Coherent Configurations and Association Schemes

Part 3: Galois correspondence between coherent configurations and permutation groups

Christian Pech
Technische Universität Dresden
Institut für Algebra
01062 Dresden

(Finite) Relational Structures

Given:

- a finite set X,
- a finite index set $\mathcal{I}$,
- a family $\{R_i\}_{i\in\mathcal{I}}$ where $R_i \subseteq X^2$ $(i \in \mathcal{I})$.

The pair $(X, \{R_i\})_{i\in\mathcal{I}}$ is called (binary) *relational structure* over X.

We will drop the attribute binary and we say only "relational structure".

Coherent Configurations

Given:

- a finite set X,
- a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$ over X.

$(X, \{R_i\}_{i \in \mathcal{I}})$ is called

coherent configuration of rank $|\mathcal{I}|$ and of degree $|X|$

if:

CC1. $\forall i, j : R_i \cap R_j = \emptyset \iff i \neq j$,

CC2. $\bigcup_{i \in \mathcal{I}} R_i = X^2$,

CC3. $\forall i \exists j : R_j = R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\}$,

CC4. $\exists \mathcal{J} \subseteq \mathcal{I} : \bigcup_{j \in \mathcal{J}} R_j = \Delta = \{(x, x) \mid x \in X\}$,

CC5. For $(x, y) \in R_k$ the number $|\{z \mid (x, z) \in R_i, (z, y) \in R_j\}|$ does not depend on (x, y) but only on i, j, k. This number is denoted by $p_{i,j}^k$.

Automorphisms of Relational Structures

Given:

- a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$,
- a permutation $\varphi \in \mathrm{Sym}(X)$,

φ is called *automorphism* of $(X, \{R_i\}_{i \in \mathcal{I}})$ if

$$\forall i \in \mathcal{I}\, \forall (x, y) \in X^2 : (x, y) \in R_i \iff (x^\varphi, y^\varphi) \in R_i .$$

By $\mathrm{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$ we denote the set of all automorphisms of $(X, \{R_i\}_{i \in \mathcal{I}})$.

Permutation Groups

Given:

- a set X,
- a subset $G \subseteq \mathrm{Sym}(X)$.

(G, X) is called *permutation group* on X if:

1. $() \in G$,
2. $\varphi \in G \Rightarrow \varphi^{-1} \in G$,
3. $\varphi, \psi \in G \Rightarrow \varphi \cdot \psi \in G$.

For given $(X, \{R_i\}_{i \in \mathcal{I}})$

$$\mathrm{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$$

is always a permutation group.

2-closed Permutation Groups

Given: a permutation group (G, X).

(G, X) is called *2-closed* if:

there exists a relational structure $(X, \{R_i\}_{i\in\mathcal{I}})$ such that

$$G = \mathrm{Aut}(X, \{R_i\}_{i\in\mathcal{I}}).$$

Let us give a characterization of 2-closed permutation groups using coherent configurations.

Contexts

A context is a triple (G, M, I) where

- a set G of objects (Gegenstände),
- a set M of attributes (Merkmale),
- a relation $I \subseteq G \times M$

For $A \subseteq G$ we define $A' := \{m \in M \mid \forall g \in A : gIm\}$.
For $B \subseteq M$ we define $B' := \{g \in G \mid \forall m \in B : gIm\}$.

The two operators define a Galois-correspondence between $\mathcal{P}(G)$ and $\mathcal{P}(M)$, i.e.

1. $\forall A_1, A_2 \subseteq G : A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ (antitonicity 1),
2. $\forall B_1, B_2 \subseteq M : B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$ (antitonicity 2),
3. $\forall A \subseteq G : A \subseteq A''$ (extensivity 1)
4. $\forall B \subseteq M : B \subseteq B''$ (extensivity 2)

Following we will treat a particular Galois-correspondence between *sets of permutations* and *sets of binary relations*.

The context of permutations and binary relations

Given a finite set X.

We consider the context $\mathbb{K}_X = (\mathrm{Sym}(X), \mathcal{P}(X^2), \triangleright)$.

For $\varphi \in \mathrm{Sym}(X)$, $R \subseteq X^2$ we define

$$\varphi \triangleright R :\Longleftrightarrow \ (x, y) \in R \Rightarrow (x^\varphi, y^\varphi) \in R$$

(φ preserves R / R is invariant for φ.)

Let us study the Galois-closed sets of permutations and of binary relations.

Galois-closed sets

Closed sets of permutations:

$G \subseteq \mathrm{Sym}(X)$ is Galois-closed $\iff$

$$\exists \{R_i \mid i \in \mathcal{I}\} \subseteq \mathcal{P}(X^2) : G = \mathrm{Aut}(X, \{R_i\}_{i \in \mathcal{I}})$$

i.e. (G, X) is a 2-closed permutation group.

Closed sets of binary relations:

$\{R_i\}_{i \in \mathcal{I}}$ is Galois-closed $\iff$ there exists a 2-closed permutation group (G, X) such that $\{R_i\}_{i \in \mathcal{I}}$ is the set of all binary invariant relations of G.

Let us study these sets in more detail.

Galois closed sets of binary relations

Given:

- a finite set X
- a 2-closed permutation group (G, X)

The binary invariant relations of G are closed under union, intersection and complement (i.e. they form a Boolean lattice).

Since the lattice is finite, it is completely determined by its atoms.

The atoms are called *2-orbits* of (G, X)

If $\{R_i\}_{i\in\mathcal{I}}$ is the complete set of 2-orbits of G, then $(X, \{R_i\}_{i\in\mathcal{I}})$ is a coherent configuration.

Such coherent configurations are called *Schurian coherent configurations*. Schurian coherent configurations inherit the order from the Galois-closed sets of binary relations.

Order on coherent configurations

We extend the natural order on Schurian coherent configurations to all coherent configurations:

Given: $\mathcal{C}_1 = (X, \{R_i\}_{i\in\mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j\in\mathcal{J}})$ coherent configurations.

Let

- $\mathbb{B}(\mathcal{C}_1)$ be all relations that are obtained from $\{R_i\}_{i\in\mathcal{I}}$ by union,
- $\mathbb{B}(\mathcal{C}_2)$ be all relations that are obtained from $\{S_j\}_{j\in\mathcal{J}}$ by union.

$$\text{We define } \mathcal{C}_1 \leq \mathcal{C}_2 \text{ iff } \mathbb{B}(\mathcal{C}_1) \subseteq \mathbb{B}(\mathcal{C}_2).$$

Question: Is $\leq$ a lattice ordering?

From coherent configurations to coherent algebras

Given: a coherent configuration $(X, \{R_i\}_{i \in \mathcal{I}})$

W.l.o.g. $I = \{1, \dots, r\}$, $X = \{1, \dots, n\}$.

To every R_i we associate $A_i = (a_{x,y}) \in \mathbb{C}^{n \times n}$ by

$$a_{x,y} = \begin{cases} 1 & (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Consider the linear span

$$W = \langle A_1, \dots, A_r \rangle$$

Then W is a self-adjoint unitary matrix-algebra that is closed with respect to the Hadamard-product: $(a_{x,y}) \circ (b_{x,y}) = (a_{x,y} b_{x,y})$.

Selfadjoint unitary matrix algebras that are closed w.r.t. Hadamard-product are called *coherent algebras*.

From coherent algebras to coherent configurations

Every coherent algebra W has a (canonical) basis of zero-one matrices $\langle A_1, \ldots, A_r \rangle$ such that

$$A_i \circ A_j = \begin{cases} 0 & i \neq j \\ A_i & \text{otherwise.} \end{cases}$$

This basis is called *(first) standard basis* of W.

To each matrix $A_i = (a_{x,y})$ we associate $R_i \subseteq X^2$ by

$$R_i = \{(x, y) \mid a_{x,y} = 1\}.$$

With $X = \{1, \ldots, n\}$ and $\mathcal{I} = \{1, \ldots, r\}$ we have that $(X, \{R_i\}_{i \in \mathcal{I}})$ is a coherent configuration.

Note: The correspondence between coherent configurations and coherent algebras is one-to-one (up to isomorphism).

Order on coherent algebras

Coherent algebras are naturally ordered by inclusion.

Clearly, this order is a lattice ordering.

The infimum in this lattice is the setwise-intersection.

The supremum is the coherent closure.

If W_1 and W_2 are coherent algebras und $\mathcal{C}_1$ und $\mathcal{C}_2$ are the associated coherent configurations then

$$W_1 \subseteq W_2 \iff \mathcal{C}_1 \leq \mathcal{C}_2.$$

In particular the coherent configurations over a set X form a complete lattice.

Infimum of coherent configurations (direct construction)

Given $\mathcal{C}_1 = (X, \{R_i\}_{i\in\mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j\in\mathcal{J}})$ coherent configurations.

Consider the bipartite graph Γ that has as vertex set $\mathcal{I}\dot{\cup}\mathcal{J}$
where $i \in \mathcal{I}$ is connected to $j \in \mathcal{J}$ if $R_i \cap S_j \neq \emptyset$.

Let $\{\mathcal{I}_1 \cup \mathcal{J}_1, \mathcal{I}_2 \cup \mathcal{J}_2, \ldots, \mathcal{I}_k \cup \mathcal{J}_k\}$ be the connected components of Γ.

Define

$$T_l = \bigcup_{i\in\mathcal{I}_l} R_i = \bigcup_{j\in\mathcal{J}_l} S_j \qquad (l = 1, \ldots, k).$$

Then

$$\mathcal{C} = (X, \{T_l\}_{l=1}^{k}) = \mathcal{C}_1 \wedge \mathcal{C}_2.$$

Coherent closure

Given a set $M \subseteq \mathbb{C}^{n \times n}$.

There is a smallest coherent algebra $\langle\langle M \rangle\rangle$ containing M.

In particular

$$\langle\langle M \rangle\rangle = \bigcap \{W \mid M \subseteq W,\, W \text{ coherent algebra}\}.$$

This is well-defined since $\mathbb{C}^{n \times n}$ is itself a coherent algebra.

Coherent closure (constructive version)

Given a relational structure $(X, \{R_i\}_{i \in \mathcal{I}})$.

The goal is to find the smallest coherent configuration $\mathcal{C}$ such that $\{R_i\}_{i \in \mathcal{I}} \subseteq \mathbb{B}(\mathcal{C})$.

Step 1) Construct the coarsest partition $\{S_j\}_{j \in \mathcal{J}}$ of X^2 such that each R_i is a union of parts.

Step 2) Construct the coarsest refinement $\{T_k\}_{k \in \mathcal{K}}$ of $\{S_j\}_{j \in \mathcal{J}}$ such that for each T_k we have either $T_k \subseteq \Delta$ or $T_k \cap \Delta = \emptyset$.

Step 3) Construct the coarsest refinement $\{U_l\}_{l \in \mathcal{L}}$ of $\{T_k\}_{k \in \mathcal{K}}$ such that each U_l is either symmetric or asymmetric.

Step 4) Perform the Weisfeiler-Leman algorithm on $\{U_l\}_{l \in \mathcal{L}}$.

Weisfeiler Leman Algorithm (Ideas)

After Step 1 to 4 the relational structure $(X, \{U_l\}_{l \in \mathcal{L}})$ already fulfills almost all axioms of coherent configurations. The following axiom might still not be fulfilled:

CC5. For $(x, y) \in U_k$ the number $|\{z \mid (x, z) \in U_i,\ (z, y) \in U_j\}|$ does not depend on (x, y) but only on i, j, k.

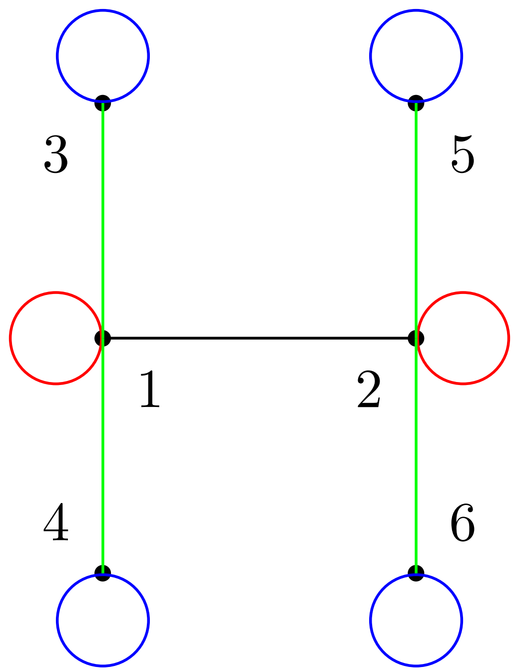
The algorithm proceeds stepwise refining the partition $\{U_l\}_{l \in \mathcal{L}}$ in every step getting closer to the fulfillment of Axiom CC5 without spoiling the other axioms.

The algorithm stops when a stabile partition is reached.

WL-stabilization algorithm (Example)

We start with a relational structure that fulfills CC1,. . . ,CC4.

We consider it as colored graph.



$$
\begin{pmatrix}
0 & 3 & 2 & 2 & 4 & 4 \\
3 & 0 & 4 & 4 & 2 & 2 \\
2 & 4 & 1 & 4 & 4 & 4 \\
2 & 4 & 4 & 1 & 4 & 4 \\
4 & 2 & 4 & 4 & 1 & 4 \\
4 & 2 & 4 & 4 & 4 & 1
\end{pmatrix}
$$

Example continued

We replace the entries of the adjacency matrix by non-commuting variable symbols:

$$\begin{pmatrix} 0 & 3 & 2 & 2 & 4 & 4 \\ 3 & 0 & 4 & 4 & 2 & 2 \\ 2 & 4 & 1 & 4 & 4 & 4 \\ 2 & 4 & 4 & 1 & 4 & 4 \\ 4 & 2 & 4 & 4 & 1 & 4 \\ 4 & 2 & 4 & 4 & 4 & 1 \end{pmatrix} \mapsto \begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix}$$

Example continued

We compute the square of this matrix:

$$\begin{pmatrix} t_0 & t_3 & t_2 & t_2 & t_4 & t_4 \\ t_3 & t_0 & t_4 & t_4 & t_2 & t_2 \\ t_2 & t_4 & t_1 & t_4 & t_4 & t_4 \\ t_2 & t_4 & t_4 & t_1 & t_4 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_1 & t_4 \\ t_4 & t_2 & t_4 & t_4 & t_4 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix}$$

where

$$x_0 = t_0^2 + 2t_2^2 + t_3^2 + 2t_4^2$$

$$x_1 = t_1^2 + t_2^2 + 4t_3^2$$

$$x_2 = t_0 t_3 + 2t_2 t_4 + t_3 t_0 + 2t_4 t_2, \qquad \text{etc.}$$

Example continued

We replace the entries with numbers:

$$\begin{pmatrix} x_0 & x_2 & x_3 & x_3 & x_4 & x_4 \\ x_2 & x_0 & x_4 & x_4 & x_3 & x_3 \\ x_5 & x_6 & x_1 & x_7 & x_8 & x_8 \\ x_5 & x_6 & x_7 & x_1 & x_8 & x_8 \\ x_6 & x_5 & x_8 & x_8 & x_1 & x_7 \\ x_6 & x_5 & x_8 & x_8 & x_7 & x_1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 2 & 3 & 3 & 4 & 4 \\ 2 & 0 & 4 & 4 & 3 & 3 \\ 5 & 6 & 1 & 7 & 8 & 8 \\ 5 & 6 & 7 & 1 & 8 & 8 \\ 6 & 5 & 8 & 8 & 1 & 7 \\ 6 & 5 & 8 & 8 & 7 & 1 \end{pmatrix}$$

Apply this procedure again until no further colors are introduced.

(In fact the above obtained matrix is already stabile)

A remark on the complexity of the WL-stabilization

The above algorithm was implemented directly using relational structures by L. Babel, I. V. Chuvaeva, and M. Klin. It is called STABIL.

The worst case-complexity of this algorithm is roughly $O(n^7)$.

From the point of view of complexity, the best known algorithm is STABCOL (L. Babel, S. Baumann, M. Lüdecke, and G. Tinhofer). It has complexity $O(n^3 \log n)$.

Though STABIL has a much worse theoretical worst-case complexity, it performs in most applications much quicker than STABCOL.

Enumeration of fusions of coherent configurations

Given $\mathcal{C}_1 = (X, \{R_i\}_{i\in\mathcal{I}})$, $\mathcal{C}_2 = (X, \{S_j\}_{j\in\mathcal{J}})$ such that $\mathcal{C}_1 \le \mathcal{C}_2$.

Then $\mathcal{C}_1$ is determined by a partition $\{\mathcal{J}_i\}_{i\in\mathcal{I}}$ of $\mathcal{J}$ according to

$$R_i = \bigcup_{j\in J_i} S_j.$$

$\mathcal{C}_1$ is a fusion of $\mathcal{C}_2$ with respect to $\{\mathcal{J}_i\}_{i\in\mathcal{I}}$.

In order to find all fusions of $\mathcal{C}_2$, we have to find all partitions of $\mathcal{J}$ whose fusions are coherent configurations.

Strategy for the enumeration of partitions of $\mathcal{J}$

Step 1 Find feasible candidates for classes $\mathcal{J}_i$.

Step 2 Construct all partitions of $\mathcal{J}$ that consist only of feasible blocks.

Step 3 Test each such partition if its fusion is a coherent configuration.

A possible (weak) feasibility-condition for blocks is:

$\mathcal{K} \subseteq \mathcal{J}$ is feasible if

$$\forall k_1, k_2 \in \mathcal{K} : \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} p_{i,j}^{k_1} = \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}} p_{i,j}^{k_2}.$$

How to test a feasible partition?

Given a partition $\{\mathcal{J}_1, \ldots, \mathcal{J}_p\}$ of $\mathcal{J}$ into feasible classes.

We have to test if $\forall l, m, o \in \{1, \ldots, p\}, \forall k_1, k_2 \in \mathcal{J}_o$:

$$\sum_{i \in \mathcal{J}_l} \sum_{j \in \mathcal{J}_m} p_{i,j}^{k_1} = \sum_{i \in \mathcal{J}_l} \sum_{j \in \mathcal{J}_m} p_{i,j}^{k_2}$$

This number is equal to the structure constant $\hat{p}_{l,m}^{o}$ of the fusion.

Enumeration of fusions as problem in symbolic computation

Let us consider the following scenario: Given an infinite series $(\mathcal{C}_{s_1,\ldots,s_l})$ of coherent configurations — all of the same rank.

Suppose the structure constants of the coherent configurations are given as polynomials in $\mathbb{Z}[s_1,\ldots,s_l]$. More precisely, for all i, j, k we have a polynomial $p_{i,j}^k \in \mathbb{Z}[s_1,\ldots,s_l]$ such that the corresponding structure constant of $\mathcal{C}_{s_1,\ldots,s_l}$ is $p_{i,j}^k(s_1,\ldots,s_l)$.

We want to find all fusions of all coherent configurations simultaneously.

Checking the feasibility-conditions involves solving Diophantine systems of polynomial equations.

Let us consider this scenario on hand of an extended example.

Steiner 2-Designs

A *Steiner 2-design* of order (s, t) is an incidence structure $(\mathcal{P}, \mathcal{B}, I)$ where

- $\mathcal{P}$ is a set of points,
- $\mathcal{B}$ is a set of lines,
- each line has $s + 1$ points,
- each point lies on $t + 1$ lines,
- through any two points goes exactly one line.

Flag-algebras

Given a Steiner 2-design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ of order (s, t).

Consider the set of flags $\mathcal{F} = \{(P, b) \mid P \in \mathcal{P},\ b \in \mathcal{B},\ PIb\}$

Define two binary relations on $\mathcal{F}$:

$$R_L = \{((P_1, b_1), (P_2, b_2)) \mid P_1 \neq P_2,\ b_1 = b_2\},$$
$$R_N = \{((P_1, b_1), (P_2, b_2)) \mid P_1 = P_2,\ b_1 \neq b_2\}.$$

Let L and N be the adjacency-matrices of R_L and R_N, respectively.

The *Flag-algebra* of $\mathcal{D}$ is defined as:

$$\mathcal{W}_{\mathcal{F}}(\mathcal{D}) = \langle\langle L, N \rangle\rangle \qquad \text{(coherent closure).}$$

This algebra has rank 7. It defines a dihedral scheme in the sense of Zieschang. In particular

$$\mathcal{W}_{\mathcal{F}}(\mathcal{D}) = \langle A_0, \ldots, A_6 \rangle \quad \text{where}$$

$A_0 = I$, $A_1 = L$, $A_2 = N$, $A_3 = LN$, $A_4 = NL$, $A_5 = LNL$,
$A_6 = NLN - LNL$

i/j	0	1	2	3	4	5	6
	1	0	0	0	0	0	0
	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
	0	0	0	1	0	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	1	0
	0	0	0	0	0	0	1
	0	s	0	0	0	0	0
	1	$s-1$	0	0	0	0	0
1	0	0	0	s	0	0	0
	0	0	1	$s-1$	0	0	0
	0	0	0	0	0	s	0
	0	0	0	0	1	$s-1$	0
	0	0	0	0	0	0	s
	0	0	t	0	0	0	0
	0	0	0	0	t	0	0
2	1	0	$t-1$	0	0	0	0
	0	0	0	0	0	s	$t-s$
	0	1	0	0	$t-1$	0	0
	0	0	0	1	0	$s-1$	$t-s$
	0	0	0	1	0	s	$t-s-1$
	0	0	0	0	ts	0	0
	0	0	t	0	$t(s-1)$	0	0
3	0	0	0	0	0	s^2	$s(t-s)$
	1	0	$t-1$	0	0	$(s-1)s$	$(s-1)(t-s)$
	0	0	0	s	0	$(s-1)s$	$s(t-s)$
	0	1	0	$s-1$	$t-1$	$(s-1)^2$	$(s-1)(t-s)$
	0	0	0	s	0	s^2	$s(t-s-1)$
	0	0	0	ts	0	0	0
	0	0	0	0	0	ts	0
4	0	s	0	$(t-1)s$	0	0	0
	0	0	0	0	s	$(s-1)s$	$s(t-s)$
	1	$s-1$	0	0	0	$(t-1)s$	0
	0	0	1	$s-1$	$s-1$	$(s-1)^2$	$s(t-s)$
	0	0	1	$s-1$	s	$(s-1)s$	$s(t-s-1)$

Continued on next slide...

i/j	0	1	2	3	4	5	6
	0	0	0	0	0	ts^2	0
	0	0	0	ts	0	$ts(s-1)$	0
5	0	0	0	0	s^2	$(s-1)s^2$	$s^2(t-s)$
	0	s	0	$(t-1)s$	$(s-1)s$	$(s-1)^2 s$	$s(s-1)(t-s)$
	0	0	s	$(s-1)s$	$(s-1)s$	$(s-1)^2 s$	$s^2(t-s)$
	1	$s-1$	$s-1$	$(s-1)^2$	$(s-1)^2$	$p^5_{5,5}$	$s(s-1)(t-s)$
	0	0	s	$(s-1)s$	s^2	$(s-1)s^2$	$s^2(t-s-1)$
	0	0	0	0	0	0	$st(t-s)$
	0	0	0	0	0	0	$st(t-s)$
6	0	0	0	0	$s(t-s)$	$s^2(t-s)$	$s(s-t+1)(s-t)$
	0	0	0	0	$s(t-s)$	$s^2(t-s)$	$s(s-t+1)(s-t)$
	0	0	$t-s$	$s(t-s)$	$(s-1)(t-s)$	$s(s-1)(t-s)$	$s(s-t+1)(s-t)$
	0	0	$t-s$	$s(t-s)$	$(s-1)(t-s)$	$s(s-1)(t-s)$	$s(s-t+1)(s-t)$
	1	s	$t-s-1$	$s(t-s-1)$	$s(t-s-1)$	$s^2(t-s-1)$	$p^6_{6,6}$

where $p^6_{6,6} = s^3 - 2ts + 2s - 2ts^2 + 3s^2 - t + t^2 s$ and $p^5_{5,5} = ts + 2s - 1 - 3s^2 + s^3$

Table 4.1: The table of structure constants for Steiner 2-designs

Feasible sets

Let us check if $\{1,2\}$ can be a class of a partition of a fusion.

We take the vectors:

$$(p_{1,1}^k)_{k=0}^6, \quad (p_{1,2}^k)_{k=0}^6, \quad (p_{2,1}^k)_{k=0}^6, \quad \text{and} \quad (p_{2,2}^k)_{k=0}^6.$$

We sum them up:

$$\begin{bmatrix} s \\ s-1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 0 \\ t-1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s+t \\ \mathbf{s-1} \\ \mathbf{t-1} \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

So we get that $\{1,2\}$ is feasible only if $s = t$.

Another example of a feasibility test

Let us check if $\{3,4,5\}$ can be a class of a partition of a fusion.

We take the vectors:

$$(p_{i,j}^{k})_{k=0}^{6}, \quad \text{for } i,j \in \{3,4,5\}.$$

We sum them up and obtain:

$$\begin{bmatrix} st(s+2) \\ t(s^2+2s-1) \\ s(s^2+s+t-1) \\ \mathbf{s(s^2+s+t-2)} \\ \mathbf{s(s^2+s+t-2)} \\ \mathbf{s^3+s^2+st-3s+t-1} \\ s^3+3s^2+s-1 \end{bmatrix}$$

Fusions of the flag-algebras os Steiner 2-Designs

$$
\begin{aligned}
W_0 &= \langle (A_0),(A_1),(A_2),(A_3),(A_4),(A_5),(A_6)\rangle \\
W_1 &= \langle (A_0),(A_1),(A_2+A_3+A_4+A_5),(A_6)\rangle \\
W_2 &= \langle (A_0),(A_1),(A_2+A_3+A_4+A_5+A_6)\rangle \\
W_3 &= \langle (A_0),(A_1),(A_2+A_5),(A_3+A_4),(A_6)\rangle \quad \text{for } (s,t)=(1,t) \\
W_4 &= \langle (A_0),(A_1+A_2+A_3+A_4+A_5+A_6)\rangle \\
W_5 &= \langle (A_0),(A_1+A_2+A_3+A_4+A_6),(A_5)\rangle \quad \text{for } (s,t)=(1,t) \\
W_6 &= \langle (A_0),(A_1+A_3+A_4+A_5+A_6),(A_2)\rangle \\
W_7 &= \langle (A_0),(A_1+A_5),(A_2+A_3+A_4+A_6)\rangle \quad \text{for } (s,t)=(2,t) \\
W_8 &= \langle (A_0),(A_1+A_6),(A_2),(A_3+A_4+A_5)\rangle \quad \text{for } (s,t)=(s,s+1) \\
W_9 &= \langle (A_0),(A_1+A_6),(A_2+A_3+A_4+A_5)\rangle \quad \text{for } (s,t)=(s,s+1) \\
W_{10} &= \langle (A_0),(A_1+A_6),(A_2+A_3+A_4),(A_5)\rangle \quad \text{for } (s,t)=(1,2) \\
W_{11} &= \langle (A_0),(A_1+A_6),(A_2+A_5),(A_3+A_4)\rangle \quad \text{for } (s,t)=(3,4)
\end{aligned}
$$

Isomorphisms between Coherent Configurations

Given coherent configurations $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (Y, \{S_j\}_{j \in \mathcal{J}})$.

We say that $\mathcal{C}_1$ is isomorphic to $\mathcal{C}_2$ ($\mathcal{C}_1 \cong \mathcal{C}_2$) if

$$\exists \varphi : X \to Y : \varphi \text{ bijective}, \forall i \in \mathcal{I}\, \exists j \in \mathcal{J}\, \forall (x, y) \in R_i : (x^\varphi, y^\varphi) \in S_j.$$

φ is called (combinatorial) isomorphism from $\mathcal{C}_1$ to $\mathcal{C}_2$.

An isomorphism of $\mathcal{C}_1$ to itself is called *weak automorphism*.

A weak automorphism φ of $\mathcal{C}_1$ is called *(strong) automorphism* if

$$\forall i \in \mathcal{I}\, \forall (x, y) \in R_i : (x^\varphi, y^\varphi) \in R_i.$$

$$\mathrm{CAut}(\mathcal{C}_1) = \{\text{all weak automorphisms of } \mathcal{C}_1\}$$

$$\mathrm{Aut}(\mathcal{C}_1) = \{\text{all automorphisms of } \mathcal{C}_1\}$$

Normalizers of 2-closed permutation groups

Given a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i\in\mathcal{I}})$.

In general we have that

$$\mathrm{Aut}(\mathcal{C}) \trianglelefteq \mathrm{CAut}(\mathcal{C}).$$

If $\mathcal{C}$ is Schurian, then $\mathrm{CAut}(\mathcal{C})$ is the normalizer of $\mathrm{Aut}(\mathcal{C})$ in $\mathrm{Sym}(X)$.

By definition, $\mathrm{CAut}(\mathcal{C})$ acts naturally on $\mathcal{I}$. The kernel of this action is $\mathrm{Aut}(\mathcal{C})$.

If $\varphi \in (\mathrm{CAut}(\mathcal{C}), \mathcal{I})$, then

$$\forall i, j, k \in \mathcal{I} :\ p^k_{i,j} = p^{k^\varphi}_{i^\varphi, j^\varphi}.$$

Algebraic isomorphisms

We generalize the property of weak automorphisms to preserve structure-constants.

Let $\mathcal{C}_1 = (X, \{R_i\}_{i \in \mathcal{I}})$, $\mathcal{C}_2 = (Y, \{S_j\}_{j \in \mathcal{J}})$ be coherent configurations with structure constants $(p_{i,j}^k)$ and $(\hat{p}_{i,j}^k)$, respectively.

A function $\psi : \mathcal{I} \to \mathcal{J}$ is called *algebraic isomorphism* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is it is bijective and

$$\forall i, j, k \in \mathcal{I} : \; p_{i,j}^k = \hat{p}_{i^\psi, j^\psi}^{k^\psi}.$$

Algebraic automorphisms

An algebraic isomorphism of a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$ to itself is called *algebraic automorphism*.

$$\mathrm{AAut}(\mathcal{C}) = \{\text{all algebraic automorphisms of } \mathcal{C}\}.$$

Generally we have:

$$(\mathrm{CAut}(\mathcal{C}), \mathcal{I}) \leq (\mathrm{AAut}, \mathcal{I}).$$

Elements of $(\mathrm{CAut}(\mathcal{C}), \mathcal{I})$ are called *realizable algebraic automorphisms* of $\mathcal{C}$.

An idea of a normalizer algorithm for 2-closed permutation groups

Given a 2-closed permutation group (G, X).

Let $\mathcal{C} = (X, \{R_i\}_{i\in\mathcal{I}})$ be the corresponding Schurian coherent configuration of 2-orbits of (G, X).

Step 1 Find a set of generators of $\mathrm{Aut}(\mathcal{C})$,

Step 2 Find a set of generators of $\mathrm{AAut}(\mathcal{C})$,

Step 3 Filter the elements of $\mathrm{AAut}(\mathcal{C})$ for realizable algebraic automorphisms.

If we want to test if an algebraic automorphism φ is realizable, we try to construct a color-preserving isomorphism between the colored graphs $(X, \{R_i\}_{i\in\mathcal{I}})$ and $(X, \{R_{i^\varphi}\}_{i\in\mathcal{I}})$.

Such an isomorphism is an element of $\mathrm{CAut}(\mathcal{C})$ that realizes φ.

Comparison with the general normalizer algorithm

There is an efficient normalizer algorithm implemented in GAP4.

For some examples this normalizer algorithm performs (somewhat) slowly.

In these cases usually our normalizer-algorithm can solve the problem in a reasonable time.

The main advantage is that most of the time our normalizer algorithm works on the level of structure constants.

This is good because this is a combinatorial object whose size depends on the *rank* and not on the *degree* of the coherent configurations.

Algebraic fusions

Given a coherent configuration $\mathcal{C} = (X, \{R_i\}_{i \in \mathcal{I}})$.

Let $(G, \mathcal{I}) \leq (\mathrm{AAut}(\mathcal{C}), \mathcal{I})$.

Let $\{\mathcal{I}_1, \ldots, \mathcal{I}_k\}$ be the orbit-partition of $(G, \mathcal{I})$.

Then the fusion of $\mathcal{C}$ with respect to $\{\mathcal{I}_1, \ldots, \mathcal{I}_k\}$ is a fusion of $\mathcal{C}$.

Fusions of $\mathcal{C}$ that are obtained in this way are called *algebraic fusions of $\mathcal{C}$ with respect to $(G, \mathcal{I})$*.

The algebraic fusion for $(G, \mathcal{I})$ is an association scheme if and only if $(G, \mathcal{I})$ acts transitively on the elements of $\mathcal{I}$ that correspond to reflexive relations of $\mathcal{C}$.

If $\mathcal{C}$ is Schurian and $(G, \mathcal{I}) \leq \mathrm{CAut}(\mathcal{C})$, then the algebraic fusion of $\mathcal{C}$ with respect to $(G, \mathcal{I})$ is also Schurian.

Algebraic twins

Let $\mathcal{C} = (X, \{R_i\}_{i\in\mathcal{I}})$ be a coherent configuration.

Let $\Pi = \{\mathcal{I}_1, \ldots, \mathcal{I}_k\}$ be a partition of $\mathcal{I}$ such that the fusion of $\mathcal{C}$ with respect to Π is a fusion of $\mathcal{C}$.

If $\varphi \in (\mathrm{AAut}(\mathcal{C}), \mathcal{I})$, then $\Pi^\varphi = \{\mathcal{I}_1^\varphi, \ldots, \mathcal{I}_k^\varphi\}$ another partition of $\mathcal{I}$ such that the corresponding fusion is a fusion of $\mathcal{C}$.

The fusions Π and Π^φ are called *algebraic twins* in the lattice of fusions of $\mathcal{C}$.

If φ is realizable, then the twins are isomorphic.

Otherwise, it can happen that the twins are ***non-isomorphic!***

In general, algebraic twins are algebraically isomorphic.

Demonstration of the concepts (Example)

Consider the group $E_8 \cong \mathbb{Z}_2^3$.

Let $X = \{\text{all 2-element subsets of } E_8\}$.

E_8 acts on X by $\{x, y\} \mapsto \{x + a, y + a\}$.

Let $AP(2) = (X, \{R_i\}_{i \in \mathcal{I}})$ be the Schurian coherent configuration of 2-orbits of (E_8, X).

It has

- degree 28,
- rank 112,
- 7 fibres.

Groups related to $AP(2)$

$$\mathrm{Aut}(AP(2)) = (E_8, X) \qquad \text{i.e. } (E_8, X) \text{ is 2-closed}$$

$$(\mathrm{CAut}(AP(2)), X) \cong E_{2^{14}} \rtimes GL(3,2) \qquad \text{order: } 2^{17} \cdot 3 \cdot 7$$

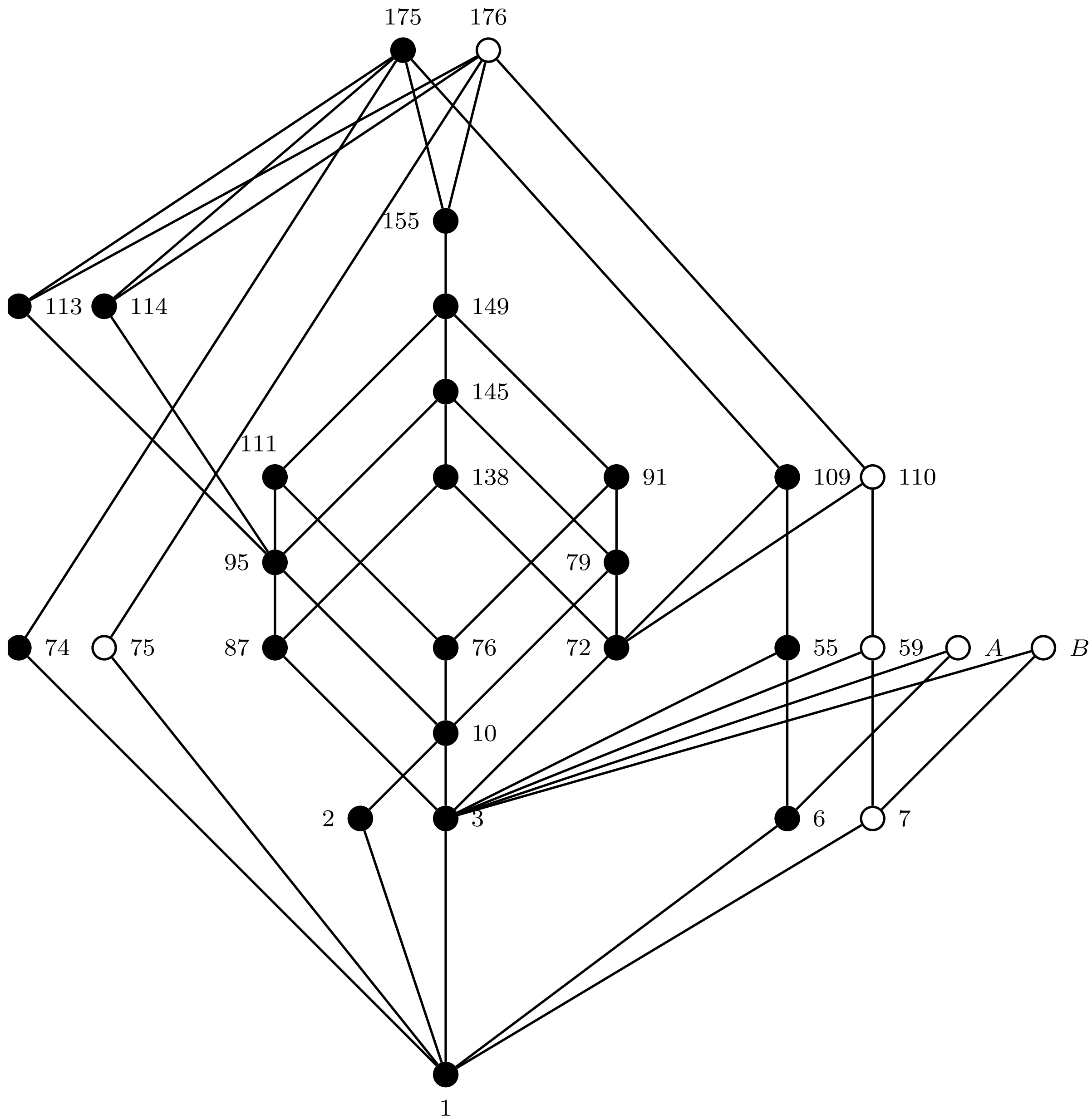
$$(\mathrm{CAut}(AP(2)), \mathcal{I}) \qquad \text{has order } 2^{14} \cdot 3 \cdot 7$$

$$(\mathrm{AAut}(AP(2)), \mathcal{I}) \cong E_{2^{14}} \rtimes GL(3,2) \qquad \text{order: } 2^{17} \cdot 3 \cdot 7$$

The index of $(\mathrm{CAut}(AP(2)), \mathcal{I})$ in $(\mathrm{AAut}(AP(2)), \mathcal{I})$ is **8**.

Hence there are algebraic automorphisms that are not realizable.

Homogeneous fusions of $AP(2)$



Some interesting twins

#175, #176 These are the two only quasithin schemes on 28 points. #176 is non-Schurian.

#74, #75 #74 is the famous pseudocyclic scheme found by Mathon #75 was discovered by Hollmann. It is non-Schurian.

#109, #110 ***This pair of twins is new!*** Both schemes arrise as algebraic fusions of $AP(2)$.

#6, #7 These are the most famous twins. #6 is the triangular graph $T(8)$ and #7 is one of the Chang-graphs.

COCO

COCO is a computer-algebra system for computing with the Galois-correspondence between coherent configurations and permutation groups.

Core functionality:

CGR: a function that takes as input a permutation group and computes the corresponding Schurian coherent configuration,

INM: a function that takes a coherent configuration and computes its structure constants,

SUB: a function that takes the structure constants and computes all fusions of the coherent configuration

AUT: a function that takes a set of fusions of a coherent configuration and computes their automorphism-groups.

Improvements to COCO

Meanwhile a few experimental improvements were made to COCO.

In particular a function for computing the normalizer of a 2-closed group was added.

Some additional functionality was created outside of COCO using the Computer Algebra System GAP.

This functions were used, e.g. to treat the coherent configuration $AP(2)$ from the example above.

COCO-II

A new version of COCO is now in the process of development. It is planned to be released as a share-package of GAP

In this new version will be contained improved version of the algorithm of COCO.

Main improvements will be the *use of algebraic automorphisms* while computing fusions of a coherent configuration.

This allows to handle configurations of much higher rank than before.

Main developers are *Sven Reichard*, *Misha Klin*, and *Ch. Pech*.

References

[BabBLT97] L. Babel, S. Baumann, M. Lüdecke, and G. Tinhofer, *STABCOL: Graph isomorphism testing based on the Weisfeiler-Leman algorithm*. Preprint TUM-M9702, TU-München, 1997.

[BabCK97] L. Babel, I. V. Chuvaeva, and M. Klin, *Algebraic combinatorics in mathematical chemistry II. Program implementation of the Weisfeiler-Leman algorithm*. Preprint TUM-M9701, TU-München, 1997.

[GanW99] Bernhard Ganter and Rudolf Wille, *Formal concept analysis*. Springer-Verlag, Berlin, 1999, ISBN 3-540-62771-5, Mathematical foundations, Translated from the 1996 German original by Cornelia Franzke.

[KliMPWZ05] M. Klin, M. Muzychuk, Ch. Pech, A. Woldar, and P.-H. Zieschang, *Association schemes on 28 points as fusions of a half-homogeneous coherent configuration*, 2005, to appear.

[KliPZ98] M. Klin, Ch. Pech, and P. H. Zieschang, *Flag algebras of block designs, I. Initial notions. Steiner 2-designs and generalized quadrangles*. Preprint MATH-AL-10-1998, TU Dresden, 1998.

[WeiL68] B. J. Weisfeiler and A. A. Leman, *A reduction of a graph to a canonical form and an algebra arising during this reduction*. Nauchno - Technicheskaja Informatsia 9(Seria 2), (1968), 12–16, (Russian).