IDEAL TURAEV–VIRO INVARIANTS — OUTLINE

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Abstract. Turaev–Viro invariants are homeomorphism invariants of 3–dimensional manifolds. Their definition yields to the so-called Biedenharn–Elliott equations defining a series of polynomial ideals. In this outline, we will define the Biedenharn–Elliott equations without to explain their relevance in applications. We propose it as a challange to compute a Gröbner basis for the occuring ideals. In several examples, it turned out that different computer algebra systems differ widely in their performance.
This paper is an outline of [2].

1. The Biedenharn–Elliott ideal

Let $\mathcal{F}$ be a finite set, to whose elements we will refer by **2-strata colours**, and let $-$ be an involution on $\mathcal{F}$. Let $\mathcal{G}$ be another finite set, whose elements are called **edge colours** (these names are motivate by the topological application, see [2]). For any $f \in \mathcal{F}$, the symbol $w(f)$ is referred to as the **weight** of f. To any 6-tuple $a, \ldots, f \in \mathcal{F}$ of 2-strata coulours and any 4-tuple $A, \ldots, D \in \mathcal{G}$ of edge coulours, we associate a symbol $\left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|_{C,D}^{A,B}$, called **$6j4k$-symbol**. If $\mathcal{G} = \{*\}$ contains only one element, we just have a **$6j$-symbol** $v^{\varphi} := \left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right| = \left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|_{*,*}^{*,*}$, to simplify the notation.

We want that the following **symmetry conditions** are satisfyed by the colour weights and $6j4k$-symbols. First, we assume that $w(f) = w(-f)$ for all $f \in \mathcal{F}$. Secondly, we assume that for all $a, b, c, d, e, f \in \mathcal{F}$ and all $A, B, C, D \in \mathcal{G}$ holds

$$\begin{aligned} \left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|_{C,D}^{A,B} &= \left|\begin{smallmatrix} b & c & a \\ -e & -d & f \end{smallmatrix}\right|_{B,D}^{C,A} \\ &= \left|\begin{smallmatrix} a & -d & -e \\ -f & -c & -b \end{smallmatrix}\right|_{D,C}^{A,B} \end{aligned}$$

To keep notations simple, we make no notational difference between a colour weight respectively a $6j4k$-symbol and its equivalence class. Let R be the polynomial ring over some field $\mathbb{F}$ whose variables are the equivalence classes of colour weights and $6j4k$-symbols. In this outline we will always assume $\mathbb{F} = \mathbb{Q}$, but in related applications it sometimes turned out to be reasonable to choose for $\mathbb{F}$ a finite field. Let $m = |\mathcal{F}|$ and $n = |\mathcal{G}|$.

In our applications in low-dimensional topology, one would read off a polynomial from a triangulation of a compact 3-manifolds. This polynomial would depend on the triangulation, but when one passes to the quotient of R by an appropriate ideal (the Biedenharn–Elliott ideal), the result is independent of the triangulation, and hence yields a homeomorphism invariant of 3-manifolds; we call this an **ideal Turaev–Viro invariant**. If the polynomial is evaluated at some point of the variety of the Biedenharn–Elliott ideal, we also get a homeomorphism invariant, calld **numerical** Turaev–Viro invariant. We are going to define the Biedenharn–Elliott ideal.

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Recall $m = |\mathcal{F}|$, $n = |\mathcal{G}|$. The **Biedenharn–Elliott ideal** $I_{m,n} \subset R$ of type (m,n) is the ideal in R that is generated by

$$\sum_{A\in\mathcal{G}} \left|\begin{smallmatrix} j_1 & j_2 & j_3 \\ j_9 & j_8 & j_7 \end{smallmatrix}\right|^{k_1,k_2}_{k_3,A} \cdot \left|\begin{smallmatrix} j_4 & j_5 & j_6 \\ -j_9 & -j_8 & -j_7 \end{smallmatrix}\right|^{k_4,k_5}_{k_6,A} -$$

$$\sum_{A_1,A_2,A_3\in\mathcal{G}} \sum_{j\in\mathcal{F}} w(j) \cdot \left|\begin{smallmatrix} j & j_1 & j_2 \\ j_7 & -j_5 & -j_4 \end{smallmatrix}\right|^{A_1,A_2}_{k_1,k_4} \cdot \left|\begin{smallmatrix} j & j_2 & j_3 \\ j_9 & -j_6 & -j_5 \end{smallmatrix}\right|^{A_2,A_3}_{k_3,k_6} \cdot \left|\begin{smallmatrix} j & j_3 & j_1 \\ -j_8 & -j_4 & -j_6 \end{smallmatrix}\right|^{A_3,A_1}_{k_2,k_5},$$

for all $j_1,\dots,j_9 \in \mathcal{F}$ and all $k_1,\dots,k_6 \in \mathcal{G}$. These generators are known from quantum mechanics and are known as "Biedenharn–Elliott equations" [3]. Note that $I_{m,n}(M)$ depends on the involution that we chose for $\mathcal{F}$. But for simplicity we did not include the involution in our notation.

There is a systematic way to construct some non-trivial complex solutions of the Biedenharn–Elliott equations for arbitrarily large colour sets, based on the representation theory of Quantum Groups [7]; first examples had been presented by V. Turaev and O. Viro [6]. However, this does not yiels *all* points of the variety of $I_{m,n}$.

We conclude this section with some remarks on the computational complexity. The number of $6j4k$-symbols is $m^6 \cdot n^4$, and we have m colour weights. The number of variables of R is slightly less since we take equivalence classes, though it still grows rapidly with the number of colours. We have roughly $m^9 \cdot n^6$ generators of $I_{m,n}$ (some of them coincide by symmetry), which are polynomials of degree 4 involving up to $n + m \cdot n^3$ monomials. Apart from the sheer size of the generating system of $I_{m,n}$, the generation of the Biedenharn–Elliott equations is easy. But of course it is challanging to compute a Gröbner basis for such a huge polynomial system.

2. Simplifying assumptions

One way to reduce the complexity of the problem is to introduce simplifying assumptions. For instance, we can restrict the set of colourings by evaluating some equivalence classes of $6j4k$-symbols at zero. This strategy is supported by the examples obtained from quantum groups. E.g., in the invariants constructed in [6], we have colour sets $\mathcal{F} = \{0,\dots,k-1\}$ and $\mathcal{G} = \{*\}$, and the $6j$-symbol $\left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|$ vanishes unless the four triples (a,b,d), (a,c,e), (b,c,f) and (d,e,f) satisfy the triangle inequalities and have even sum. We will refer to this as the *admissibility condition*. It is also reasonable to pick one colour $z \in \mathcal{F}$ and impose $w(z) = \left|\begin{smallmatrix} z & z & z \\ z & z & z \end{smallmatrix}\right| = 1$.

In order to deal with non-trivial $\mathcal{G}$, one might reduce the number of variables of R by providing the $6j4k$-symbols with additional symmetries. For instance, we could assume

$$\left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|^{A,B}_{C,D} = \left|\begin{smallmatrix} a & b & c \\ f & e & d \end{smallmatrix}\right|^{\sigma(A),\sigma(B)}_{\sigma(C),\sigma(D)},$$

for any $a,b,c,d,e,f \in \mathcal{F}$, any $A,B,C,D \in \mathcal{G}$ and any permutation σ of $\{A,B,C,D\}$.

Because all mentioned simplifying assumptions are known to hold for some numerical Turaev–Viro invariants, we can be sure that *ideal* Turaev–Viro invariants subject to these assumptions are nontrivial as well.

We present here an example of type $(2,1)$ with simplifying assumptions inspired by Matveev's ϵ-invariant [5]. We have 2-strata colours $\mathcal{F} = \{1,2\}$ and trivial edge colours $\mathcal{G} = \{*\}$. So we have $6j$-symbols rather than $6j4k$-symbols. We work under the simplifying assumption that the $6j$-symbol of a vertex v vanishes in one of the four triples (a,b,d), (a,c,e), (b,c,f) and (d,e,f) the colour 2 is assumed precisely once. We also assume $w(1) = 1$ and $\left|\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right| = 1$, which holds for the ϵ-invariant as well. So the variables of R are four equivalence classes $\left|\begin{smallmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{smallmatrix}\right|, \left|\begin{smallmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \end{smallmatrix}\right|, \left|\begin{smallmatrix} 2 & 1 & 2 \\ 2 & 2 & 2 \end{smallmatrix}\right|, \left|\begin{smallmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{smallmatrix}\right|$ of $6j$-symbols, and the colour weight $w(2)$. The Biedenharn–Elliott ideal in this setting is generated by 12 polynomials (see [2]), and one obtains a complex solution of the equations as follows:

$$\left|\begin{smallmatrix}1&1&2\\2&2&1\end{smallmatrix}\right| = \epsilon^{-\frac{1}{2}}, \left|\begin{smallmatrix}2&1&2\\2&1&2\end{smallmatrix}\right| = \epsilon^{-1}, \left|\begin{smallmatrix}2&1&2\\2&2&2\end{smallmatrix}\right| = \epsilon^{-1}, \left|\begin{smallmatrix}2&2&2\\2&2&2\end{smallmatrix}\right| = -\epsilon^{-2}, w(2) = \epsilon,$$

where ϵ is any root of $\epsilon^2 - \epsilon - 1$.

3. The Challange

I collected seven examples, translated into Singular language [1]. We did not erase duplicate generators. The file names mean the following: If there is an 'Or' in the name, then we have $\mathcal{F} = \{-m, \ldots, m\}$ with the natural involution; otherwise the name contains 'Unor', and then $\mathcal{F} = \{0, \ldots, m-1\}$ with trivial involution. We have $\mathcal{G} = \{*\}$ (i.e., $n = 1$) in all files. The rest of the name refers to the simplifying assumption that was used. The files contain a Singular [1] description of the ring R (r in the files) and the Biedenharn–Elliott ideal (i in the files). The file `m2n1MatUnor.dat` contains the example presented in Section 2, which should be easy on any computer algebra system.

(1) Compute a Gröbner basis for the examples presented in the files, which can be considered as a benchmark test. For instance, `m1n1AllOr.dat` should be at most 2 minutes with `slimgb` in Singular or some hours with `std` in Singular.
(2) Create your own examples. This involves definition of R (hence, determination of the equivalence classes of $6j4k$-symbols) and generation of the Biedenharn–Elliott equations, choice of a monomial order, and computation of a Gröbner basis. Of course, it would be nice to make $\mathcal{F}$ as large as possible, to have a non-trivial involution on $\mathcal{F}$, or to have non-trivial edge colours $\mathcal{G}$.
(3) Formulate your own reasonable simplifying assumptions. The admissibility condition certainly is reasonable, since it is known from the represenation theory of Quantum Groups that the Biedenharn–Elliott ideal is non-trivial, and moreover the resulting 3-manifold invariants are known to be very interesting. But other assumptions could be interesting as well. We remark that the assumptions used to create `m3n1OrAnsA.dat` and `m3n1OrAnsB.dat` yield a non-trivial Gröbner basis, but unfortunately the associated 3-manifold invariant turns out to be trivial.
(4) Compute the radical of the Biedenharn–Elliott ideals, in order to be able to compare the strength of numerical and ideal Turaev–Viro invariants.

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