Bounds for Algorithms in Differential Algebra

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1 Contents

- Introduction: basic notions of diffalg
- Usual Rosenfeld-Gröbner algorithm
- Modification of the Rosenfeld-Gröbner
- A bound for the new version and examples

2 Introduction

- **Differential ring** has differentiation

$$\Delta = \delta_1, \dots, \delta_m.$$

When $\Delta = \delta$ we say that we are in the *ordinary* case.

- For

$$\Theta = (\delta_1^{k_1} \delta_2^{k_2} \cdots \delta_n^{k_m}, k_i \geq 0)$$

the **ring of differential polynomials** in $y_1, \dots, y_n = Y$ is

$$k[\Theta Y] =: k\{y_1, \dots, y_n\}.$$

- **Differential ranking** is a *total ordering* on $\{\theta y_i \mid \theta \in \Theta, 1 \leqslant i \leqslant n\}$ satisfying

$$\theta u \geqslant u, \quad u \geqslant v \Longrightarrow \theta u \geqslant \theta v.$$

- For $f \in k\{y_1, \ldots, y_n\} \setminus k$,
  - **Leader** of f is $\mathbf{u}_f$.
  - **Initial** of f is $\mathbf{I}_f$
  - For $\delta \in \Theta$ the initial $\frac{\partial f}{\partial \mathbf{u}_f} =: \mathbf{S}_f$ of δf is called its **separant**.
- **Rank on differential polynomials**. We say that

$$\operatorname{rk} f > \operatorname{rk} g$$

if $\mathbf{u}_f > \mathbf{u}_g$ or ($\mathbf{u}_f = \mathbf{u}_g$ and $\deg_{\mathbf{u}_g} f > \deg_{\mathbf{u}_g} g$).

- – f is **partially reduced** w.r.t. g if no $\delta_i^k \mathbf{u}_g$ is in f,
  - f is **reduced** w.r.t. g if f is partially reduced and $\deg_{\mathbf{u}_g} f < \deg_{\mathbf{u}_g} g$.
- A finite subset $\mathbb{A} \subset k\{y_1, \ldots, y_n\}$ is **autoreduced** if $\mathbb{A} \cap k = \emptyset$ and each element of $\mathbb{A}$ is reduced w.r.t. all the others.

- $I_{\mathbb{A}}$ and $S_{\mathbb{A}}$ are the sets of initials and separants, $I_{\mathbb{A}} \cup S_{\mathbb{A}} =: H_{\mathbb{A}}$.
- For $S \subset k\{Y\}$ we denote S^∞ the multiplicative set generated by S. Let $I \subset k\{Y\}$. Then

$$I : S^\infty = \{a \in k\{Y\} \mid \exists s \in S^\infty : s \cdot a \in I\}.$$

- If $\mathbb{A} = A_1 < \ldots < A_r$ and $\mathbb{B} = B_1 < \ldots < B_s$ autoreduced sets then one can define what $\mathbb{A} < \mathbb{B}$ means.
- For $F \subset k\{y_1, \ldots, y_n\}$ the **differential** and **radical differential ideal** generated by F are denoted by $[F]$ and $\{F\}$, respectively.
- Let f, $g \in k\{Y\}$. Applying differentiations and pseudo divisions:
  - **differential partial remainder** f_1, $sf = f_1 \mod [g]$,
  - **differential remainder** f_2, $hf = f_2 \mod [g]$,

  $s \in S_g^\infty$, $h \in H_g^\infty$.

- An autoreduced set of the lowest rank in an ideal I is called a **characteristic set** of I.

  Theorem 1. *An autoreduced set $\mathbb{A}$ is a characteristic set of a differential ideal I iff each element of I is reducible to 0 w.r.t. $\mathbb{A}$.*

- **Characterizable ideals**:

  $$I = [\mathbb{C}] : H_{\mathbb{C}}^{\infty},$$

  where $\mathbb{C}$ is a characteristic set of I.

- One can decompose $\{F\}$ using **Rosenfeld-Gröbner**:

  $$\{F\} = [\mathbb{C}_1] : H_{\mathbb{C}_1}^{\infty} \cap \ldots \cap [\mathbb{C}_k] : H_{\mathbb{C}_k}^{\infty}.$$

  One also uses here regular systems like $[\mathbb{C}_i] : H_i^{\infty}$:

  - the set $\mathbb{C}_i$ is a coherent (do not need in the ordinary case) autoreduced set
  - the set H_i is partially reduced w.r.t. $\mathbb{C}_i$ and contains $H_{\mathbb{C}_i}$

- **Factorization-free algorithms**:
  - Boulier F., Lazard D., Ollivier F., Petitot M., 1995 — the first algorithm
  - Hubert E., 2000 — clear solution separating algebraic and differential operations
  - Bouziane D., Kandri Rodi A., Maârouf H., 2001 — approach uses invertibility

Algorithm 1. Rosenfeld-Gröbner

Input: *a finite set of differential polynomials* F_0.

Output: *a finite set* T *of regular systems such that*

$$\{F_0\} = \bigcap_{(\mathbb{A},H)\in T} [\mathbb{A}] : H^{\infty}.$$

- $T := \varnothing$, $U := \{(F_0, \varnothing, \varnothing)\}$
- **while** $U \neq \varnothing$ **do**
  - *Take and remove any* $(F, \mathbb{C}, H) \in U$; $R :=$ d-rem$(F, \mathbb{C}) \setminus \{0\}$
  - **if** $R = \varnothing$ **then** $T := T \cup (\mathbb{C}, H \cup H_{\mathbb{C}})$ **else**
    - $\bar{\mathbb{C}} :=$ *characteristic set of* $\mathbb{C} \cup R$; $\bar{F} := (\mathbb{C} \cup R) \setminus \bar{\mathbb{C}}$
    - $U := U \cup \{(\bar{F}, \bar{\mathbb{C}}, H \cup H_{\bar{\mathbb{C}}})\}$
  - *Let* $U = U \cup \{(F \cup \{h\}, \mathbb{C}, H) \mid h \in H_{\bar{\mathbb{C}}},\ h \notin k\}$
- **return** T

Example.

- $F = \{y + z, x, x^2 + z\}$, $x > y > z$
- $\mathbb{C} := \{y + z, x\}$, the leading variables of $\mathbb{C}$ are $\{y, x\}$
- $R := \mathsf{d\text{-}rem}(F \setminus \mathbb{C}, \mathbb{C}) = \{z\}$
- $F_1 := \mathbb{C} \cup R = \{z, y + z, x\}$
- As radical differential ideals:

$$\{y + z, x, x^2 + z\} = \{z, y + z, x\} : 1^\infty \cap \{y + z, x, x^2 + z, 1\}$$

- New $\mathbb{C} = \{z, x\}$ and the leading variables have changed!
- $\ldots$
- Finally,

$$\{y + z, x, x^2 + z\} = [z, y, x] : 1^\infty = [z, y, x]$$

Algorithm 2. Modified Rosenfeld-Gröbner

- $T := \varnothing,\ U := \{(F_0,\ \varnothing,\ \varnothing)\}$
- **while** $U \neq \varnothing$ **do**
  - *Take and remove any* $(F,\ \mathbb{B},\ H) \in U$
  - $R := \mathsf{algrem}(F,\ \mathbb{B}) \setminus \{0\}$
  - **if** $R = \varnothing$ **then** $T := T \cup (\mathbb{B}^{(0)},\ H)$ **else**
    * $\mathbb{C} :=$ *weak d-triangular subset of* $\mathbb{B}^{(0)} \cup R$ *of the lowest rank*
    * $\bar{F} := (\mathbb{B}^{(0)} \cup R) \setminus \bar{\mathbb{C}}$
    * $\bar{\mathbb{B}} := \mathsf{Differentiate\&Autoreduce}(\mathbb{C}, \{m_i(R \cup \mathbb{B}^{(0)} \cup H)\}_{i=1}^{n})$
    * $U := U \cup \{(\bar{F},\ \bar{\mathbb{B}},\ \mathsf{algrem}(H, \bar{\mathbb{B}}) \cup H_{\bar{\mathbb{B}}})\}$
  - $U := U \cup \{(F \cup \{h\},\ \mathbb{B},\ H) \mid h \in H_{\bar{\mathbb{B}}},\ h \notin k\}$
- **return** T

At each step we have

$$\{F_0\} = \bigcap_{(\mathbb{A},\, H) \in T} [\mathbb{A}] : H^\infty \cap \bigcap_{(F,\, \mathbb{B},\, H) \in U} \{F,\, \mathbb{B}\} : H^\infty$$

Example for Differentiate&Autoreduce

- Let $F = \{x, y^2 + x', y'\}$, elimination ranking $x < y$
- $\mathbb{C} = x, y^2 + x'$, $\mathbb{B} := \varnothing$
- $m_1 = m_x = 1$, $m_2 = m_y = 1$
- Then $\mathbb{B} := \mathbb{B} \cup \{x\}$ and $\mathbb{C} := \mathbb{C} \setminus \{x\}$
- Differentiate x and put the answer x' into $\mathbb{B}$
- Take and remove $y^2 + x'$ from $\mathbb{C}$
- Before putting it into $\mathbb{B}$ we reduce $y^2 + x'$ w.r.t. x'
- So, $\mathbb{B} := \{x, x', y^2\}$
- We differentiate y^2 and put in $\mathbb{B} = \{x, x', y^2, 2yy'\}$
- The "zero level" set $\mathbb{B}^{(0)} = \{x, y^2\}$

3 Bounds for the orders.

For $F \subset k\{y_1, \ldots, y_n\}$ we let

$$m_i(F) = \max\{\operatorname{ord}_{y_i} f \mid f \in F\}$$

and

$$M(F) = \sum_{i=1}^{n} m_i(F).$$

Theorem 2. *If* F_0 *is the input of* Modified Rosenfeld-Gröbner *then the output satisfies the following bound:*

$$M(\mathbb{A}) \leqslant (n-1)! M(F_0)$$

for all regular systems $(\mathbb{A}, H) \in T$.