Coherent Configurations and Association Schemes. Part II. Spectral Properties and Merging of Classes

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Notations

$[x, y]$ is an integer interval;

Let X be a finite set. Then $M_X(\mathbb{F})$ is the algebra of $X \times X$-matrices over the field $\mathbb{F}$;

The x-th row (column) of a matrix $A \in M_X(\mathbb{F})$ is denoted by A_x (resp., $A^{(x)}$). A^t is the matrix transposed to A.

If $\mathcal{L} \subseteq M_X(\mathbb{F})$, then $\langle \mathcal{L} \rangle$ is a $\mathbb{F}$-linear span of $\mathcal{L}$.

If $R \subseteq X \times X$, then $\widehat{R} \in M_X(\mathbb{F})$ is defined as

$$\widehat{R}_{xy} = \begin{cases} 1, (x, y) \in R \\ 0, (x, y) \notin R \end{cases}$$

If $\mathcal{R}$ is a set of binary relations on X, then $\widehat{\mathcal{R}} := \{\widehat{R} \mid R \in \mathcal{R}\}$.

Association schemes

Let X be a finite set. A partition $\mathcal{R} = \{R_0, ..., R_d\}$ of $X \times X$ is called an **association scheme** with d classes iff

(AS1) $R_0 = \{(x, x) \,|\, x \in X\}$ is a diagonal relation;

(AS2) ${R_i}^t = R_{i'}$ for some $i' \in [0, d]$;

(AS3) for all $i, j, k \in [0, d]$ there exists an integer p_{ij}^k such that for all $(x, y) \in R_k$

$$|\{z \in X \,|\, (x, z) \in R_i \wedge (z, y) \in R_j\}| = p_{ij}^k.$$

The graphs (X, R_i) are called the **basic** graphs of the scheme.

Adjacency algebra of an association scheme

The matrix $\sum_{i=0}^{d} i\widehat{R_i}$ is called the **adjacency matrix** of the scheme $(X,\mathcal{R})$.

Let $\mathcal{A}$ be the subspace of $M_X(\mathbb{C})$ consisting of all $X \times X$ matrices which are constant on the relations R_i, that is $A \in \mathcal{A}$ iff

$$(x,y) \in R_i \wedge (z,w) \in R_i \implies A_{xy} = A_{zw}.$$

The adjacency matrices $\widehat{R_i}, i = 0,...,d$ form a basis of $\mathcal{A}$ which is called the **standard** basis of $\mathcal{A}$. It follows from (AS3) that

$$\widehat{R_i}\widehat{R_j} = \sum_{i=0}^{d} p_{ij}^{k}\widehat{R_k}.$$

In particular $\mathcal{A}$ is a subalgebra of $M_X(\mathbb{C})$. It is called the **adjacency** algebra of $(X,\mathcal{R})$.

Examples

The adjacency matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$$

determines a non-symmetric association scheme on 6 points with 5 classes. Since a cyclic shift $x \mapsto x + 1, x \in \mathbb{Z}_6$ is an automorpism of this scheme, it's basic relations may be desrcibed as follows:

$$R_i = \{(x,y) \in \mathbb{Z}_6 \times \mathbb{Z}_6 \,|\, y - x = i\},\ \ i = 0, ..., 5.$$

Examples

Let H be a group. For each $h \in H$ we define a binary relation R_h as follows:

$$R_h := \{(x, y) \in H \times H \mid x^{-1}y = h\}.$$

Then the relations $\{R_h\}_{h \in H}$ form an association scheme with $|H| - 1$ classes. It is called a **thin** scheme of H. Note that each relation of this scheme is a permutation of H and $\{R_h\}_{h \in H}$ is a regular permutation group isomorphic to H.

The adjacency algebra of this scheme is isomorphic to the group algebra $\mathbb{C}[H]$.

Note that the previous example is a thin scheme of the group $\mathbb{Z}_6$.

Fusion schemes

Let $(X, \mathcal{R})$ and $(X, \mathcal{S})$ be two association schemes. The scheme $\mathcal{S}$ is called **fusion** of $\mathcal{R}$, notation $\mathcal{S} \sqsubseteq \mathcal{R}$, iff each basic relation of $\mathcal{S}$ is a union of some relations from $\mathcal{R}$.

$$\mathcal{S} \sqsubseteq \mathcal{R} \iff \langle \widehat{\mathcal{S}} \rangle \subseteq \langle \widehat{\mathcal{R}} \rangle.$$

Each scheme has a **trivial** fusion, namely the scheme with one class.

Write $\mathcal{R} = \{R_i\}_{i=0}^{d}$. Then every fusion scheme of $\mathcal{R}$ is uniquely determined by a partition Π of $[0, d]$.

Problem. Find all partitions of $[0, d]$ corresponding to fusion schemes.

Problem. Given a scheme $(X, \mathcal{R})$. Does there exist a nontrivial fusion of $\mathcal{R}$?

Fusion schemes (an example)

The scheme with the adjacency matrix

$$\begin{pmatrix} 0 & 2 & 2 & 1 & 2 & 2 \\ 2 & 0 & 2 & 2 & 1 & 2 \\ 2 & 2 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 2 & 1 & 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 2 & 2 & 0 \end{pmatrix}$$

is a fusion of the scheme with the adjacency matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$$

Fusion schemes

Theorem. A partition Π of $[0,d]$ determines a fusion of $\mathcal{R}$ iff it satisfies the following conditions

(F1) $\{0\} \in \Pi$;

(F2) $P' \in \Pi$ for each $P \in \Pi$,
where $P' := \{p' \,|\, p \in P\}$;

(F3) for each triple $P \in \Pi$, $S \in \Pi, T \in \Pi$ and any pair $t, t^* \in T$

$$\sum_{i \in P, j \in S} p_{ij}^{t} = \sum_{i \in P, j \in S} p_{ij}^{t^*}.$$

Fusions in imprimitive schemes

A subset $\mathcal{S} \subseteq \mathcal{R}$ is called **closed** iff the linear span $\langle \hat{R}_i \rangle_{R_i \in \mathcal{S}}$ is a subalgebra (in a usual sense) of $\langle \widehat{\mathcal{R}} \rangle$.
Another characterization of a closed subset is this: a subset $\mathcal{S} \subseteq \mathcal{R}$ is closed iff $\cup_{R \in \mathcal{S}}$ is an equivalence relation on X.

Proposition. If $\mathcal{S}$ is a closed subset, then $R_0 \in \mathcal{S}$ and $\mathcal{S}^t = \mathcal{S}$.

Each association scheme contains two trivial closed subsets $\{R_0\}$ and $\mathcal{R}$. A scheme is called **primitive** iff it has only the trivial closed subsets.

Proposition. Let $\mathcal{S} = \{R_0, ..., R_f\}$ be a closed subset of $\mathcal{R}$. Then the partition

$$\{0\}, ..., \{f\}, \{f+1, ..., d\}$$

determines a fusion of $\mathcal{R}$.

Fusions in thin schemes

Let H be a group and $(H, \mathcal{R} := \{R_h\}_{h \in H})$ it's thin scheme. There exists a one-to-one correspondence between fusions of $\mathcal{R}$ and **Schur partitions** of H.

A partition Π of H is called Schur partition iff

(S1) $\{e\} \in \Pi$;

(S2) $P^{-1} := \{p^{-1} \,|\, p \in P\} \in \Pi$ for each $P \in \Pi$;

(S3) for each triple $Q, R, S \in \Pi$ there exists an integer $p_{Q,R}^S$ such that

$$|\{(x, y) \in Q \times R \,|\, xy = s\}| = p_{Q,R}^S$$

holds for each $s \in S$.

If $\Pi = \{P_0, ..., P_d\}$ is a Schur partition of H, then the linear span of $\underline{P_i} := \sum_{g \in P_i} g$ is a **Schur ring (algebra)** over the group H.

Schur partitions (examples)

Theorem. Let $\Phi \leq \mathrm{Aut}(H)$ be an arbitrary subgroup and $O_0, ..., Q_d$ be the complete set of Φ-orbits on H. Then $O_0, ..., O_d$ is a Schur partition of H.

For example, if we take $H = \mathbb{Z}_6$ and $\Phi = \mathrm{Aut}(\mathbb{Z}_6)$, then the corresponding partition

$$\{0\}, \{2, 4\}, \{3\}, \{1, 5\}$$

is a Schur partition of $\mathbb{Z}_6$.

If Φ is a group of inner automorphisms of a finite group H, then Φ-orbits are nothing but the conjugacy classes of H. Thus the conjugacy classes of H always form a Schur partition of H. The corresponding scheme is known as a **group scheme** (Bannai,[B-91]).

Commutative association schemes

An association scheme is called **commutative** if it's adjacency algebra is commutative. In this case the adjacency algebra is called the **Bose-Mesnar** algebra of a scheme

Theorem 2 [Weisfeiler, Higman]. Every association scheme with at most 4 classes is commutative.

The next statement yields another source of commutative schemes.

Proposition 2. Symmetric scheme is commutative.

Proof.

If A, B are two arbitrary matrices from $\mathcal{A}$, then

$$AB \in \mathcal{A} \Rightarrow AB = (AB)^t = B^t A^t = BA.$$

□

Second standard basis

Let $(X, \{R_i\}_{i=0}^d)$ be a commutative association scheme and $A_0, ..., A_d$ the first standard basis of it's Bose-Mesner algebra $\mathcal{A}$.

Theorem. The adjacency algebra of an association scheme is semisimple.

This Theorem is true for non-commutative association schemes too.

Thus, if $\mathcal{A}$ is commutative, then $\mathcal{A} \cong \mathbb{C}^{d+1}$ and there exists a decomposition of the unity $I = A_0$ into a sum of pairwise orthogonal idempotents $I = E_0 + E_1 + ... + E_d$.

The matrices $E_0, ..., E_d$ form a basis of $\mathcal{A}$ which is called the **second** standard basis of $\mathcal{A}$.

Properties of the second standard basis

$E_0 = \frac{1}{|X|}J$;

$E_iE_j = \delta_{ij}E_i$;

$E_\imath^t = E_{\imath^*}$ for some $\imath^* \in [0, d]$;

Columns of E_i are eigenvectors for each A_j;

Since every BM-algebra is closed with respect to Schur-Hadamard product, there exist scalars $q_{ij}^k \in \mathbb{C}$, called **Krein parameters** of the scheme such that

$$E_i \circ E_j = \sum_{k=0}^{d} q_{ij}^k E_k.$$

Theorem [Krein]. q_{ij}^k are non-negative real numbers.

The eigenmatrices of an association scheme

Since $A_0, ..., A_d$ and $E_0, ..., E_d$ are bases of $\mathcal{A}$, there exist scalars $p_i(j) \in \mathbb{C}$ and $q_i(j) \in \mathbb{C}$ such that

$$A_i = \sum_{j=0}^{d} p_i(j) E_j;$$

$$E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_i(j) A_j.$$

Note that $(p_i(0), p_i(1), ..., p_i(d))$ are eigenvalues of A_i.

The matrices $P_{ij} := p_j(i), Q_{ij} := q_j(i)$ are called the **first and second eigenmatrices** of $\mathcal{A}$. They are related as follows

$$PQ = QP = |X|I.$$

The eigenmatrices of an association scheme

The eigenmatrices P and Q have the following forms

$$P = \begin{pmatrix} 1 & v_1 & \dots & v_d \\ 1 & * & \dots & * \\ . & . & \dots & . \\ 1 & * & \dots & * \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & m_1 & \dots & m_d \\ 1 & * & \dots & * \\ . & . & \dots & . \\ 1 & * & \dots & * \end{pmatrix}$$

where v_i denotes the valency of R_i and m_i is the rank of the idemotent E_i.

Fusions in commutative association schemes

The eigenmatrices of commutative association schemes may be efficiently used for fusion enumeration. The enumeration becomes easier, since we work with a matrix instead of three-dimensional tensor of structure constants. This fact was realized independently by many researches. We'd like to mention Bannai [B-91], Bridges and Mena [BM-79], Godsil [G-93], Johnson and Smith [JS-89], Mathon, Muzychuk [M-88]. But a rigorous statement was publisehd independently in [B-91] and [M-88]. To formulate the precise statement we need some additional notions.

Fusions in commutative association schemes

Let P be the first eigenmatrix of an association scheme $(X, \mathcal{R} = \{R_0, ..., R_d\})$. Given an arbitrary partition $\Pi = \{\Pi_1, ..., \Pi_\ell\}$ of $[0, d]$, define P^Π as a $(d+1) \times \ell$-matrix the columns of which are obtained by summing up the columns of P within the classes of Π:

$$(P^\Pi)^{(i)} = \sum_{j \in \Pi_i} P^{(j)}.$$

Since P is non-degenerate, $\text{rank}(P^\Pi) = \ell$. Define a **dual** partition Π^P of $[0, d]$ according to the following equivalence:

$$i \sim j \iff (\Pi^P)_i = (\Pi^P)_j.$$

Since $\text{rank}(P^\Pi) = \ell$, there are at least ℓ classes in Π^P.

Fusions in commutative association schemes

Theorem. Let $\Pi = \{\Pi_1, ..., \Pi_\ell\}$ be a partition of $[0, d]$ which satisfies (F1) and (F2) (see page 8). Then Π determines a fusion iff $|\Pi^P| = \ell$.

If Π determines a fusion, then writing $\Pi^P = \{\Sigma_1, ..., \Sigma_\ell\}$ we obtain that

The matrices $A_{\Pi_i} := \sum_{j\in\Pi_i} A_j$ form a first standard basis of a fusion algebra;

The matrices $E_{\Sigma_i} := \sum_{j\in\Sigma_i} E_j$ form a second standard basis of a fusion algebra;

The first eigenmatrix of the fusion determined by Π is obtained from P^Π by removing repeated rows.

Fusions in commutative association schemes

Thus every fusion scheme $(X, \mathcal{S})$ of $(X, \mathcal{R} = \{R_i\}_{i=0}^d)$ determines two partitions Π and Σ of the index set $[0, d]$ which are relation via

$$\Sigma = \Pi^P, \Pi = \Sigma^Q.$$

We call these partitions as the **first** and **second** partitions related to the fusion scheme.

We also say that relations R_i and R_j (idempotents E_i and E_j) are **fused** in $\mathcal{S}$ if i, j belong to the same class of Π (resp. Σ).

Note that the above Theorem has it's natural dual:

Theorem. Let $\Sigma = \{\Sigma_1, ..., \Sigma_\ell\}$ be a partition of $[0, d]$ such that $\{0\} \in \Sigma$ and $\Sigma^* = \Sigma$. Then Σ determines a fusion iff $|\Sigma^Q| = \ell$.

Example: a fusion in Johnson scheme

Let $\binom{[1,v]}{d}$ be a set of d-element subsets of a v-element set $[1, v]$. For each $i \in [0, d]$ we define

$$R_i := \left\{ (A, B) \in \binom{[1,v]}{d} \times \binom{[1,v]}{d} \middle| |A \cap B| = d - i \right\}.$$

The relations $R_i, i \in [0, d]$ form a d-class association scheme on $\binom{[1,v]}{d}$ known as **Johnson** scheme $J(v, d)$.

For example the first eigenmatrix of $J(8, 4)$ has the following form

$$\begin{pmatrix} 1 & 16 & 36 & 16 & 1 \\ 1 & 8 & 0 & -8 & -1 \\ 1 & 2 & -6 & 2 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$

Take $\Pi = \{\{0\}, \{1, 3\}, \{2\}, \{4\}\}$.

Example: a fusion in Johnson scheme

$$P^{\Pi} = \begin{pmatrix} 1 & 32 & 36 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 4 & -6 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & -8 & 6 & 1 \end{pmatrix}$$

Then $\Pi^{P} = \{\{0\},\{1,3\},\{2\},\{4\}\}$, and, consequently, Π determines a fusion scheme of $J(8,4)$. The first eigenmatrix of this fusion scheme is

$$\begin{pmatrix} 1 & 32 & 36 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 4 & -6 & 1 \\ 1 & -8 & 6 & 1 \end{pmatrix}$$

Enumeration algorithm

We start with some definitions.

Let $\vec{v} = (v_0, ..., v_d) \in \mathbb{C}^{d+1}$ be an arbitrary vector. Let us say that a subset $A \subseteq [0, d]$ is **orthogonal** to $\vec{v}$ iff $\sum_{i \in A} v_i = 0$.

Proposition. Let Π be a partition which determines a fusion scheme, say $\mathcal{S}$, of $\mathcal{R}$. If E_i and E_j (dually R_i and R_j) are fused in $\mathcal{S}$, then each $P \in \Pi$ is orthogonal to $P_i - P_j$ (resp. each $S \in \Pi^P$ is orthogonal to $Q_i - Q_j$).

Note that the sets $\{0\}$ and $[1, d]$ are always orthogonal to $P_i - P_j$ and $Q_i - Q_j$ whenever $i, j > 0$.

Enumeration algorithm

Let $\mathcal{S}$ be a non-trivial fusion of a commutative scheme $(X, \mathcal{R} = \{R_i\}_{i=0}^{d})$. Then there exist at least two idempotents E_i and E_j, $0 < i < j \leq d$ which are fused in $\mathcal{S}$. This yields us the following, rather "naive", algorithm

- **For** $0 < i < j \leq d$ **do**

  Find the set $\mathcal{O}_{ij}$ consisting of all subsets of $\{1, ..., d\}$ which are orthogonal to the vector $P_i - P_j$;

  Construct all partitions Π from $\mathcal{O}_{ij}$ which satisfy $\Pi' = \Pi$. **Select** those which determine a fusion scheme.

Note that this algorithm may be improved. For example, if one of the idempotents, say E_1, generates the whole scheme, then we need to consider only pairs of type $1, j$.

Fusions in $J(8,4)$

We illustrate this algorithm enumerating all fusions in $J(8,4)$.

The scheme $J(v,d)$ is Q-polynomial which implies that E_1 generates the whole scheme. Thus we have to analize only three possible pairs $(1,2)$, $(1,3)$ and $(1,4)$.

If E_1 is fused with E_2, then

$$P_1 - P_2 = (0,6,6,-10,-2)$$

and the only subset of $\{1,2,3,4\}$ orthogonal to $P_1 - P_2$ is $\{1,2,3,4\}$ itself. Hence the only fusion scheme where E_1 and E_2 are fused is the trivial one.

Fusions in $J(8,4)$

If E_1 is fused with E_3, then the elements of Π are orthogonal to $P_1 - P_3 = (0, 10, 0, -10, 0)$. Thus

$$\mathcal{O}_{13} = \left\{ \begin{array}{l} \{1,3\}, \{2\}, \{4\}, \{1,3,2\}, \\ \{1,3,4\}, \{2,4\}, \{1,2,3,4\} \end{array} \right\}$$

A direct check shows that one can construct only two non-trivial fusions from $\mathcal{O}_{13}$:

$$\{\{0\}, \{1,3\}, \{2\}, \{4\}\}; \ \{\{0\}, \{1,2,3\}, \{4\}\}.$$

If E_1 is fused with E_4, then

$$P_1 - P_4 = (0, 12, -6, -4, -2)$$

implying $\mathcal{O}_{14} = \{\{1,2,3,4\}\}$. Hence the only fusion scheme where E_1 and E_4 are fused is the trivial one.

Thus we got two non-trivial fusions

$$\{\{0\}, \{1,2\}, \{2\}, \{4\}\} \text{ and } \{\{0\}, \{1,2,3\}, \{4\}\}.$$

Fusions in Johnson schemes

Theorem. If $v \geq 3d + 4 \geq 13$, then $J(v, d)$ has no non-trivial fusion.

Our next goal is to enumerate fusions of q-analog of Johnson scheme which is better known as the Grassman scheme. For this purpose we need to formulate **dual Schur-Wielandt** principle.

Dual Schur-Wielandt principle

Let $S \subseteq [0, d]$ be such that the adjacency matrix A_S of the relation R_S belongs to a fusion scheme $\mathcal{S}$. Write A_S as a linear combination of the minimal idempotents

$$A_S = \sum_{i=0}^{d} P_S(i) E_i.$$

Theorem (dual Schur-Wielandt principle). For each scalar $\lambda \in \mathbb{C}$ the idempotent

$$\sum_{\{i \,|\, P_S(i)=\lambda\}} E_i$$

belongs to the BM-algebra of $\mathcal{S}$.

Dual Schur-Wielandt principle in a modular form

Sometimes it is more convinient to use this principle in modular form. To formulate it, let us denote by R the subring of $\mathbb{C}$ generated by the entries of the first eigenmatrix P (they are algebraic integers). Then we have the following

Theorem. For any ideal J of R and arbitrary $\lambda \in R/J$ the idempotent

$$\sum_{\{i \,|\, P_S(i) \equiv \lambda (\mathrm{mod}\, J)\}} E_i$$

belongs to the BM-algebra of $\mathcal{S}$.

Fusions in the Grassman scheme

Let V be an n-dimensional vector space over finite field $\mathbb{F}_q$ and $\begin{bmatrix} V \\ d \end{bmatrix}$ the set of all d-dimensional subspaces of V $(2d \leq n)$. For each $0 \leq i \leq d$ we define:

$$R_i := \left\{(U, W) \in \begin{bmatrix} V \\ d \end{bmatrix} \times \begin{bmatrix} V \\ d \end{bmatrix} \mid \dim(U \cap W) = d - i\right\}.$$

Then the relations $R_0, ..., R_d$ form an association scheme on the set $\begin{bmatrix} V \\ d \end{bmatrix}$ with d classes which is called the **Grassman** scheme. Note that this scheme is P and Q-polynomial.

Theorem. If $q \geq 2$, then the Grassman scheme has no non-trivial fusion.

In what follows $\mathcal{S}$ denotes a proper fusion of the Grassman scheme; Π and Σ are the first and the second partitions of $[1, d]$ related to $\mathcal{S}$.

Auxiliary Lemma.

Lemma. The relations R_1 and R_2 are fused in $\mathcal{S}$.

Proof. First we calculate the first eigenmatrix $\bar{P}$ of the Grassman scheme modulo q^2. We obtain the following matrix (recall that the first row is indexed by 0)

$$\begin{pmatrix} 1 & \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \ldots & \bar{v}_d \\ 1 & -1 & 0 & 0 & \ldots & 0 \\ 1 & -1-q & q & 0 & \ldots & 0 \\ . & . & . & . & \ldots & . \\ 1 & -1-q & q & 0 & \ldots & 0 \end{pmatrix}$$

That is

$$\bar{P}_2 = \ldots = \bar{P}_d = (1, -1-q, q, 0, \ldots, 0).$$

Here v_i denotes the valency of the relation R_i and $\bar{v}_i$ is the remainder of v_i modulo q^2.

Proof of the Theorem.

Let Π be the first partition of $[0, d]$ related to $\mathcal{S}$ and let $\Pi_1 \in \Pi$ be such that $1 \in \Pi_1$. Assume towards a contradiction that $2 \notin \Pi_1$. Then

$$P_{\Pi_1}(j) \equiv \begin{cases} -1 \pmod{q^2} & j = 1 \\ 0 \pmod{q^2} & j > 2 \end{cases}$$

By dual Schur-Wielandt principle E_1 belongs to the BM-algebra of $\mathcal{S}$. Since the Grassman scheme is Q-polynomial, E_1 generates it's BM-algebra. Hence $\mathcal{S}$ coincides with the Grassman scheme, contrary to being a proper fusion. □

Proof of the Theorem.

Thus R_1 and R_2 are fused in $\mathcal{S}$. This implies that each $\Sigma_i \in \Sigma$ is orthogonal to the vector $Q_1 - Q_2$. A direct calculation shows that $(Q_1 - Q_2)_i > 0$ holds for each $1 \leq i \leq d-1$. Therefore the only subset of $[1, d]$ orthogonal to $Q_1 - Q_2$ is $[1, d]$ implying $\Sigma = \{\{0\}, [1, d]\}$. That is $\mathcal{S}$ is trivial.

Fusions in dual polar schemes

Let V be one of the following vector spaces provided with a non-degenerate form, [BI-84].

Notation	dim(V)	the field	the form
B_d	$2d+1$	$\mathbb{F}_q$	quadratic
C_d	$2d$	$\mathbb{F}_q$	symplectic
D_d	$2d$	$\mathbb{F}_q$	quadratic, Witt ind. d
${}^2D_{d+1}$	$2d+2$	$\mathbb{F}_q$	quadratic, Witt ind. d
${}^2A_{2d}$	$2d+1$	$\mathbb{F}_{q^2}$	hermitian
${}^2A_{2d}$	$2d$	$\mathbb{F}_{q^2}$	hermitian

Fusions in dual polar schemes

A subspace of V is called **isotropic** if the form vanishes on this subspace. Let X be a set of all maximal isotropic subspaces (they are of dimension d). Define the following relations on X:

$$R_i := \{(U, W) \in X \times X \mid \dim(U \cap W) = d - i\}.$$

Then these relations form a symmetric association scheme which is P- and Q-polynomial. This scheme is called a **dual polar spaces** scheme.

Note that the schemes of types B_d and C_d have the same structure constants but they are not isomorphic.

Fusions in dual polar schemes

Theorem. All non-trivial proper fusions of the dual polar schemes are given in the following list (we give the first partition related to a fusion scheme).

(1) $\{0\}, \{1, 2\}, ..., \{2i - 1, 2i\}, ..., \{d - 1 + \delta(d), d\}$
if the scheme is of types B_d or C_d;

(2) $\{0\}, \{2\}, ..., \{2i\}, ...\{d - \delta(d)\}$,
$\{1, 3, ..., d - 1 + \delta(d)\}$
if the scheme is of type D_d;

(3) $\{0, 2, ..., d - \delta(d)\}$,$\{1, 3, ..., d - 1 + \delta(d)\}$
if the scheme is of type D_d.

Here $\delta(d)$ is the remainder of d modulo 2.

Fusions in dual polar schemes

Note that in the first case the graph $(X, R_1 \cup R_2)$ is distance regular. It turned out that in the case of C_d this graph wasn't known before [IMU-88].

Fusions in Hamming schemes $H(n,q)$ were studied in [M-93]. The complete list of all fusions was obtained for all $q \neq 3$. The case of $H(n,3)$ is still open.

Apart Johnson and Hamming schemes and their q-analogs, no other infinite series of P-polynomial schemes was studied. This make reasonable to formulate the following

Problem. Find fusion schemes for all known infinite series of P-polynomial schemes.

Half-homogeneous coherent configruations

Definition. A coherent configuration $(X, \{R_i\}_{i=o}^{d})$ is called **half-homogeneous** if all it's fibres have the same cardinality.

An important class of such configurations arises from Wallis-Fon-der-Flaass prolific construction of strongly regular graphs [FDF-02],[W-71]. To get a flavour of it we present the simplest examples of this configuration.

To get started we need the notion of an affine plane.

Affine plane

Affine plane is an incidence structure (P, L, I) where the incidence relation I between P (a set of points) and L (a set of lines) satisfies the following axioms:

- any two distinct points are incident to a unique line;
- given any point p and a line l not through p there exists a unique line m through p disjoint from l;
- there are at least three distinct points not on a line.

  A unique line going through two points p, q will be denoted as $\overline{pq}$.

Affine planes

Theorem. If P is finite, then there exists $n \in \mathbb{N}$ such that

(1) each line contains n points;

(2) there are $n+1$ lines through each point;

(3) $|P| = n^2$, $|L| = n^2 + n$.

The number n is called the **order** of the plane. Two lines are called parallel if they are disjoint or coincide. Being parallel is an equivalence relation with $n+1$ classes which will be denoted as $L_1, ..., L_{n+1}$. Each class contains n lines.

Association schemes from affine planes

Affine planes of order n exist for each n which is prime power.

Every affine plane (P, L, I) of order n gives rise to a commutative association scheme on P. It has $n + 1$ non-diagonal symmetric relations which correspond to parallel classes of the plane. More precisely, the relation R_i corresponding to the parallel class L_i consists of all pairs (p, q) such that $\overline{pq} \in L_i$.

The set of relations $R_0, R_1, ..., R_{n+1}$ form an association scheme which is called the **complete affine scheme of order** n.

Example: affine plane of order 2

The smallest example is the affine plane of order 2 with $P = \{1,2,3,4\}$ and
$L = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.
There are three parallel classes:

$L_1 = \{\{1,2\},\{3,4\}\}$,$L_2 = \{\{1,3\}.\{2,4\}\}$,
$L_3 = \{\{1,4\},\{2,3\}\}$.

The complete affine scheme corresponding to this plane has the following adjacency matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Note that this is a thin scheme correponding to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

WFDF configuration

We start with an affine plane (P, L, I) of order n. Recall that $|P| = n^2$.

The point set of our configuration is $X := P \times P$. We write a pair $(x, p) \in P \times P$ as x_p. The fibres of the configuration are

$$X_p := \{x_p \,|\, x \in p\}.$$

Inside each fibre X_p we define $n+1$ non-diagonal relations

$$R_i^p := \{(x_p, y_p) \,|\, \overline{xy} \in L_i\}, i = 1, ..., n+1.$$

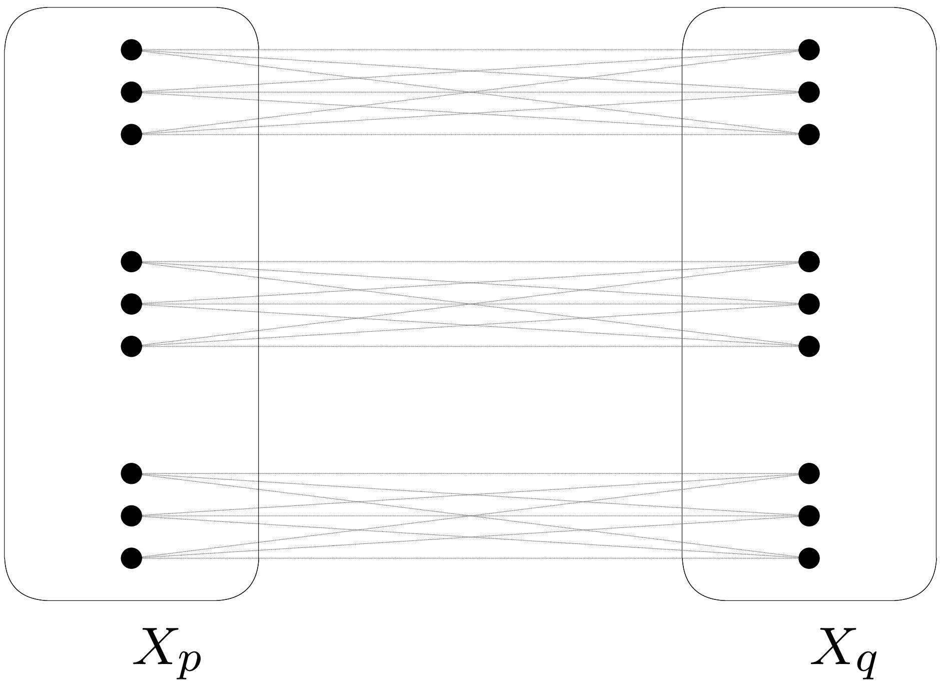
Thus on each fibre we have complete affine scheme of order n.

WFDF configuration

Between any two fibres X_p and X_q, $p \neq q$, we have only two relations denoted as S_{pq} and S^c_{pq} where

$$S_{pq} := \{(x_p, y_q) \,|\, \overline{xy} \parallel \overline{pq}\},\ S^c_{pq} := (X_p \times X_q) \setminus S_{pq}.$$

The graph of the relation S_{pq} is depictured on the next slide


X_p
X_q

Fusions in a WFDF configuration

Theorem. The relations

$$R_0 := \bigcup_{p\in P} R_0^p,\ \ S := \bigcup_{p\neq q} S_{pq},\ \ S^c := \bigcup_{p\neq q} S_{pq}^c,$$

$$T := \bigcup_{p\in P, i\in[1,n+1]} R_i^p.$$

form an amorphic association scheme with three classes. The valencies of these relations are $1, n^2-1, n(n^2-1), (n^2-n)(n^2-1)$ respectively. Each relation is a strongly regular graph of Latin square type.

Problem. Find all homogeneoue fusions in the coherent WFDF configuration defined above.

Remarks

- Each WFDF configuration has many "brothers", that is configurations with the same parameters but pairwise non-isomorphic.

- Affine plane may be replaced by an affine design.

- the structure of affine plane on the fibre set may be replaced by a partial linear space.

- the number of relations between fibres may be greater than 2.

This led to new infinite series of strongly regular graphs with new parameters.

New series of stronlgy regular graphs

If there exists an Hadamard matrix of order $4r$, then there exists a strongly regular graph with parameters [M].

$$(16r^2, (4r+1)(2r-1), 4r^2-2r-2, 4r^2-2r).$$

If $r = 7$, then we obtain a graph with parameters $(784, 377, 180, 182)$ which was mentioned as open case in Brouwer's catalog.

This graph gives rise to a symmetric design with parameters $(784, 377, 182)$. A symmetric design with these parameters appears in the infinite series of symmetric designs which was built by Ionin and Kharaghani (Designs, Codes and Cryptography, 2005). We don't know whether these designs are isomorphic or not.

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