implies $y \neq x^{-1}$ for all $(b, c, t) \in \mathbb{F}_p^3$.

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It is enough to prove that the equation

$$U_1(x, y) = U_2(x, y), \text{ i.e.}$$

$$x^{-1}yx^{-1}y^{-1}x^2 = yx^{-2}y^{-1}xy^{-1}$$

has a solution $(b, c, t) \in \mathbb{F}_p^3$.

The equations

The entries of $U_1(x,y) - U_2(x,y)$ are the following polynomials in $\mathbb{Z}[b,c,t]$ Let $I =< p_1, \ldots, p_4 >$ and $I^{(p)}$ the induced ideal over $\mathbb{Z}/p$:

$$
\begin{aligned}
p_1 = \quad & b^3c^2t^2 + b^2c^2t^3 - b^2c^2t^2 - bc^2t^3 - b^3ct + b^2c^2t + b^2ct^2 + 2bc^2t^2 \\
& +bct^3 + b^2c^2 + b^2ct + bc^2t - bct^2 - c^2t^2 - ct^3 - b^2t + bct + c^2t \\
& +ct^2 + 2bc + c^2 + bt +^2 ct + c + 1 \\
p_2 = \quad & -b^3ct^2 - b^2ct^3 + b^2c^2t + bc^2t^2 + b^3t - b^2ct - 2bct^2 - b^2c + bct \\
& +c^2t + ct^2 - bt - ct - b - c - 1 \\
p_3 = \quad & b^3c^3t^2 + b^2c^3t^3 - b^2c^2t^3 - bc^2t^4 - b^3c^2t + b^2c^3t +^2 b^2c^2t^2 \\
& +2bc^3t^2 +^2 bc^2t^3 + b^2c^2t +^2 b^2ct^2 + bc^2t^2 - c^2t^3 - ct^4 - 2b^2ct \\
& +bc^2t + c^3t + bct^2 + 2c^2t^2 + ct^3 - b^2c - b^2t + bct + c^2t + bt^2 \\
& +3ct^2 + bc - bt - b - c + 1 \\
p_4 = \quad & -b^3c^2t^2 - b^2c^2t^3 + b^2c^2t^2 + bc^2t^3 + b^3ct - b^2c^2t - b^2ct^2 - 2bc^2t^2 \\
& -bct^3 - 2b^2ct + c^2t^2 + ct^3 + b^2t - bct - c^2t - ct^2 + b^2 - bt \\
& -2ct - b - t + 1
\end{aligned}
$$

Hasse–Weil–Theorem

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Theorem von Hasse–Weil (generalized by Aubry and Perret for singulare curves):

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$$\#C(\mathbb{F}_q) \geq q + 1 - 2p_a\sqrt{q} - d$$

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We obtain $H(t) = 10t - 11 \Rightarrow d = 10$, $p_a = 12$.
Since $p + 1 - 24\sqrt{p} - 10 > 0$ if $p > 593$, we obtain the result.

absolute irreducibliy

Proposition: $V(I^{(p)})$ is absolutely irreducibel for all primes $p \geq 5$.

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$$\langle f_1, f_2 \rangle : h^2 = I.$$

$$\begin{aligned}
f_1 = \quad & t^2b^4 + (t^4 - 2t^3 - 2t^2)b^3 - (t^5 - 2t^4 - t^2 - 2t - 1)b^2 \\
& -(t^5 - 4t^4 + t^3 + 6t^2 + 2t)b + (t^4 - 4t^3 + 2t^2 + 4t + 1) \\
f_2 = \quad & (t^3 - 2t^2 - t)c + t^2b^3 + (t^4 - 2t^3 - 2t^2)b^2 \\
& -(t^5 - 2t^4 - t^2 - 2t - 1)b - (t^5 - 4t^4 + t^3 + 6t^2 + 2t) \\
h = \quad & t^3 - 2t^2 - t
\end{aligned}$$

We give explicitely matrices M and N with entries in $\mathbb{Z}[b,c,t]$ such that

$$M \begin{pmatrix} p_1 \\ \vdots \\ p_4 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \quad N \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} h^2 p_1 \\ \vdots \\ h^2 p_4 \end{pmatrix}$$

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We obtain for all fi elds K

$$IK[b,c,t] = \big(\langle f_1, f_2\rangle K[b,c,t]\big) : h^2 \,.$$

Schritt 2

f_2 is linear in c , it is enough to show, that f_1 is absolutely irreducibel.

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geometrically:
Curve $V(I)$ is irreducibel, if the projection to the b,t–plane is irreducibel.

Let $P(x) := t^2 J[1]|_{b=x/t}$ then P is monic of degree 4.

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$$x^4 + (t^3 - 2t^2 - 2t)x^3 - (t^5 - 2t^4 - t^2 - 2t - 1)x^2 - (t^6 - 4t^5 + t^4 + 6t^3 + 2t^2)x + (t^6 - 4t^5 + 2t^4 + 4t^3 + t^2).$$

We prove, that the induced polynomial $P \in \mathbb{F}_p[t, x]$ is absolutely irreducibel for all primes $p \geq 2$.
(Using the lemma of Gauß this is equivalent to P being irreducibel in $\overline{\mathbb{F}}_p(t)[x]$.)

Ansatz

$$(*) \quad P = (x^2 + ax + b)(x^2 + gx + d)$$

a, b, g, d polynomials in t with variable coeffi cients

$$a(i),\ b(i),\ g(i),\ d(i)\,.$$

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a, b, g, d polynomials in t with variable coeffi cients

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The decomposition ($*$) with `a(i), b(i), g(i), d(i)` $\in \overline{\mathbb{F}}_p$ does not exist iff the ideal `C` generated by the coeffi cients with respect to x, t of $P - (x^2 + ax + b)(x^2 + gx + d)$ has no solution in $\overline{\mathbb{F}}_p$. This is equivalent to the fact that $1 \in$ `C`.

The ideal of the coefficients of C:

```
C[1]=-b(5)*d(3)
C[2]=-b(5)*g(2)
C[3]=-b(4)*d(3)-b(5)*d(2)
C[4]=-b(4)*g(2)-b(5)*g(1)-d(3)-1
C[5]=-b(3)*d(3)-b(4)*d(2)-b(5)*d(1)+1
C[6]=-b(5)-g(2)-1
C[7]=a(0)*b(5)-a(2)*d(3)-b(3)*g(2)-b(4)*g(1)-d(2)+4
C[8]=-a(0)^2*b(5)+b(0)*b(5)-b(2)*d(3)-b(3)*d(2)-b(4)*d(1)-b(5)-4
C[9]=-a(2)*g(2)-b(4)-g(1)+2
C[10]=a(0)*b(4)-a(1)*d(3)-a(2)*d(2)-b(2)*g(2)-b(3)*g(1)-d(1)-1
C[11]=-a(0)^2*b(4)+b(0)*b(4)-b(1)*d(3)-b(2)*d(2)-b(3)*d(1)-b(4)+2
C[12]=a(0)-a(1)*g(2)-a(2)*g(1)-b(3)-d(3)
C[13]=-a(0)^2+a(0)*b(3)-a(0)*d(3)-a(1)*d(2)-a(2)*d(1)+b(0)-b(1)*g(2)-b(2)*g(1)-7
C[14]=-a(0)^2*b(3)+b(0)*b(3)-b(0)*d(3)-b(1)*d(2)-b(2)*d(1)-b(3)+4
C[15]=-a(2)-g(2)-2
C[16]=a(0)*a(2)-a(0)*g(2)-a(1)*g(1)-b(2)-d(2)+1
C[17]=-a(0)^2*a(2)+a(0)*b(2)-a(0)*d(2)-a(1)*d(1)+a(2)*b(0)-a(2)-b(0)*g(2)-b(1)*g(1)-2
C[18]=-a(0)^2*b(2)+b(0)*b(2)-b(0)*d(2)-b(1)*d(1)-b(2)+1
C[19]=-a(1)-g(1)-2
C[20]=a(0)*a(1)-a(0)*g(1)-b(1)-d(1)+2
C[21]=-a(0)^2*a(1)+a(0)*b(1)-a(0)*d(1)+a(1)*b(0)-a(1)-b(0)*g(1)
C[22]=-a(0)^2*b(1)+b(0)*b(1)-b(0)*d(1)-b(1)
C[23]=-a(0)^3+2*a(0)*b(0)-a(0)
C[24]=-a(0)^2*b(0)+b(0)^2-b(0)
```

Using SINGULAR, one shows that over $\mathbb{Z}\big[\{a(i)\},\ \{b(i)\},\ \{g(i)\},\ \{d(i)\}\big]$

$$4 = \sum_{i=1}^{24} M_i \ \mathtt{C}[i]\,.$$

Suzuki groups

Technische Universität Kaiserslautern

This case is much more complicated.
We have to prove that on a surface U any odd power of a certain endomorphism θ has fixed points.

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We have to prove that on a surface U any odd power of a certain endomorphism θ has fi xed points.
Here we use the **Lefschetz–Weil–Grothendieck trace formulae** generalized by Deligne–Lusztig, Th. Zink, Pink, Katz and Adolphson–Sperber:

$$2^n - b_1(U) \cdot 2^{\frac{3}{4}n} - b_2(U) \cdot 2^{\frac{1}{2}n} \leq \#\,\mathrm{Fix}\,(\theta^n, U)$$

for n suffi cientely large.

Model for a national economy

F. Kubler (Mannheim)
K. Schmedders (Kellogg School of Mathematics)

General problem:

- Study a computer model of a national economy
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Mathematical problem:
Find the positive real roots of a given system of polynomial equations

A small example

This example could be solved by the economists themselves

```
 ring R= 0,x(1..8),dp;
 ideal I =
  -1024+x(1)^5*x(3)*x(7),
 -1+x(2)^5*x(3)*x(8),
 -1+x(4)^5*x(6)*x(7),
 -1024+x(5)^5*x(6)*x(8),
 x(1)*x(7)+x(2)*x(8)-12*x(7)-x(8),
 x(1)+x(4)-13,
 x(2)+x(5)-13,
 x(7)+x(8)-1;
ideal J=std(I);
vdim(J);
// 25   there are 25 zeros
list L=solve(J,100,0,100);size(L);
// 25     the zeros are simple
```

A hard example

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```
ring R = 0,x(1..22),dp;
ideal I =
-1+x(1)^5*x(4)*x(13),-1+x(2)^5*x(4)*x(14),
-1+x(3)^5*x(4)*x(15),-1+x(5)^3*x(8)*x(13),
-1+x(6)^3*x(8)*x(14),-1+x(7)^3*x(8)*x(15),
-1+x(9)^4*x(12)*x(13),-1+x(10)^4*x(12)*x(14),
-1+x(11)^4*x(12)*x(15),
5+2*x(16)-x(1)*x(13)-x(2)*x(14)-x(3)*x(15),
3+5*x(16)-x(5)*x(13)-x(6)*x(14)-x(7)*x(15),
(x(1)+x(5)+x(9))^3-x(17)^2*x(18),
(x(2)+x(6)+x(10))^2-x(19)*x(20),
(x(3)+x(7)+x(11))^2-4*x(21)*x(22),
x(17)+x(19)+x(21)-10,x(18)+x(20)+x(22)-10,
8*x(13)^3*x(18)-27*x(16)^3*x(17),
x(13)^3*x(17)^2-27*x(18)^2,
x(14)^2*x(20)-4*x(16)^2*x(19),x(14)^2*x(19)-4*x(20),
x(15)^2*x(22)-x(16)^2*x(21),x(15)^2*x(21)-x(22);
```

A hard example

Problem:

- a purely numerical approach was not sucessful
- automatical symbolical preprocessing was not sucessful

A hard example

Problem:

- a purely numerical approach was not sucessful
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Computer-Human solution:

- using factorization to split the problem
- Subsitution of variables
- choose suitable fi eld extensions to simplify the problem

A degenerate example

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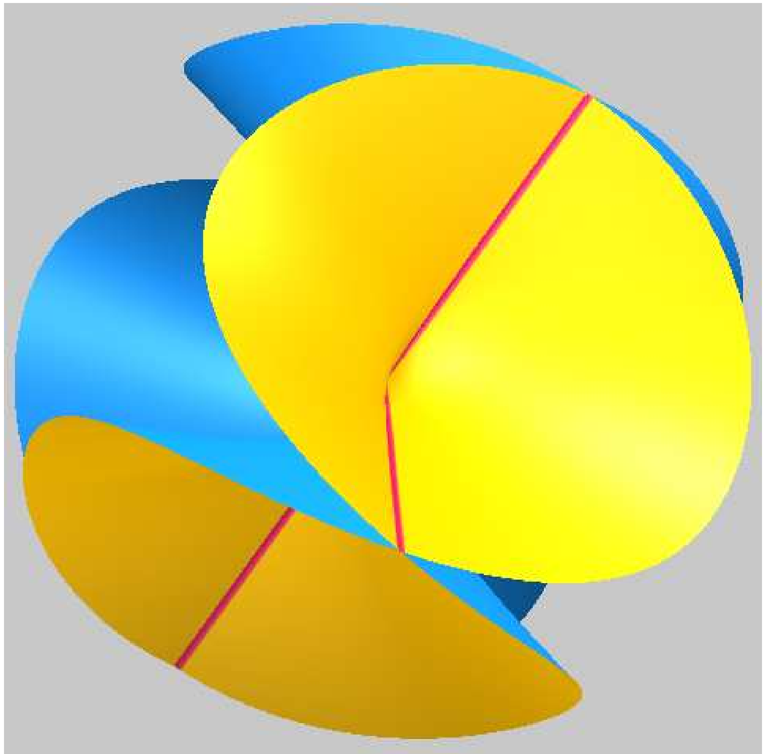
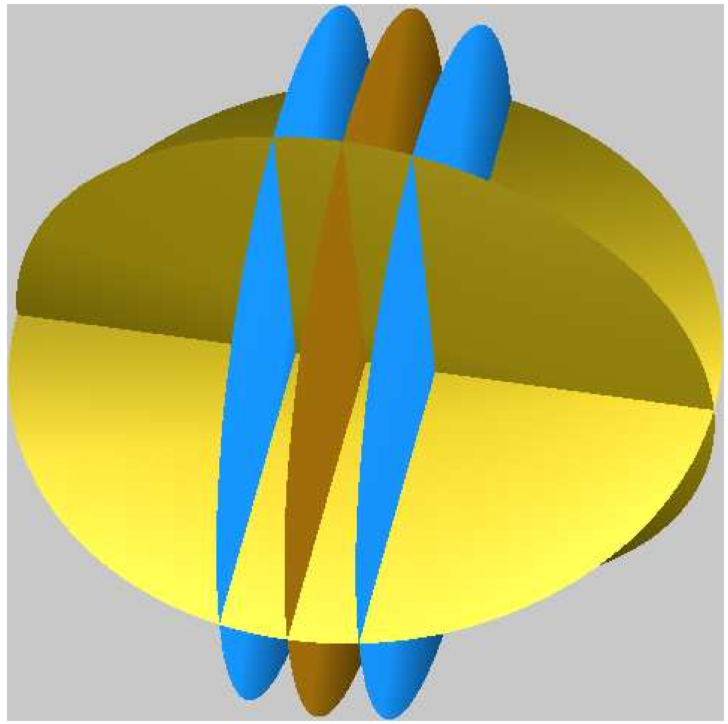
```
int i;
ring R=0,x,dp;
poly p=x+1;
for(i=2;i<=20;i++){p=p*(x+i);}
p=p+1/2^23*x19;
```

```
x20+1761607681/8388608x19+20615x18+1256850x17+53327946x16+1672280820x15
+40171771630x14+756111184500x13+11310276995381x12+135585182899530x11
+1307535010540395x10+10142299865511450x9+63030812099294896x8
+311333643161390640x7+1206647803780373360x6+3599979517947607200x5
+8037811822645051776x4+12870931245150988800x3+13803759753640704000x2
+8752948036761600000x+2432902008176640000
```

```
LIB"solve.lib";
list L=solve(p,1000,0,1000,"nodisplay");
LIB "rootsur.lib";
nrroots(p);
```

Resolution of $X = V(z^2 - x^2y^2) \subset K^3$





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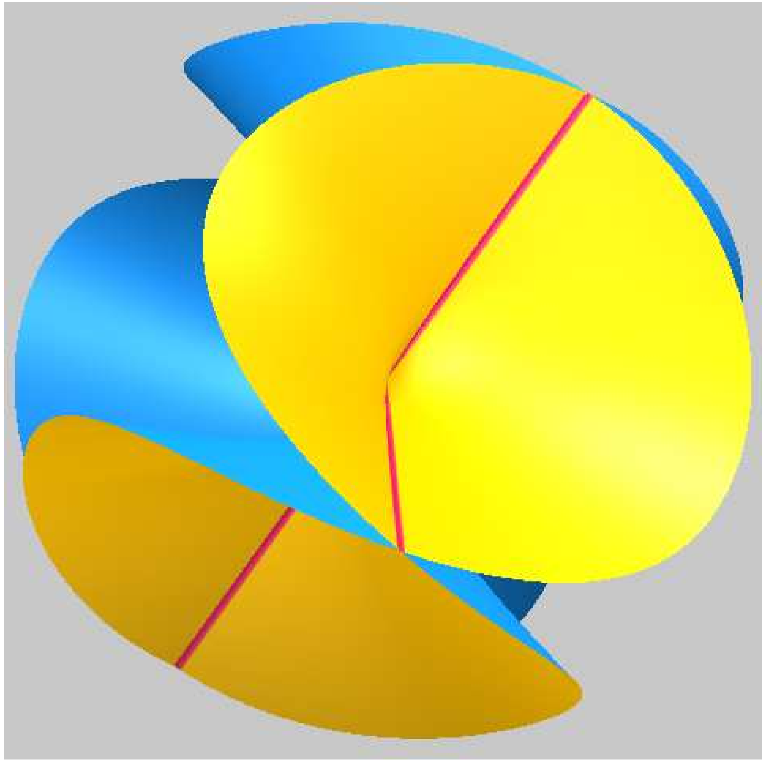

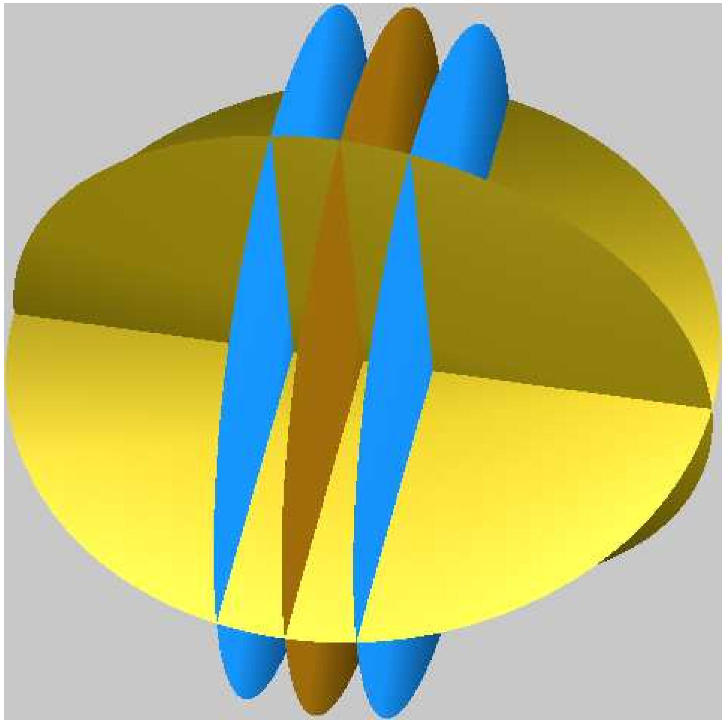

- resolve.lib

Sheaf cohomology

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- Let $S = \mathbb{K}[x_0, \ldots, x_n]$ and M be a finitely generated graded S–module.
  We want to compute

$$H^j(\widetilde{M}(k))$$

Sheaf cohomology

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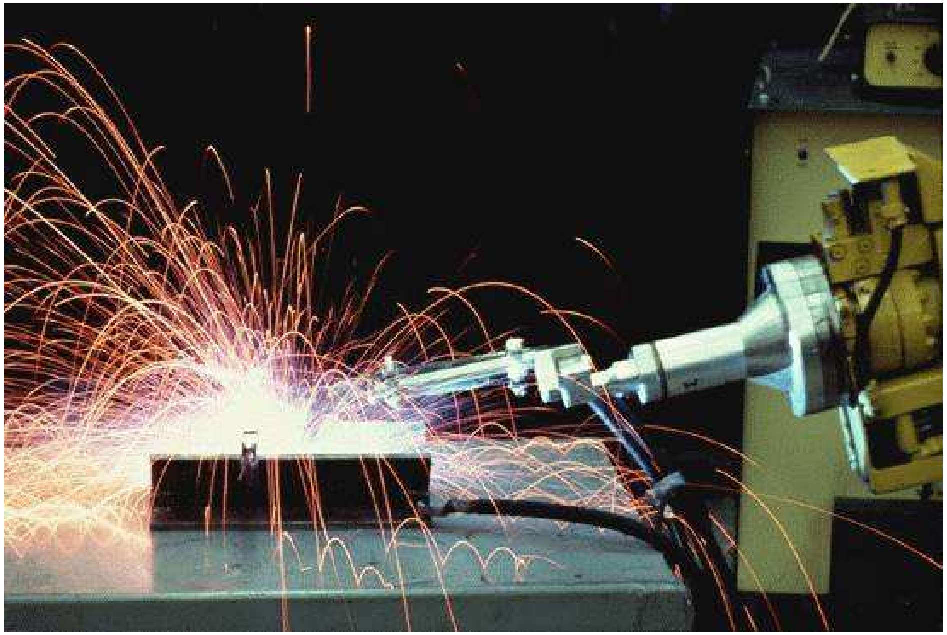
We want to compute

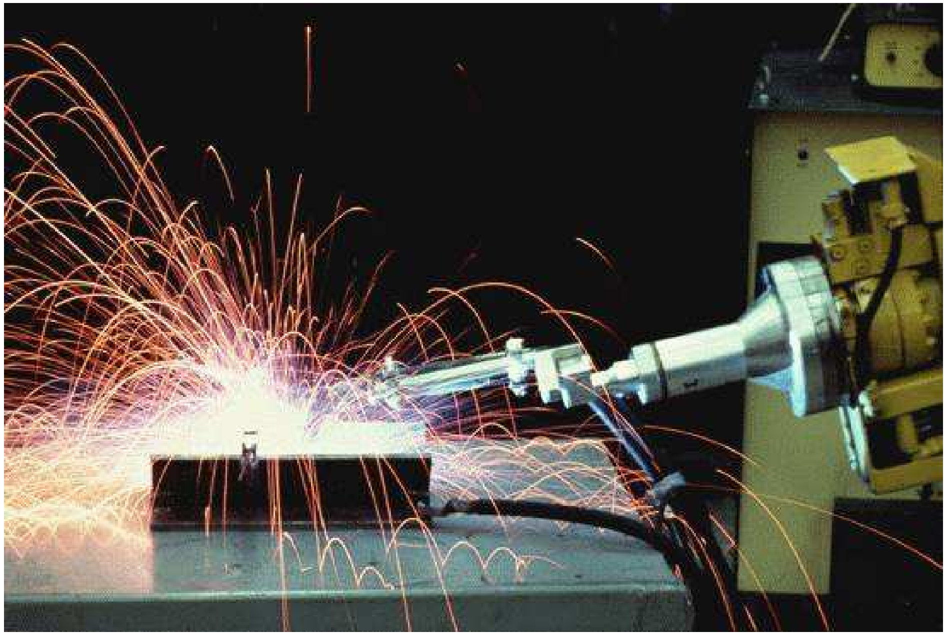
$$H^j(\widetilde{M}(k))$$

- Using non-commutative methods will be 50 times faster then the direct (commutative) approach.

Robotics and the Cycloheptane Molecule



Robotics and the Cycloheptane Molecule



A.H.M. Levelt

Robotics and the Cycloheptane Molecule

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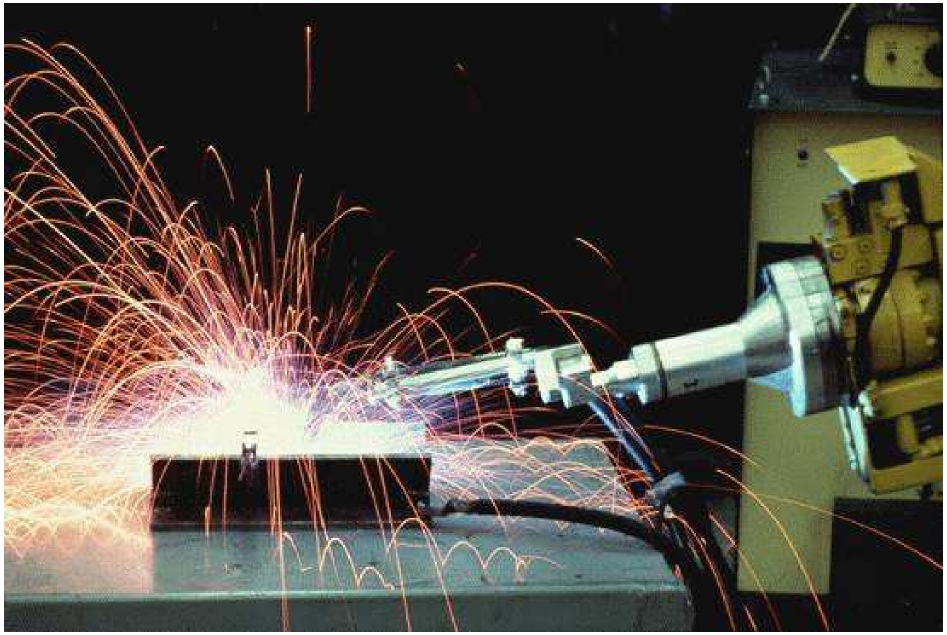

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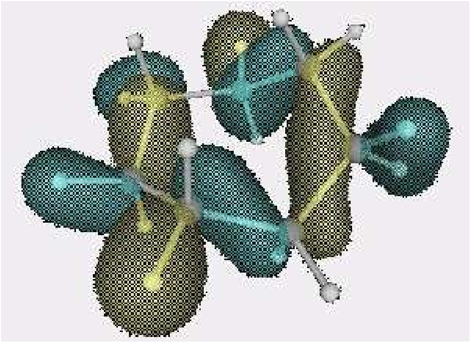
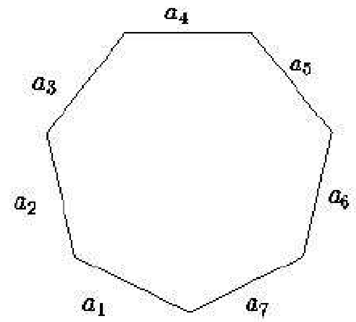
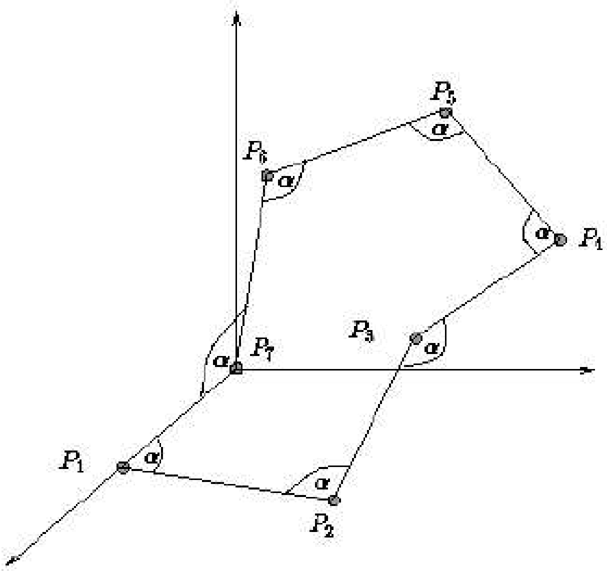




The Heptagon



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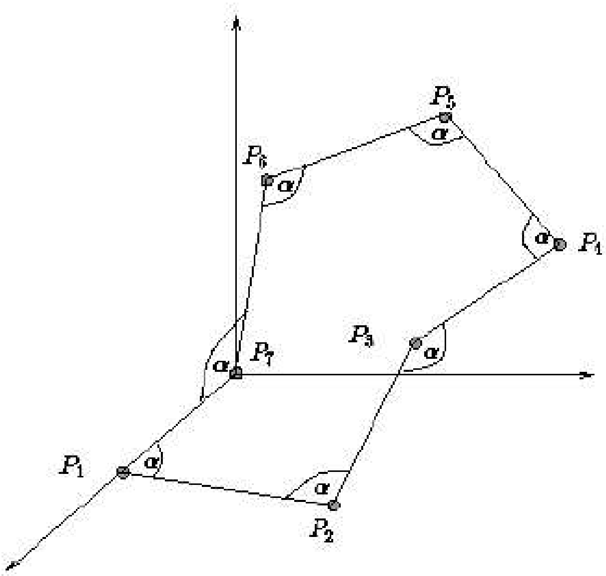
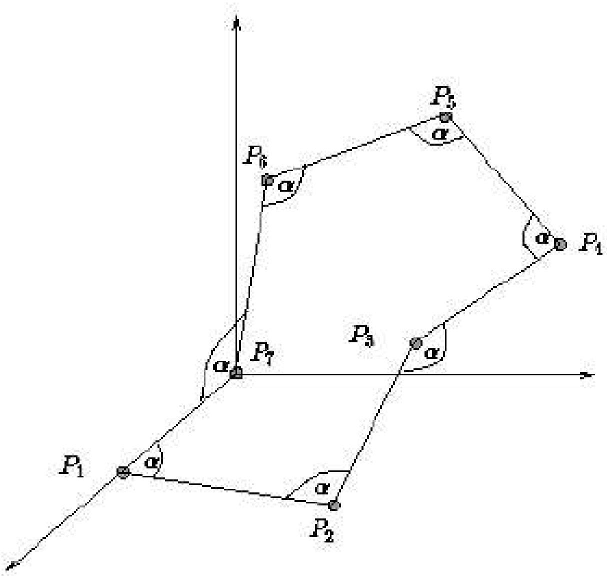


Equations for the vectors:

- $(a_1, a_2) = (a_2, a_3) = \ldots =$
  $(a_7, a_1) = c$
- $(a_1, a_1) = (a_2, a_2) = \ldots =$
  $(a_7, a_7) = 1$
- $a_1 + a_2 + \ldots + a_7 = 0$

$c = cos(\alpha)$ and $(,)$ is the scalar product.

The Heptagon



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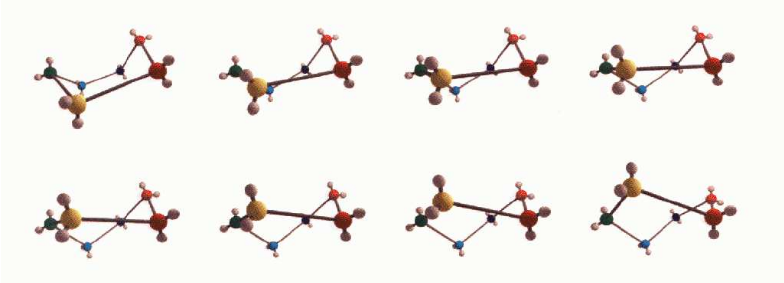
- $a_1 + a_2 + \ldots + a_7 = 0$

$c = cos(\alpha)$ and $(,)$ is the scalar product.

- For $c = 0$: equations for the configuration space of a robot
- For $c = \frac{1}{3}$: equations for the configurations space of a molecule

The Molecule

Equations for the confi guration space (in SINGULAR):



```
ring R=0,(v,w,x,y,z),dp;
ideal I=
81y2z2-54wyz+54y2z+54yz2-72w2+198wy-207y2+198wz-225yz-207z2+114w-141y-141z+10,
81w2x2+54w2x+54wx2-54wxz-207w2-225wx-207x2+198wz+198xz-72z2-141w-141x+114z+10,
324vw2x+432vw2+540vwx+432w2x-432wxy-432vwz+324wyz+180vw+846w2-576vx+180wx-
306wy+144xy+144vz-306wz-36yz+12v+585w+12x-318y-318z-79,
81v2w2+54v2w+54vw2-54vwy-207v2-225vw-207w2+198vy+198wy-72y2-141v-141w+114y+10;
```

The Projection

The equations describe a curve in $\mathbb{R}^5$. The projection to the w, x–plane is difficult to compute:

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13343098629642274643741505w20x16+18458805154059402163602552w20x15
+12528539096440613433050772w19x16-307469543636682571308498792w20x14
-308745089273555811810514188w19x15-335770469789305978523636514w18x16

.

.

.

-57603722394732542788396875000w2x-56209703485755917382271875000wx2
-29459059311819369252628125000x3-3456386878638867977468750000w2
-388065077492910629437500000wx-3500955605594366547468750000x2
+1264097844032306972500000000w+1126578705265908772500000000x
+240658492841196850000000000

Projection of the curve to the w, x–plane

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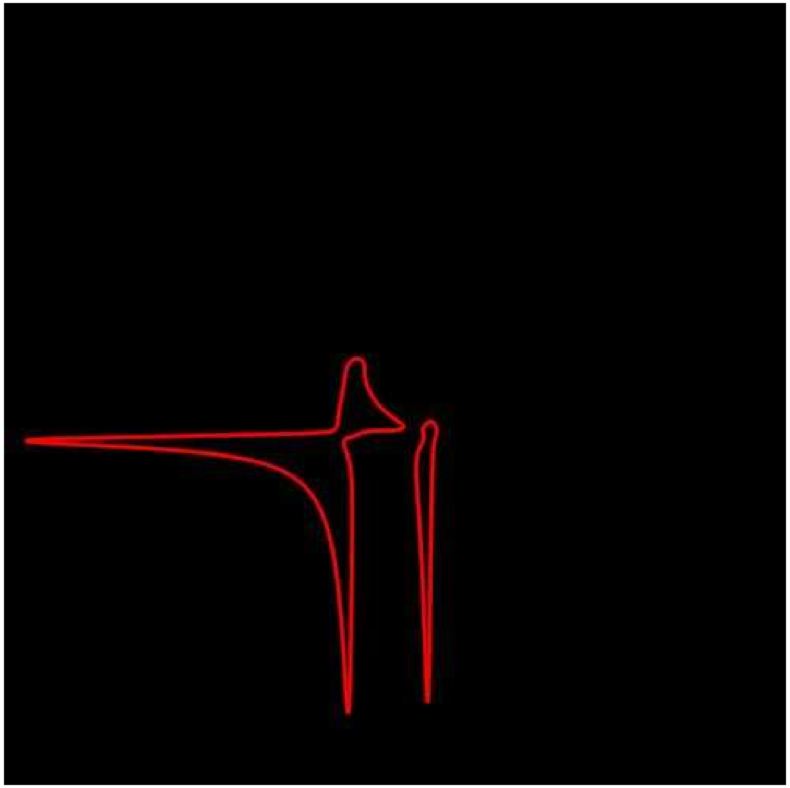

```
ideal K = eliminate(I,vyz);
LIB''surf.lib'';
plot(K[1]);
```

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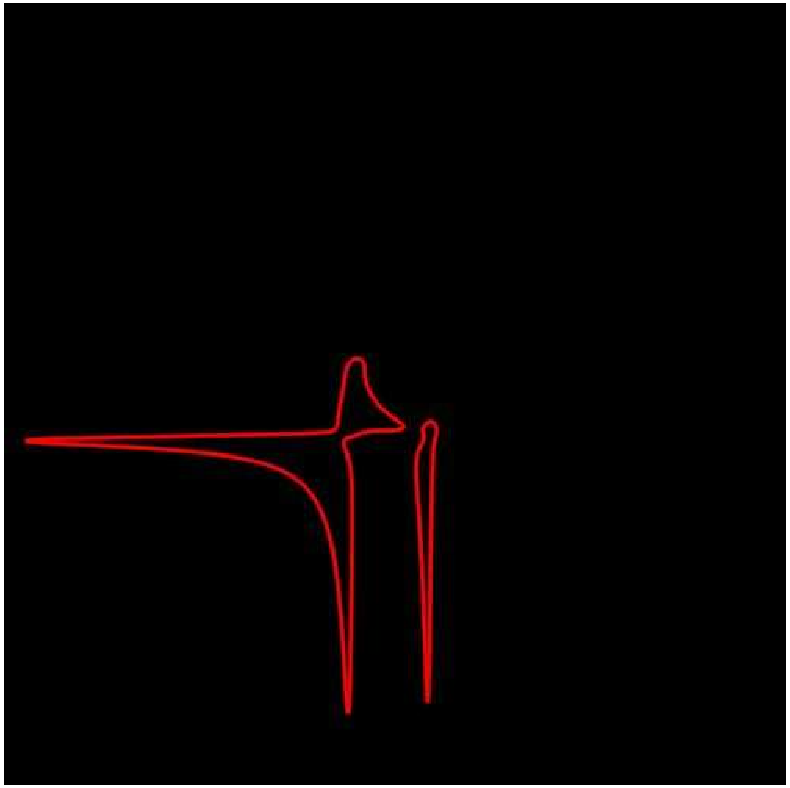

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plot(K[1]);
```

The curve shows the possible w, x–coordinates of the molecule.

Electronics: Analog circuits

The sizing leads to the following system of equations:

```
ring R=(0,VCC,vLF,p,VBE1,IB1,VCE1,IC1,VBE2,IB2,VCE2,IC2,vHF),
       (V_Vin, V_C1, V_BE_Q1, V_CE_Q1, V_R1, V_R2, V_R3, V_C3, V_R4, V_R5,
        V_R6, V_R7, V_R8, V_BE_Q2, V_CE_Q2, V_C2, V_VCC, I_Vin, I_C1,
        I_BE_Q1, I_CE_Q1, I_R1, I_R2, I_R3, I_C3, I_R4, I_R5, I_R6, I_R7,
        I_R8, I_BE_Q2, I_CE_Q2, I_C2, I_VCC,
        R1,R2,R3,R4,R5,R6,R7,R8,C3),lp;
ideal I=
  V_BE_Q1 + V_C1 + V_R1 - V_Vin,
  V_C3 - V_R2 + V_R3,
  -V_CE_Q1 - V_R2 - V_R4 + V_R5,
  V_C1 - V_R7 + V_R8 - V_Vin,
  V_BE_Q1 + V_BE_Q2 - V_CE_Q1 + V_R6 + V_R7,
  V_BE_Q1 + V_CE_Q2 + V_R2 + V_R6 + V_R7,
  V_C1 + V_C2 - V_R6 - V_R7 - V_Vin,
        .
        .
        .
  -R2 + R1*vLF,
  -R2*R3 + R1*(R2 + R3)*vHF,1 + C3*p*(R2 + R3),
  -VCC + 3*V_R5,R6 - R8;
```

Electronics

Elimination of variables: The resistors $R_1, \ldots, R_8$ and the capacities C_3 in terms of currents and voltages:.

```
option(redSB);
ideal J=std(I);
J=J[1..9];
simplify(J,1);
==>
_[1]=C3+(3*vLF*IB1+3*vLF*IC1-3*IB1*vHF-3*IC1*vHF)/(2*VCC*vLF^2*p-
3*vLF^2*p*VCE1+3*vLF^2*p*VBE2-3*vLF^2*p*VCE2-3*vLF*p*VCE1+3*vLF*p*VBE2-
3*vLF*p*VCE2)
                                .
                                .
                                .
_[6]=R4+(-VCC+3*VBE2-3*VCE2)/(3*IC1+3*IB2)
                                .
                                .
R6 = -(2*VCC-3*VCE2)/(3*IB1-6*IB2-6*IC2).
```

Control theory

$\mathcal{D} = K[\partial_1, \ldots, \partial_n] \qquad R \in \mathcal{D}^{g \times q} \qquad \mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, K)$

$$\mathcal{B} = \mathrm{Ker}(R) = \{\omega \in \mathcal{A}^q \mid R\omega = 0\}$$

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$\mathcal{B}$ controllable $\iff B = \text{im}(M)$ for some $M \in \mathcal{D}^{q\times l}$

system module $\mathcal{M} = \mathcal{D}^q/\mathcal{D}^g R\,,\ \mathcal{N} = \mathcal{D}^g/R\mathcal{D}^q$
the transposed one

$\mathcal{B}$ controllable $\iff \mathcal{M}$ torsion free

$\iff \text{Ext}^1_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) = 0$

Genus of a Curve

TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

- Let C be a projective curve, and let $H_C(t) = d(C)t - p_a(C) + 1$ be its Hilbert polynomial, then
  - $d(C)$ =: degree of the curve C
  - $p_a(C)$ =: arithmetic genus of the curve.
- The geometric genus $g(C)$ is the arithmetic genus of the normalization C_n of C:
  - $g(C) := p_a(C_n)$

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- The geometric genus $g(C)$ is the arithmetic genus of the normalization C_n of C:
  - $g(C) := p_a(C_n)$
- If we are able to compute the normalization, we can compute the geometric genus. But this is very time consuming.

Genus of a Curve

The procedure we implemented is based on the following knowledge:

- $p_a(C) = g(C) + \delta(C)$, where $\delta(C)$ is the sum over the local δ-invariants in the singular points.
- There exist a projection $C \longrightarrow D$ to a plane curve D with degree $d(D) = d(C)$, such that $C_n = D_n$. Then
  - $g(C) = p_a(C_n) = p_a(D_n) = g(D)$.
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  - $g(C) = p_a(C_n) = p_a(D_n) = g(D)$.
  - Almost every projection has this property.
- Let C be a plane projective curve. We compute the geometric genus by a local analysis of the singularities.

Genus of a Curve

Assume the plane curve C is defi ned by f=0.
We know:

- the ideal $Sing(f) := < f, f_x, f_y >$ defi nes the singular locus
- the ideal $Sing(Sing(f)) := < f, f_x, f_y, det(Hess(f)) >$ defi nes the non-nodal locus
- the ideal $S := Sing(Sing(Sing(f)))$ defi nes the non-nodal-cuspidal locus
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- $\delta(C, x) = 1$ in nodal or cuspidal singularities, so we just have to count them.
- the singular points different from cusps and nodes are obtained by a primary decomposition of S.
- The primary decomposition is done over Q. To obtain the points, we have to extend the fi eld.

Genus of a Curve

We know:

- $\mu(f) = dim(\mathbb{C}[[x,y]]/ < f_x, f_y >) = 2\delta$ - number of branches + 1,
- To compute the number of (local) branches, we proceed as follows:
  - Test for A_k- and D_k-singularities.
  - Compute the Newton Polygon.
  - If the Newton Polygon is non-degenerate, then the number of branches can be computed combinatorically from the faces.
  - If the Newton Polygon is degenerate and has more than one face, then f can be splitted (modulo analytic equivalence) into a product.
  - If the Newton Polygon is degenerate and has only one face, then we use the Puiseux expansion to compute the number of branches.

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- Needs:
  - Puiseux expansion
  - Primary decomposition
  - Field extensions
  - Newton polygon.