Stability and Stabilization of Multidimensional Input/Output Systems and Difficult Open Problems

Talk at Gröbner-Semester
Universität Linz
May 17th, 2006, 17.40-18.45h
after a paper of
essentially the same title, to appear in
SIAM J. Control and Optimization

Ulrich Oberst
Institut für Mathematik
Universität Innsbruck

April 2006

INGREDIENTS

1. Multidimensional Constant Linear Systems or Behaviors

2. Algebraic Analysis

3. Algebraic Geometry

4. Open problems for Gröbner bases theory

5. Not in this talk: *Discrete* case of partial *difference* equations

History since 1985: Stabilization of *discrete* multidimensional transfer matrices or *input/output maps*:
N.K. Bose, J.P.Guiver, Z. Lin, A. Quadrat, S. Shankar, V.R. Sule, L. Xu, J.-Q. Ying, E. Zerz, et.al.

DATA

$\mathbb{C}$ −algebra of operators

$$A := \mathbb{C}\,[s] = \mathbb{C}\,[s_1, \cdots, s_r] \subset \mathbb{C}\,(s)$$

function spaces

$$C^\infty(\mathbb{R}^{\,r}) :=$$
$$\{y : \mathbb{R}^{\,r} \longrightarrow \mathbb{C}\,,\ z = (z_1, \cdots, z_r) \mapsto y(z);$$
$$y \text{ infinitely often differentiable}\} \subset$$
$$\mathcal{D}'(\mathbb{R}^{\,r}) := \{\text{distributions}\}$$

$\mathbb{C}[s]-$module structure or action on $\mathcal{D}'$:

$$s_\rho \circ y := \partial y / \partial z_\rho$$

$$y \in \mathcal{D}'(\mathbb{R}^r) \text{ locally finite} \quad \Leftrightarrow$$
$$\dim_{\mathbb{C}} (\mathbb{C}[s] \circ y) < \infty$$

function space of *polynomial-exponential* functions:

$$\mathcal{F} := \{y \in \mathcal{D}'(\mathbb{R}^r);\ y \text{ locally finite }\} =$$
$$\oplus_{\lambda \in \mathbb{C}^r} \mathbb{C}[z_1, \cdots, z_r] \exp(\lambda \bullet z)$$
$$\lambda \bullet z := \lambda_1 z_1 + \cdots + \lambda_r z_r,$$

STABILITY DECOMPOSITION

arbitrary disjoint decomposition

$$\mathbb{C}^r := \Lambda_1 \uplus \Lambda_2$$

Λ_1 resp. $\Lambda_2 :=$ *stable* resp. *unstable* region

Standard examples: 1.

$r = 1,\ \Lambda_2 := \overline{\mathbb{C}_+} := \{\lambda \in \mathbb{C};\ \Re(\lambda) \geq 0\}.$

2.

$r \geq 1,\ \Lambda_2 := \overline{\mathbb{C}_+}^r$ or $\Lambda_2 := \overline{\mathbb{C}_+} \times i\mathbb{R}^{r-1}$

direct decomposition

$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2,\ \mathcal{F}_i := \oplus_{\lambda\in\Lambda_i}\mathbb{C}\,[z]\exp(\lambda \bullet z)$$

$$\mathcal{F}_1 =: \{\ \textit{stable}\ \text{functions}\}$$

$$y = y_1 + y_2$$

$$y_1 =: \textit{negligible}\ \text{part of}\ y$$

$$y_2 =: \textit{steady}\ \text{state of}\ y$$

Main motivation

$$r = 1,\ \Lambda_1 := \{\lambda \in \mathbb{C}\,;\ \Re(\lambda) < 0\}$$

$$\mathcal{F}_1 = \{y \in \mathcal{F};\ \lim_{z\to\infty} y(z) = 0\}$$

multiplicatively closed set of
stable polynomials :

$$T := \{f \in \mathbb{C}[s];\ \forall \lambda \in \Lambda_2 :\ f(\lambda) \neq 0\} \subset A = \mathbb{C}[s]$$

Open problem : Decide $f \in T$.

Standard example: $r = 1$
$\Lambda_1 := \{z \in \mathbb{C}\,;\ \Re(z) < 0\}$
$f \in \mathbb{R}\,[s] \subset \mathbb{C}\,[s]$: *Routh-Hurwitz test*

algebra of *stable rational functions* or
SISO stable plants :

$$A_T := \{\frac{f}{t};\ f \in \mathbb{C}\,[s],\ t \in T\} \subset \mathbb{C}\,(s).$$

Open problem:

$$\text{ideal } \mathfrak{a} := \mathbb{C}[s]f_1 + \cdots + \mathbb{C}[s]f_m \subset \mathbb{C}[s]$$
$$\mathfrak{a}_T := \{\frac{a}{t};\ a \in \mathfrak{a}, t \in T\} = \mathbb{C}[s]_T\mathfrak{a} \subset \mathbb{C}[s]_T$$

Decide and *find* constructively

(i) $\mathfrak{a}_T = \mathbb{C}[s]_T$, ie. $\mathfrak{a} \cap T \neq \emptyset$
and $t \in \mathfrak{a} \cap T$

(ii) generators of the $T-$ closure
$\mathbb{C}[s] \cap \mathfrak{a}_T =$
$\{f \in \mathbb{C}[s]; \exists t \in T \text{ with } tf \in \mathfrak{a}\}$

Application to torsion modules :

${}_{\mathbb{C}[s]}M$ finitely generated

$t(M) := \{x \in M;\ \exists 0 \neq f :\ fx = 0\} =:$ torsion submodule, via Gröbner bases

$0 \neq \mathfrak{a} := \mathrm{ann}_{\mathbb{C}[s]}(t(M))$ via Gröbner bases. Then

$M_T := \{\frac{x}{t};\ x \in M, t \in T\}$ torsionfree $\Leftrightarrow$ $\mathfrak{a} \cap T \neq \emptyset$

INPUT/OUTPUT-SYSTEMS or BEHAVIORS

ALGEBRA

matrices

$$R \in \mathbb{C}[s]^{k\times l},\ p := \mathrm{rank}(R),\ m := l - p$$
$$R = (P, -Q) \in \mathbb{C}[s]^{k\times(p+m)}$$
$$\mathrm{rank}(P) = p\ \Rightarrow$$
$$\exists_1\ H \in \mathbb{C}(s)^{p\times m},\ PH = Q$$
$$H = \textit{transfer matrix}$$

modules

$$U := \mathbb{C}[s]^{1\times k}R \subset \mathbb{C}[s]^{1\times(p+m)} \text{ row module}$$
$$M := \mathbb{C}[s]^{1\times(p+m)}/U \text{ factor module}$$
$$\mathrm{rank}(M) := \dim_{\mathbb{C}(s)} \mathbb{C}(s) \otimes_{\mathbb{C}[s]} M = m$$

$$U^0 := \mathbb{C}[s]^{1\times k}P \subset \mathbb{C}[s]^{1\times p},\ M^0 := \mathbb{C}[s]^{1\times p}/U^0$$
$$\mathbb{C}(s) \otimes_{\mathbb{C}[s]} M^0 = 0,\ \text{ie. } M^0 = \text{ torsion module}$$

ANALYSIS

input/output (IO-) behavior or

solution space

$$\mathcal{B} := \{w = \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};$$

$$R \circ w = 0 \text{ or } P \circ y = Q \circ u\} \cong \mathrm{Hom}_{\mathbb{C}[s]}(M, \mathcal{F})$$

after B. Malgrange 1962

$$\mathcal{B}^0 := \{y \in \mathcal{F}^p;\ P \circ y = 0\} \cong \mathrm{Hom}_{\mathbb{C}[s]}(M^0, \mathcal{F})$$

categorical duality

$$ {}_{\mathbb{C}[s]}M = \text{finitely generated polynomial module} $$
$$ \leftrightarrow \ \mathcal{B} \cong \mathrm{Hom}_{\mathbb{C}[s]}(M, \mathcal{F}) \text{ behavior} $$

Exact sequence

$$ 0 \longrightarrow \mathcal{B}^0 \xrightarrow{\text{inj}} \mathcal{B} \xrightarrow{\text{proj}} \mathcal{F}^m \longrightarrow 0 $$
$$ y \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}, \ \begin{pmatrix} y \\ u \end{pmatrix} \mapsto u $$

$$ \forall \ \ \textit{input } u \ \exists \ \ \textit{output } y : \ P \circ y = Q \circ u $$

$$\mathcal{B}\ autonomous :\ \Leftrightarrow$$
$$M = M^0\ torsion \text{ module } \Leftrightarrow$$
$$\operatorname{rank}(M) = 0\ \Leftrightarrow$$
$$\exists 0 \neq f \in \mathbb{C}\,[s] \text{ with } f \circ \mathcal{B} = 0$$

$$\mathcal{B}^0 := \text{ autonomous part } of\mathcal{B}$$

$$\mathcal{B}\ controllable :\ \Leftrightarrow\ M\ torsionfree\ \Leftrightarrow$$
$$\exists \text{ monomorphism } M \longrightarrow \mathbb{C}\,[s]^{1\times m}\ \Leftrightarrow$$
$$\exists \text{ epimorphism } \phi : \mathcal{F}^m \longrightarrow \mathcal{B}$$

ϕ =: parametrization (J.F. Pommaret) of $\mathcal{B}$,
image representation (J.C. Willems) of $\mathcal{B}$

$$t(M) = U_{\rm cont}/U \subset M = \mathbb{C}\left[s\right]^{1\times(p+m)}/U$$

$$U_{\rm cont} = \mathbb{C}\left[s\right]^{1\times k_{\rm cont}}(P_{\rm cont}, -Q_{\rm cont})$$

via Gröbner bases, $P_{\rm cont}H = Q_{\rm cont}$

$$\mathcal{B}_{\rm cont} :=$$

$$\{\begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{p+m};\ P_{\rm cont} \circ y = Q_{\rm cont} \circ u\} =$$

least IO-behavior with transfer matrix H =
largest controllable subbehavior of $\mathcal{B}$ =
unique *controllable realization* of H.

ALGEBRAIC GEOMETRY

maximal ideal

$$\mathfrak{m}_\lambda := \{f \in \mathbb{C}\,[s];\ f(\lambda) = 0\},\ \lambda \in \mathbb{C}^{\,r}$$

local ring

$$A_{\mathfrak{m}_\lambda} = \{\frac{a}{t};\ t(\lambda) \neq 0\} \subset \mathbb{C}\,(s)$$

algebra of stable rational functions

$$A_T = \cap_{\lambda \in \Lambda_2} A_{\mathfrak{m}_\lambda}$$

variety of an ideal $\mathfrak{a} \subset A$:

$$V(\mathfrak{a}) := \{\lambda \in \mathbb{C}^{\,r} :\ \mathfrak{a}(\lambda) = 0\}$$

Open problem: Decide constructively
$V(\mathfrak{a}) \cap \Lambda_2 = \emptyset$.

$$\mathfrak{a} \cap T \neq \emptyset \stackrel{\text{trivial}}{\Rightarrow} V(\mathfrak{a}) \cap \Lambda_2 = \emptyset$$
$$t \in T \;\Leftrightarrow\; V(t) \cap \Lambda_2 = \emptyset$$

Special case : $\mathfrak{a}$ Krull zero-dimensional $\Leftrightarrow$
$V(\mathfrak{a})$ finite, via Buchberger's algorithm $\Rightarrow$
$V(\mathfrak{a}) \cap \Lambda_2$ finite, constructive

Λ_2 *ideal-convex* after S. Shankar, V. Sule :
$\Leftrightarrow\; \forall \mathfrak{a} : V(\mathfrak{a}) \cap \Lambda_2 = \emptyset \;\Rightarrow\; \mathfrak{a} \cap T \neq \emptyset$.

Examples

1. compact, polynomially convex $\Rightarrow$ ideal convex via theorems A and B by H. Cartan, J.P. Serre!!

2. for instance closed unit polydisc $\overline{U}^r$

$$\overline{U} := \{z \in \mathbb{C}\,; \mid z \mid \leq 1\}$$

3. **not** ideal-convex :

$$\Lambda_2 := \mathbb{C}^{\,r} \setminus \{\lambda \in \mathbb{C}^{\,r} :\ \forall \rho : \Re(\lambda_\rho) < 0\}$$

Open problem: Decide ideal convexity constructively and find $t \in T \cap \mathfrak{a}$.

solution for two-dimensional closed unit polydisc by N.K. Bose, J.P. Guiver, Z. Lin, L. Xu et al *singular variety*

of a behavior or module:

$$\operatorname{sing}(\mathcal{B}) := \operatorname{sing}(M) :=$$
$$\{\lambda \in \mathbb{C}^r;\ \operatorname{rank}(R(\lambda)) < p = \operatorname{rank}(R)\}$$

characteristic variety

of an autonomous behavior:

$$\operatorname{char}(\mathcal{B}^0) := \operatorname{char}(M^0) := \operatorname{sing}(\mathcal{B}^0) =$$
$$\{\lambda \in \mathbb{C}^r;\ \operatorname{rank}(P(\lambda)) < p = \operatorname{rank}(P)\} =$$
$$V(\operatorname{ann}_{\mathbb{C}[s]}(M^0)) \supset \operatorname{sing}(\mathcal{B}).$$

Standard example

$$r = 1,\ F \in \mathbb{C}^{\,p\times p},\ P := s\,\mathrm{id}_p - F$$

$$\mathcal{B}^0 = \{x \in \mathcal{F}^p;\ P \circ x = 0\} =$$
$$\{x \in \mathcal{F}^p;\ \dot{x} = Fx\}$$

$$\mathrm{char}(\mathcal{B}^0) = \{\text{eigenvalues of } F\}$$

STABILITY

Theorem and Definition 0.1. *(1) Equivalent:*

(i) Analysis: $\mathcal{B}$ *is* stable, *ie. (definition)*

$$\mathcal{B}^0 \subset \oplus_{\lambda \in \Lambda_1} \mathbb{C}\,[z]^p \exp(\lambda \bullet z)$$

(ii) Geometry: $\operatorname{char}(\mathcal{B}^0) \cap \Lambda_2 = \emptyset$.

(iii) Algebra:
(a) H is stable, ie. $H \in A_T^{p\times m}$.

(b) For each $\lambda \in \Lambda_2$ the $A_{\mathfrak{m}_\lambda}-$module $M_{\mathfrak{m}_\lambda}$ is torsionfree.

If $r = 1,\ \Lambda_1 = \{\lambda \in \mathbb{C}\,;\ \Re(\lambda) < 0\}$*:*

(i) $\mathcal{B}^0$ *asymptotically stable, ie.*

$$\forall y \in \mathcal{B}^0 : \lim_{z \to \infty} y(z) = 0$$

(iii)(b) $\mathrm{sing}(\mathcal{B}) \cap \Lambda_2 = \emptyset$.

(2) **If** $\mathcal{B}$ *is stable then* external stability*:*

$$\mathcal{F}_2 = \oplus_{\lambda\in\Lambda_2}\mathbb{C}\left[z\right]\exp(\lambda \bullet z) = A_T - module$$

$$\mathcal{B} \cap \mathcal{F}_2^{p+m} \cong \mathcal{F}_2^m,\ \begin{pmatrix} y \\ u \end{pmatrix} \leftrightarrow u,\ y = H \circ u$$

$y =$ *unique solution of* $P \circ y = Q \circ u$ *in* $\mathcal{F}_2^p$

If

$$u = \sum_{\lambda\in\Lambda_2} u_\lambda(z)\exp(\lambda \bullet z)$$

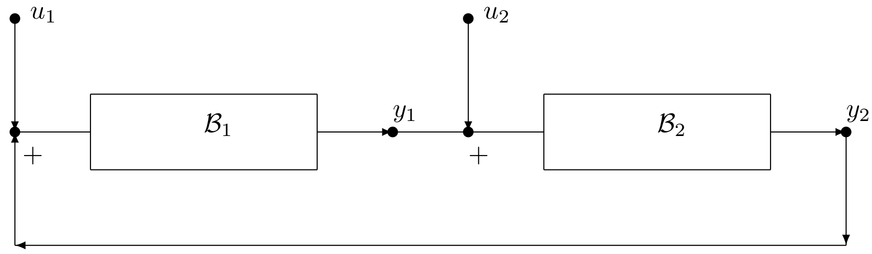
$$u_\lambda \in \mathbb{C}\left[z\right]^m,\ \deg(u_\lambda) \leq d\ \Rightarrow$$

$$y := H \circ u \textit{ has the same form}$$

Analogue of BIBO-stability.

STABILIZABILITY

output feedback connection of two IO-behaviors $\mathcal{B}_1$ (*plant*) and $\mathcal{B}_2$ (*compensator*):



where

$$\mathcal{B}_1 := \{\begin{pmatrix} y_1 \\ u_1 \end{pmatrix} \in \mathcal{F}^{p+m};\ P_1 \circ y_1 = Q_1 \circ u_1\}$$

$$\mathcal{B}_2 := \{\begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2\}$$

$$P_1H_1 = Q_1,\ P_1 \in \mathbb{C}\left[s\right]^{k_1\times p}, H_1 \in \mathbb{C}\left(s\right)^{p\times m}$$

$$R_1 := (P_1, -Q_1),\ U_1 := \mathbb{C}\left[s\right]^{1\times k_1} R_1.$$

feedback behavior

$$\mathcal{B} := \{ \begin{pmatrix} y \\ u \end{pmatrix} \in \mathcal{F}^{2(p+m)};$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, u = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \in \mathcal{F}^{p+m}$$

$$P_1 \circ y_1 = Q_1 \circ (u_1 + y_2)$$

$$P_2 \circ y_2 = Q_2 \circ (u_2 + y_1)\}$$

$$\mathcal{B}_1 \ \textit{stabilizable}{:}\ \Leftrightarrow\ \exists \mathcal{B}_2 \text{ such that}$$

$$\mathcal{B} = \text{stable IO-behavior}$$

$$\text{with input } u = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \in \mathcal{F}^{p+m}$$

$$\mathcal{B}_2 := \textit{stabilizing compensator} \text{ or controller}$$

GABRIEL LOCALIZATION

$$A_T = \cap_{\lambda\in\Lambda_2} \subset A_{\mathfrak{m}_\lambda},\ \lambda \in \Lambda_2.$$

standard localization $U_{1,T} = A_T^{1\times k_1} R_1 \subset$
Gabriel localization $U_{1,\mathfrak{T}} :=$
$A_T^{1\times(p+m)} \cap \cap_{\lambda\in\Lambda_2} U_{1,\mathfrak{m}_\lambda} =$
$A_T^{1\times(p+m)} \cap \cap_{\lambda\in\Lambda_2} A_{\mathfrak{m}_\lambda}^{1\times k} R_1$

with respect to the linear topology

$$\mathfrak{T} := \{\mathfrak{a};\ \text{ideal of } A,\ \forall\lambda\in\Lambda_2 : (A/\mathfrak{a})_{\mathfrak{m}_\lambda} = 0\}$$

$$\Lambda_2 \text{ ideal-convex} \quad \Leftrightarrow \; \forall M: \; M_T = M_{\mathfrak{T}}$$

Open problem

Construct via Gröbner bases!?

$R_{\mathrm{st}} \in A^{k_{\mathrm{st}} \times (p+m)}$ such that

$$U_{1,\mathfrak{T}} = A_T^{1 \times k_{st}} R_{\mathrm{st}}.$$

Theorem 0.2. (Special cases) *Compute* $\mathfrak{a}_1 := \mathrm{ann}_{\mathbb{C}[s]}(t(M_1)$ *via Gröbner bases.*
(1) Then

$$\forall \lambda \in \Lambda_2 : \ M_{1,\mathfrak{m}_\lambda} \ \textit{torsionfree} \ \Leftrightarrow$$
$$V(\mathfrak{a}_1) \cap \Lambda_2 = \emptyset,$$
$$\textit{and then } U_{1,\mathfrak{T}} = U_{1,\mathrm{cont},T}$$
$$(\text{ constructive!})$$

(2)

$$M_{1,T} \ \textit{torsionfree} \ \ \Leftrightarrow \ \mathfrak{a}_1 \cap T \neq \emptyset \ \Leftrightarrow$$
$$M_{1,T} \cong \mathcal{M} := A_T^{1\times p} H_1 + A_T^{1\times m}$$
$$(A_T - \textit{lattice in } \mathbb{C}(s)^{1\times m})$$
$$\textit{and then } U_{1,\mathfrak{T}} = U_{1,\mathrm{cont},T} = U_{1,T}$$
$$(\text{ constructive!})$$

Theorem 0.3. (Stabilization)
(1) Equivalent for $\mathcal{B}_1$:

(i) $\mathcal{B}_1$ stabilizable.

(ii) $U_{1,\mathfrak{T}} = A_T^{1\times k_{st}} R_{st} =$ direct summand of $A_T^{1\times(p+m)}$.

(iii)

$$\exists G_1 \in A_T^{(p+m)\times k_{\mathrm{st}}} \text{ such that}$$
$$R_{\mathrm{st}} = R_{\mathrm{st}} G_1 R_{\mathrm{st}}. \text{ Then}$$
$$E_1 := G_1 R_{\mathrm{st}} = E_1^2 \in A_T^{(p+m)\times(p+m)} \text{ projection}$$
$$U_{1,\mathfrak{T}} = A_T^{1\times(p+m)} E_1.$$

Computation *via Buchberger's algorithm for $\mathbb{C}[s]$ and test $\mathfrak{a} \cap T \neq \emptyset$.*

(iv) With G_1 *and* E_1 *from (iii):*

$$\exists X \in A_T^{(p+m)\times k_{\rm st}} \text{ with } R_{\rm st} X R_{\rm st} = 0 \text{ such that}$$
$$E := E_1 + X R_{\rm st} = E^2 \in A_T^{(p+m)\times(p+m)},$$
$$U_{1,\mathfrak{T}} = A_T^{1\times(p+m)} E \text{ and}$$
$$\operatorname{rank}((\mathrm{id}_{p+m} - E)\begin{pmatrix} 0 \\ \mathrm{id}_m \end{pmatrix}) = m.$$

(v) **If** $M_{1,T}$ *is torsionfree, in particular if* $\mathcal{B}_1$ *=unique controllable realization of* H_1*:*

$M_{1,T} (\cong \mathcal{M} := A_T^{1\times p} H_1 + A_T^{1\times m})$ *projective or* H_1 *stabilizable as usual [A. Quadrat 2006].*

(vi) **If** Λ_2 *ideal-convex :*

$M_{1,T} \cong \mathcal{M}$ *projective or* $\operatorname{sing}(\mathcal{B}) \cap \Lambda_2 = \emptyset$.

(2) Construction of one stabilizing compensator $\mathcal{B}_2$*: With E from (1)(iv) choose*

$$t \in T \ \textit{with}\ tE \in \mathbb{C}\,[s]^{(p+m)\times(p+m)}$$

$$R_2 := (-Q_2, P_2) := t(\mathrm{id}_{p+m} - E)$$

$$\mathcal{B}_2 := \{ \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathcal{F}^{p+m};\ P_2 \circ y_2 = Q_2 \circ u_2 \}$$

(3) Parametrization*: Obtain* all *stabilizing compensators according to (1)(iv) and (2) via Gröbner bases.*