The Structure of Differential Invariant Algebras of Lie Groups and Lie Pseudo–Groups

Peter J. Olver
University of Minnesota
http://www.math.umn.edu/∼olver

$\Longrightarrow$ Juha Pohjanpelto
Jeongoo Cheh

Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan . . . Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, . . .
- Relativity
- Noether's Second Theorem
- Gauge theory and field theories
  Maxwell, Yang–Mills, conformal, string, . . .
- Fluid Mechanics, Metereology
  Euler, Navier–Stokes,
  boundary layer, quasi-geostropic , . . .
- Linear and linearizable PDEs
- Solitons (in $2+1$ dimensions)
  K–P, Davey-Stewartson, . . .
- Image processing
- Numerical methods — geometric integration
- Kac–Moody symmetry algebras
- *Lie groups*!

What's New?

Direct constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Constructive Basis Theorem
- Syzygies and recurrence formulae
- Gröbner basis constructions
- Further applications

$\Longrightarrow$ Symmetry groups of differential equations $\Longrightarrow$ Vessiot group splitting
$\Longrightarrow$ Gauge theories
$\Longrightarrow$ Calculus of variations

Differential Invariants

$\mathcal{G}$ — transformation group acting on p-dimensional submanifolds $N = \{ u = f(x) \} \subset M$

$\mathcal{G}^{(n)}$ — prolonged action on the submanifold jet space $\mathrm{J}^n = \mathrm{J}^n(M,p)$

Differential invariant $\quad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$\Longrightarrow$ arc length derivative

$\star\star$ $\mathcal{I}(\mathcal{G})$ — the algebra of differential invariants $\star\star$

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(\mathcal{G})$ is generated by a finite number of differential invariants $I_1, \ldots, I_\ell$, meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ *Tresse, Kumpera*

♢ ♢ functional independence ♢ ♢

⋆⋆ Constructive Version ⋆⋆

$\Longrightarrow$ Computational algebra & Gröbner bases

The Algebra of Differential Invariants

Key Issues:

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for
  the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} A^i_{j,k}\,\mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among
  the differentiated invariants:

$$\Phi(\;\ldots\;\mathcal{D}_J I_\kappa\;\ldots\;) \equiv 0$$

$\Longrightarrow$ Gauss–Codazzi relations

Applications:

- Equivalence and signatures of submanifolds and characterization of moduli spaces
- Computation of invariant variational problems:

$$\int L(\;\ldots\;\mathcal{D}_J I_\kappa\;\ldots\;)\,d\boldsymbol{\omega}$$

- Group splitting of PDEs

Basic Themes

- The structure of a (connected) pseudo-group is fixed by its Lie algebra of *infinitesimal generators*: $\mathfrak{g}$
- The infinitesimal generators satisfy an overdetermined system of linear partial differential equations — the *determining equations*: $F = 0$
- The basic structure of an overdetermined system of PDEs is fixed by the algebraic structure of its *symbol module*: $\mathcal{I}$

- The structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ is fixed by the *prolonged infinitesimal generators*: $\mathfrak{g}^{(\infty)}$
- These satisfy an overdetermined system of partial differential equations — the *prolonged determining equations*: $H = 0$
- The basic structure of the prolonged determining equations is fixed by the algebraic structure of its *prolonged symbol m*

Jets and Prolongation

Jet $=$ Taylor polynomial/series (Ehresmann)

$\mathrm{j}_n \mathbf{v}$ — n^{th} order jet of a vector field

$\mathrm{J}^n = \mathrm{J}^n(M, p)$

— jets of p-dimensional submanifolds $N \subset M$

$\mathbf{v}^{(n)}$ — prolonged vector field on J^n

The *prolongation map* takes (jets of) vector fields on M to vector fields on the jet space J^n.

$$\mathbf{p}\colon\ \mathrm{j}_\infty \mathbf{v} \longmapsto \mathbf{v}^{(\infty)}$$

There is an induced dual prolongation map

$$\mathbf{p}^*\colon\ \mathcal{J} \longrightarrow \mathcal{I}$$

on the symbol modules, which is *algebraic*
(at sufficiently high order).

With this "algebraic prolongation" in hand, the structure of the prolonged symbol module, and hence the differential invariants, is algorithmically determined by the structure of the pseudo-group's symbol module.

Pseudo-groups

Definition. A *pseudo-group* is a collection of local diffeomorphisms $\varphi\colon M \to M$ such that

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses*: $\varphi^{-1} \in \mathcal{G}$,
- *Restriction*: $U \subset \operatorname{dom}\varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Composition*: $\operatorname{im}\varphi \subset \operatorname{dom}\psi \implies \psi \circ \varphi \in \mathcal{G}$.

Definition. A *Lie pseudo-group* $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

- Nonlinear determining equations $\implies$ *analytic* (*Cartan–Kähler*)

$\star\star$ Key complication: $\nexists$ Abstract object $\mathcal{G}$ $\star\star$

A Non-Lie Pseudo-group

Acting on $M = \mathbb{R}^2$:

$$\boxed{X = \varphi(x), \qquad Y = \varphi(y)}$$

where $\varphi \in \mathcal{D}(\mathbb{R})$.

♠ Cannot be characterized by a system of partial differential equations

$$\Delta(x, y, X^{(n)}, Y^{(n)}) = 0$$

Theorem. (Johnson, Itskov)

Any non-Lie pseudo-group can be completed to a Lie pseudo-group with the same differential invariants.

Completion of previous example:

$$\boxed{X = \varphi(x), \qquad Y = \psi(y)}$$

where $\varphi, \psi \in \mathcal{D}(\mathbb{R})$.

Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of
the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on M

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\#A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Infinitesimal (Linearized) Determining Equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0 \qquad (*)$$

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

Prolonged vector field:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \widehat{\varphi}\,_J^\alpha \frac{\partial}{\partial u_J^\alpha}$$

where

$$\widehat{\varphi}\,_J^\alpha = D_J\left(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha\, \xi^i\right) + \sum_{i=1}^{p} u_{J,i}^\alpha\, \xi^i$$

$$\equiv \Phi_J^\alpha(x, u^{(n)}; \xi^{(n)}, \varphi^{(n)})$$

Infinitesimal invariance:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

$$\mathcal{L}(\,\ldots\,, x^i,\,\ldots\,, u^\alpha,\,\ldots\,, \xi_A^i,\,\ldots\,, \varphi_A^\alpha,\,\ldots\,) = 0$$

$\Longrightarrow$ involutive completion

The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \ \cdots\ + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\begin{aligned}
\varphi^t &= \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u \\
\varphi^x &= \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u \\
\varphi^{xxx} &= \varphi_{xxx} + 3\,u_x\varphi_u + \ \cdots
\end{aligned}$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\, u \tau_t \qquad \varphi_u = -\,\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \ \cdots \ = \varphi_{uu} = 0$$

General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2 c_4 u.$$

Basis for symmetry algebra:

$$\partial_t, \qquad \partial_x, \qquad t\,\partial_x + \partial_u, \qquad 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \ \longmapsto \ (\lambda^3 t + a, \lambda x + c\,t + b, \lambda^{-2} u + c)$$

Moving Frames

For a finite-dimensional Lie group action, a moving frame is defined to be a (right) equivariant map

$$\rho^{(n)} \; : \mathrm{J}^n \;\longrightarrow\; G$$

where

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot g^{-1}$$

Moving frames are explicitly constructed by choosing a cross-section to the group orbits, and solving the normalization equations for the group parameters.

The existence of a moving frame requires that the group action be *free*, i.e., that there is no isotropy, or (locally) that the group orbits have the same dimension as the group itself.

Moving Frames for Pseudo–Groups

- The role of the group parameters is played by the pseudo-group jet coordinates, while the group is replaced by a certain principal bundle
$$\mathcal{H}^{(n)} \;\simeq\; \mathrm{J}^n \times \mathcal{G}^{(n)} \;\longrightarrow\; \mathrm{J}^n.$$
- There are an infinite number of pseudo-group parameters to normalize, which are to be done order by order. (Or else use the Taylor series approach.) However, at each finite order, the algebraic manipulations in the moving frame normalization procedure are *identical* to the finite-dimensional calculus.
- Since the pseudo-group $\mathcal{G}$ is infinite-dimensional, classical freeness is impossible! Rather, a pseudo-group action is said to act *freely* at a jet $z^{(n)} \in \mathrm{J}^n$ if the only pseudo-group elements which fix $z^{(n)}$ are those whose n-jet coincides with the identity n-jet.

Theorem. If $\mathcal{G}$ acts freely on J^n for $n \geq 1$, then it acts freely on all J^k for $k \geq n$.

$n^\star$ — order of freeness.

Normalization

♠ To construct a moving frame :

I. Choose a cross-section to the pseudo-group orbits:

$$u^{\alpha_\kappa}_{J_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

II. Solve the normalization equations

$$F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

III. Invariantization maps differential functions to differential invariants:

$$\iota : F(x, u^{(n)}) \longmapsto I(x, u^{(n)}) = F(\,\rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})\,)$$

$\Longrightarrow$ an algebra morphism and a projection:

$$\iota \circ \iota = \iota$$

$$I(x, u^{(n)}) = \iota(\, I(x, u^{(n)})\,)$$

Invariantization

A moving frame induces an invariantization process, denoted ι, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.

Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

$$\iota(x^i) = H^i \qquad \iota(u^\alpha_J) = I^\alpha_J$$

- Phantom differential invariants: $I^{\alpha_\kappa}_{J_\kappa} = c_\kappa$
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra $\mathcal{I}(\mathcal{G})$
- Replacement Theorem

$$\begin{aligned} I(\,\ldots\; x^i\;\ldots\; u^\alpha_J\;\ldots\,) &= \iota(\,I(\,\ldots\; x^i\;\ldots\; u^\alpha_J\;\ldots\,)\,) \\ &= I(\,\ldots\; H^i\;\ldots\; I^\alpha_J\;\ldots\,) \end{aligned}$$

♢ Differential forms $\Longrightarrow$ invariant differential forms

$$\iota(\,dx^i) = \omega^i \qquad i = 1, \ldots, p$$

♢ Differential operators $\Longrightarrow$
invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i \qquad i = 1, \ldots, p$$

Recurrence Formulae

⋆⋆ Invariantization and differentiation do not commute ⋆⋆

The *recurrence formulae* connect the differentiated invariants with their invariantized counterparts:

$$\boxed{\boxed{\mathcal{D}_i I^\alpha_J = I^\alpha_{J,i} + M^\alpha_{J,i}}}$$

$\Longrightarrow$ $M^\alpha_{J,i}$ — correction terms

♡ Once established, they completely prescribe the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ — thanks to the functional independence of the non-phantom normalized differential invariants.

⋆⋆ The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!

The Key Formula

$$dI^\alpha_J \;=\; \sum_{i=1}^{p} (\,\mathcal{D}_i I^\alpha_J\,)\,\omega^i \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}{}^\alpha_J$$

where

$$\widehat{\psi}{}^\alpha_J = \iota(\,\widehat{\varphi}{}^\alpha_J\,) = \Phi^\alpha_J(\;\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\;;\;\ldots\;\gamma^b_A\;\ldots\;)$$

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

$\gamma^b_A \;=\; \iota(\zeta^b_A)$ — invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.

Proposition.

The invariantized Maurer–Cartan forms are subject to the *invariantized determining equations*:

$$\mathcal{L}(H^1,\ldots,H^p,I^1,\ldots,I^q,\;\ldots\;,\gamma^b_A,\;\ldots\;) = 0$$

$$d_H\, I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,{}^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)$$

Step 1: Solve the phantom recurrence formulas

$$0 = d_H\, I^\alpha_J \;=\; \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,{}^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)$$

for the invariantized Maurer–Cartan forms:

$$\gamma^b_A = \sum_{i=1}^{p} J^b_{A,i}\,\omega^i \qquad (*)$$

Step 2: Substitute $(*)$ into the non-phantom recurrence formulae to obtain the explicit correction terms.

♢ Only uses linear differential algebra based on the specification of cross-section.

♡ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

Korteweg–deVries equation

Symmetry Group Action:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

Prolonged Action:

$$\begin{aligned}
U_T &= e^{-5\lambda_4}(u_t - \lambda_3 u_x), \\
U_X &= e^{-3\lambda_4} u_x, \\
U_{TT} &= e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}), \\
U_{TX} &= D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}), \\
U_{XX} &= e^{-4\lambda_4} u_{xx}, \\
&\vdots
\end{aligned}$$

Cross Section:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + u u_x)$$

Phantom Invariants:

$$H^1 = \iota(t) = 0, \qquad I_{00} = \iota(u) = 0,$$
$$H^2 = \iota(x) = 0, \qquad I_{10} = \iota(u_t) = 1.$$

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2 u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + u u_{xx}}{(u_t + u u_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + u u_x)^{4/5}}$$

$$I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + u u_x}$$

$$\vdots$$

Replacement Theorem:

$$0 = \iota(u_t + u u_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + u u_x + u_{xxx}}{u_t + u u_x}.$$

Invariant horizontal one-forms:

$$\omega^1 = \iota(dt) = (u_t + u u_x)^{3/5}\, dt,$$
$$\omega^2 = \iota(dx) = -u(u_t + u u_x)^{1/5}\, dt + (u_t + u u_x)^{1/5}\, dx.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + u u_x)^{-3/5} D_t + u(u_t + u u_x)^{-3/5} D_x,$$
$$\mathcal{D}_2 = \iota(D_x) = (u_t + u u_x)^{-1/5} D_x.$$

Recurrence formula:

$$dI_{jk} = I_{j+1,k}\omega^1 + I_{j,k+1}\omega^2 + \iota(\varphi^{jk})$$

Invariantized Maurer–Cartan forms:

$$\iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \psi = \nu, \quad \iota(\tau_t) = \psi^t = \lambda_t, \quad \ldots$$

Invariantized determining equations:

$$\lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0$$

$$\nu = \mu_t \qquad \nu_u = -\,2\,\mu_x = -\,\tfrac{2}{3}\,\lambda_t$$

$$\lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \ \cdots \ = \nu_{uu} = \ \cdots \ = 0$$

Invariantizations of prolonged vector field coefficients:

$$\iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \nu, \quad \iota(\varphi^t) = -I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\iota(\varphi^x) = -I_{01}\lambda_t, \quad \iota(\varphi^{tt}) = -2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \quad \ldots$$

Phantom recurrence formulae:

$$0 = \; d_H\, H^1 = \omega^1 + \lambda,$$

$$0 = \; d_H\, H^2 = \omega^2 + \mu,$$

$$0 = \; d_H\, I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu,$$

$$0 = \; d_H\, I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$\Longrightarrow$ Solve for $\lambda = -\omega^1, \quad \mu = -\omega^2, \quad \nu = -\omega^1 - I_{01}\omega^2,$

$$\lambda_t = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2.$$

Non-phantom recurrence formulae:

$$d_H\, I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t,$$

$$d_H\, I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t,$$

$$d_H\, I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t,$$

$$d_H\, I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\lambda_t,$$

$$\vdots$$

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5}I_{01}^2 - \tfrac{3}{5}I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}\,I_{01}^3 - \tfrac{3}{5}\,I_{01}I_{11},$$

$$\mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \tfrac{8}{5}I_{01}I_{20} - \tfrac{8}{5}I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2I_{01}I_{11} - \tfrac{8}{5}I_{01}^2 I_{20} - \tfrac{8}{5}I_{11}I_{20},$$

$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5}I_{01}I_{11} - \tfrac{6}{5}I_{11}I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01}I_{02} - \tfrac{6}{5}I_{01}^2 I_{11} - \tfrac{6}{5}I_{11}^2,$$

$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5}I_{01}I_{02} - \tfrac{4}{5}I_{02}I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5}I_{01}^2 I_{02} - \tfrac{4}{5}I_{02}I_{11},$$

$$\vdots \qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + u u_x)^{-3/5} D_t + u (u_t + u u_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + u u_x)^{-1/5} D_x.$$

Commutation formula:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_{01}\,\mathcal{D}_1$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01}\right) \mathcal{D}_1 I_{01}$$

$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

The Symbol Module

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^b(z)\,\frac{\partial}{\partial z^b}$$

Vector field jet:

$$\mathrm{j}_\infty \mathbf{v} \quad\Longleftrightarrow\quad \zeta^{(\infty)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\# A}\zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1}\cdots \partial z^{a_k}}$$

Determining Equations for $\mathbf{v} \in \mathfrak{g}$

$$\mathcal{L}(z;\ \ldots\ \zeta^b_A\ \ldots\ ) = 0 \qquad (*)$$

Duality

$$t = (t_1, \ldots, t_m) \qquad T = (T_1, \ldots, T_m)$$

Polynomial module:

$$\mathcal{T} = \left\{ P(t,T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T]$$

$$\mathcal{T} \;\simeq\; (\mathrm{J}^\infty TM|_z)^*$$

Dual pairing:

$$\left\langle\ \mathrm{j}_\infty \mathbf{v}\,;\, t_A T^b\ \right\rangle = \zeta^b_A.$$

Each polynomial

$$\eta(z;t,T) = \sum_{b=1}^{m} \sum_{\#A \le n} h^A_b(z)\, t_A T^b \ \in \mathcal{T}$$

induces a linear partial differential equation

$$L(z, \zeta^{(n)}) = \left\langle\ \mathrm{j}_\infty \mathbf{v}\,;\, \eta(z;t,T)\ \right\rangle$$

$$= \sum_{b=1}^{m} \sum_{\#A \le n} h^A_b(z)\, \zeta^b_A = 0$$

The Linear Determining Equations

Annihilator:

$$\mathcal{L} = (\mathrm{J}^\infty \mathfrak{g})^\perp$$

Determining Equations

$$\Big\langle \, \mathrm{j}_\infty \mathbf{v} \, ; \, \eta \, \Big\rangle = 0 \quad \text{for all} \quad \eta \in \mathcal{L} \qquad \Longleftrightarrow \qquad \mathbf{v} \in \mathfrak{g}$$

Symbol = highest degree terms:

$$\mathbf{\Sigma}[\, L(z, \zeta^{(n)}) \,] = \mathbf{H}[\, \eta(z; t, T) \,] = \sum_{b=1}^{m} \sum_{\#A=n} h_b^A(z) \, t_A T^b.$$

Symbol submodule:

$$\mathcal{I} = \mathbf{H}(\mathcal{L})$$

$\Longrightarrow$ Formal integrability (involutivity)

Prolonged Duality

Prolonged vector field:

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha,J} \widehat{\varphi}\,{}_J^{\alpha}(x,u^{(k)})\,\frac{\partial}{\partial u_J^\alpha}$$

$$\widetilde{s} = (\widetilde{s}_1,\ldots,\widetilde{s}_p), \quad s = (s_1,\ldots,s_p), \quad S = (S_1,\ldots,S_q)$$

"Prolonged" polynomial module:

$$\widehat{\mathcal{S}} = \left\{ \sigma(s,S,\widetilde{s}) = \sum_{i=1}^{p} c_i\widetilde{s}_i + \sum_{\alpha=1}^{q} \widehat{\sigma}_\alpha(s)\,S^\alpha \right\} \simeq \mathbb{R}^p \oplus (\mathbb{R}[s] \otimes \mathbb{R}^q)$$

$$\widehat{\mathcal{S}} \;\simeq\; T^*\mathrm{J}^\infty\big|_{z^{(\infty)}}$$

Dual pairing:

$$\left\langle\ \mathbf{v}^{(\infty)}\,;\ \widetilde{s}_i\ \right\rangle = \xi^i$$

$$\left\langle\ \mathbf{v}^{(\infty)}\,;\ S^\alpha\ \right\rangle = Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha\,\xi^i$$

$$\left\langle\ \mathbf{v}^{(\infty)}\,;\ s_J S^\alpha\ \right\rangle = \widehat{\varphi}\,{}_J^{\alpha} = \Phi_J^\alpha(u^{(n)};\zeta^{(n)})$$

Algebraic Prolongation

Prolongation of vector fields:

$$\begin{aligned} \mathbf{p}:\ \mathrm{J}^\infty\mathfrak{g} &\longmapsto \mathfrak{g}^{(\infty)} \\ \mathrm{j}_\infty\mathbf{v} &\longmapsto \mathbf{v}^{(\infty)} \end{aligned}$$

Dual prolongation map:

$$\mathbf{p}^*:\ \mathcal{S} \longrightarrow \mathcal{T}$$

$$\Big\langle\ \mathrm{j}_\infty\mathbf{v}\,;\,\mathbf{p}^*(\sigma)\ \Big\rangle = \Big\langle\ \mathbf{p}(\mathrm{j}_\infty\mathbf{v})\,;\,\sigma\ \Big\rangle = \Big\langle\ \mathbf{v}^{(\infty)}\,;\,\sigma\ \Big\rangle$$

$\star\star$ On the symbol level, $\mathbf{p}^*$ is algebraic $\star\star$

Prolongation Symbols

Define the linear map $\boldsymbol{\beta} : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^m$

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha \, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S^\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha \, T_i, \qquad \alpha = 1, \ldots, q.$$

Pull-back map

$$\boldsymbol{\beta}^*[\, \sigma(s_1, \ldots, s_p, S_1, \ldots, S_q) \,]$$
$$= \sigma(\, \beta_1(t), \ldots, \beta_p(t), B_1(T) \ldots, B_q(T) \,)$$

Lemma. The symbols of the prolonged vector field coefficients are

$$\boldsymbol{\Sigma}(\xi^i) = T^i \qquad \boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}) = T^{\alpha+p}$$

$$\boldsymbol{\Sigma}(Q^\alpha) = \boldsymbol{\beta}^*(S^\alpha) = B_\alpha(T)$$

$$\boldsymbol{\Sigma}(\widehat{\varphi}^{\,\alpha}_J) = \boldsymbol{\beta}^*(s_J S^\alpha) = \boldsymbol{\beta}^*(s_{j_1} \cdots s_{j_n} S^\alpha)$$
$$= \beta_{j_1}(t) \, \cdots \, \beta_{j_n}(t) \, B_\alpha(T)$$

Prolonged annihilator:

$$\mathcal{Z} = (\mathbf{p}^*)^{-1}\mathcal{L} = (\mathfrak{g}^{(\infty)})^{\perp}$$

$$\langle\, \mathbf{v}^{(\infty)} ; \sigma \,\rangle = 0 \qquad \text{for all} \quad \mathbf{v} \in \mathfrak{g} \quad \Longleftrightarrow \quad \sigma \in \mathcal{Z}$$

Prolonged symbol subbundle:

$$\mathcal{U} \;=\; \mathbf{H}(\mathcal{Z}) \;\subset\; \mathrm{J}^{\infty}(M, p) \times \mathcal{S}$$

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$

Warning:

$$\mathcal{U} \;\subsetneq\; \mathcal{J}$$

But

$$\mathcal{U}^n \;=\; \mathcal{J}^n \qquad \text{when} \qquad n > n^\star$$

$n^\star$ — order of freeness.

Algebraic Recurrence

Polynomial:

$$\widetilde{\sigma}(\mathbf{I}^{(k)}; s, S) = \sum_{\alpha, J} h_a^J(\mathbf{I}^{(k)})\ s_J S^\alpha \in \widehat{\mathcal{S}}$$

Differential invariant:

$$I_{\tilde{\sigma}} = \sum_{\alpha, J} h_a^J(\mathbf{I}^{(k)})\, I_J^\alpha$$

Recurrence:

$$\boxed{\boxed{\mathcal{D}_i\, I_{\tilde{\sigma}} = I_{\mathcal{D}_i \tilde{\sigma}} \equiv I_{s_i \tilde{\sigma}} + R_{i,\tilde{\sigma}}}}$$

$$\tilde{\sigma} \in \widetilde{\mathcal{J}}^n,\ n > n^\star \quad \Longrightarrow \quad \begin{aligned} \text{order}\, I_{\tilde{\sigma}} &= n \\ \text{order}\, I_{\mathcal{D}_i \tilde{\sigma}} &= n+1 \\ \text{order}\, R_{i,\tilde{\sigma}} &\leq n \end{aligned}$$

Algebra $\Longrightarrow$ Invariants

$\mathcal{I}$ — symbol module

- determining equations for $\mathfrak{g}$

$\mathcal{M} \simeq \mathcal{T} \,/\, \mathcal{I}$ — complementary monomials $t_A T^b$

- pseudo-group parameters
- Maurer–Cartan forms

$\mathcal{N}$ — leading monomials $s_J S^\alpha$

- normalized differential invariants I_J^α

$\mathcal{K} = \mathcal{S} \,/\, \mathcal{N}$ — complementary monomials $s_K S^\beta$

- cross-section coordinates $u_K^\beta = c_K^\beta$
- phantom differential invariants I_K^β

$$\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})$$

Freeness: $\boldsymbol{\beta}^* \colon\ \mathcal{K} \xrightarrow{\ \sim\ } \mathcal{M}$

Generating Differential Invariants

Theorem. The differential invariant algebra is generated by differential invariants that are in one-to-one correspondence with the Gröbner basis elements of the prolonged symbol module plus, possibly, a finite number of differential invariants of order $\leq n^\star$.

Syzygies

Theorem. Every differential syzygy among the generating differential invariants is either a syzygy among those of order $\leq n^\star$, or arises from an algebraic syzygy among the Gröbner basis polynomials in $\widetilde{\mathcal{J}}$.