Computation of central elements in G- and GR-algebras with SINGULAR:PLURAL

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Zassenhaus variety

$\mathfrak{g}$– fin. dim. Lie algebra over alg. closed field of char. $p > 0$.

- The *Zassenhaus variety* of $\mathfrak{g}$ is an algebraic variety $M_{\mathfrak{g}}$ with the coordinate ring $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$.
- $M_{\mathfrak{g}}$ is a normal irreducible affine variety, $\dim M_{\mathfrak{g}} = \dim \mathfrak{g}$
- irred. representations of $\mathfrak{g} \twoheadrightarrow$ points of $M_{\mathfrak{g}}$

Known facts

- $\mathfrak{g}$– complex simple Lie algebra, $r = \operatorname{rank}(\mathfrak{g})$, m_k exponents of $\mathfrak{g}$, $p_k := m_k + 1$
  - $\exists I_k \in \mathcal{Z}_{p_k}(\mathcal{U}(\mathfrak{g})), \quad 1 \leqslant k \leqslant r$ such that $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathbb{C}\,[I_1, \ldots, I_r]$ is a polynomial algebra in r generators
- $\mathfrak{g}$– Lie algebra of semisimple alg. group G over alg. closed field $\Bbbk$ of char $p > 0$
  - Chevalley Basis of $\mathfrak{g}$: $H_1, \ldots, H_r$ and $X_\alpha, \alpha \in \Sigma$. $H_i^p - H_i$ and X_α^p are alg. indep. and belong $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$. $\mathcal{O} := \Bbbk\left[H_i^p - H_i, X_\alpha^p \mid 1 \leqslant i \leqslant r, \alpha \in \Sigma\right]$
  - $T_1, \ldots, T_l$ – alg. indep. generators of invariants of $\mathcal{U}(\mathfrak{g})$ under the adjoint action of G. If $p > h(G)$ then $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathcal{O}\,[T_1, \ldots, T_l]$.

Known facts

Type	h	Alg	Rank	Dim	$p_1; \ldots; p_r$
A_r	$(r+1)$	$\mathfrak{sl}_{r+1}$	r	$r(r+2)$	$2; \ldots; (r+1)$
B_r	$2r$	$\mathfrak{so}_{2r+1}$	r	$r(2r+1)$	$2; 4; \ldots; 2r$
C_r	$2r$	$\mathfrak{sp}_r$	r	$r(2r+1)$	$2; 4; \ldots; 2r$
D_r	$2(r-1)$	$\mathfrak{so}_{2r}$	r	$r(2r-1)$	$2; 4; \ldots; 2(r-1); r$
G_2	6	$\mathfrak{g}_2$	2	14	$2; 6$
F_4	12	$\mathfrak{f}_4$	4	52	$2; 6; 8; 12$
E_6	12	$\mathfrak{e}_6$	6	78	$2; 5; 6; 8; 9; 12$
E_7	18	$\mathfrak{e}_7$	7	133	$2; 6; 8; 10; 12; 14; 18$
E_8	30	$\mathfrak{e}_8$	8	248	$2; 6; 12; 14; 18; 20; 24; 30$

Overview of `center.lib`

Procedure	Description
centralizeSet(F, V)	v.s. basis of centralizer of F within v.s. V
centralizerVS(F, D)	v.s. basis of centralizer of F
centralizerRed(F, D[, k])	reduced basis of centralizer of F
centerVS(D)	v.s. basis of center
centerRed(D[, k])	reduced basis of center
sa_reduce(V)	reduction of pairwise commuting elements
sa_poly_reduce(p, V)	reduction of p by pairwise commuting elements
center(D[, k])	reduced basis of center
centralizer(F, D[, k])	reduced basis of centralizer of F

Benchmarks

SINGULAR 3–0–0, GNU/Linux, Athlon MP 2000+ (1666MHz), 3Gb

Algebra	r	n.gens.	$p_1; \ldots; p_r$	Bound	Time in sec.
$\mathcal{U}(\mathfrak{sl}_5)$	4	24	$2, 3, 4, 5$	$\leqslant 5$	1680.47
$\mathcal{U}(\mathfrak{so}_7)$	3	21	$2, 4, 6$	$\leqslant 6$	216276.33
$\mathcal{U}(\mathfrak{sp}_3)$	3	21	$2, 4, 6$	$\leqslant 6$	14355.36
$\mathcal{U}(\mathfrak{so}_6)$	3	15	$2, 4, 3$	$\leqslant 4$	1.80
$\mathcal{U}(\mathfrak{g}_2)$	2	14	$2, 6$	$\leqslant 6$	875.20
$\mathcal{U}(\mathfrak{f}_4)$	4	52	$2, [6, 8, 12]$	$\leqslant 4$	15734.95
$\mathcal{U}(\mathfrak{e}_6)$	6	78	$2, [5, 6, 8, 9, 12]$	$\leqslant 3$	1029.31
$\mathcal{U}(\mathfrak{e}_7)$	2	133	$2, [6, 8, 10, 12, 14, 18]$	$\leqslant 3$	26735.82
$\mathcal{U}(\mathfrak{e}_8)$	2	248	$2, [6, 12, 14, 18, 20, 24, 30]$	$\leqslant 2$	306.23
$\mathcal{U}(\mathfrak{gl}_5)$	5	25	$1, 2, 3, 4, 5$	$\leqslant 5$	2095.78

Examples

- $\mathcal{U}(\mathfrak{sl}_3(\mathbb{Q}))$
- $\mathcal{U}(\mathfrak{sl}_3(\mathbb{F}_5))$
- $\mathcal{U}_q(\mathfrak{sl}_2)$, q - free parameter
- $\mathcal{U}_q(\mathfrak{sl}_2)$, q - 7-th primitive root of unity