I. Duality

- Macaulay F. S., On the Resolution of a given Modular System into Primary Systems including some Properties of Hilbert Numbers, *Math. Ann.* **74** (1913), 66–121;

- Macaulay F. S. , *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press (1916);

- Gröbner W., *Moderne Algebraische Geometrie*, Springer (1949);

- Möller H.M., Systems of Algebraic Equations Solved by Means of Endomorphisms, *L. N. Comp. Sci.* **673** (1993), 43–56, Springer;

- Marinari M.G., Möller H.M., On multiplicities in Polynomial System Solvin. *Trans. AMS*, **348** (1996), 3283–3321;

- Alonso M.E., Marinari M.G., The big Mother of all Dualities 2: Macaulay Bases, *J AAECC* To appear

$\mathcal{P} := k[X_1, \ldots, X_n]$,

$\mathbb{L} := \{\ell_1, \ldots, \ell_r\} \subset \mathcal{P}^*$ be a linearly indipendent set of k-linear functionals such that

$L :=$ Span$_k(\mathbb{L})$ is a $\mathcal{P}$-module so that

I $:= \mathfrak{P}(L)$ is a zero-dimensional ideal;

$\mathbf{N}(\mathsf{I}) := \{t_1, \ldots, t_r\}$,

$\mathbf{q} := \{q_1, \ldots, q_r\} \subset \mathcal{P}$ the set triangular to $\mathbb{L}$, obtained via Möller's Algorithm;

$\left(q_{ij}^{(h)}\right) \in k^{r^2}, 1 \leq k \leq r$ be the matrices defined by $X_h q_i = \sum_j q_{ij}^{(h)} q_j$ mod I,

$\Lambda := \{\lambda_1, \ldots, \lambda_r\}$ be the set biorthogonal to $\mathbf{q}$, which can be trivially deduced by Gaussian reduction

Then

$$X_h \lambda_j = \sum_{i=1}^{r} q_{ij}^{(h)} \lambda_i, \forall i, j, h.$$

$\mathcal{P} := k[X_1, \ldots, X_n]$;

$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\}$;

$\mathsf{m} := (X_1, \ldots, X_n)$ be the maximal at the origin;

$\mathsf{I} \subset \mathcal{P}$ an ideal;

the m-*closuse* of I is the ideal $\bigcap_d \mathsf{I} + \mathsf{m}^d$;

I is m-*closed* iff $\mathsf{I} = \bigcap_d \mathsf{I} + \mathsf{m}^d$;

For each $\tau \in \mathcal{T}$, denote $M(\tau) : \mathcal{P} \to k$ the morphism defined by

$$M(\tau) = c(f, \tau), \forall f = \sum_{t \in \mathcal{T}} c(f, t) t \in \mathcal{P}.$$

Denoting $\mathbb{M} := \{M(\tau) : \tau \in \mathcal{T}\}$ for all

$$f := \sum_{t \in \mathcal{T}} a_t t \in \mathcal{P} \text{ and } \ell := \sum_{\tau \in \mathcal{T}} c_\tau M(\tau) \in k[[\mathbb{M}]] \cong \mathcal{P}^*$$

it holds $\ell(f) = \sum_{t \in \mathcal{T}} a_t c_t$.

$$\forall \tau \in \mathcal{T}, X_i \cdot M(\tau) = \begin{cases} M(\frac{\tau}{X_i}) & \text{if } X_i \mid \tau \\ 0 & \text{if } X_i \nmid \tau \end{cases}$$

A k-vector subspace $\Lambda \subset \mathrm{Span}_k(\mathbb{M})$ is called *stable* if $\lambda \in \Lambda \implies X_i \cdot \lambda \in \Lambda$ i.e. Λ is a $\mathcal{P}$-module.

Clealy $\mathcal{P}^* \cong k[[\mathbb{M}]]$; however in order to have reasonable duality we must restrict ourselves to $\mathsf{Span}_k(\mathbb{M}) \cong k[\mathbb{M}]$.

For each k-vector subspace $\Lambda \subset \mathsf{Span}_k(\mathbb{M})$ denote

$$\mathfrak{I}(\Lambda) := \mathfrak{P}(\Lambda) = \{f \in \mathcal{P} : \ell(f) = 0, \forall \ell \in \Lambda\}$$

and for each k-vector subspace $P \subset \mathcal{P}$ denote

$$\begin{aligned}\mathfrak{M}(P) &:= \mathfrak{L}(P) \cap \mathsf{Span}_K(\mathbb{M}) \\ &= \{\ell \in \mathsf{Span}_K(\mathbb{M}) : \ell(f) = 0, \forall f \in P\}.\end{aligned}$$

The mutually inverse maps $\mathfrak{I}(\cdot)$ and $\mathfrak{M}(\cdot)$ give a biunivocal, inclusion reversing, correspondence between the set of the m-closed ideals $\mathsf{I} \subset \mathcal{P}$ and the set of the stable k-vector subspaces $\Lambda \subset \mathsf{Span}_k(\mathbb{M})$.

They are the restriction of, respectively, $\mathfrak{P}(\cdot)$ to m-closed ideals $\mathsf{I} \subset \mathcal{P}$, and $\mathfrak{L}(\cdot)$ to stable k-vector subspaces $\Lambda \subset \mathsf{Span}_k(\mathbb{M})$.

Moreover, for any m-primary ideal $\mathfrak{q} \subset \mathcal{P}$, $\mathfrak{M}(\mathfrak{q})$ is finite k-dimensional and we have

$$\deg(\mathfrak{q}) = \dim_K(\mathfrak{M}(\mathfrak{q}));$$

conversely for any finite k-dim. stable k-vector subspace $\Lambda \subset \mathsf{Span}_K(\mathbb{M})$, $\mathfrak{I}(\Lambda)$ is an m-primary ideal and we have

$$\dim_k(\Lambda) = \deg(\mathfrak{I}(\Lambda)).$$

II. Macaulay Bases

- Macaulay F. S., On the Resolution of a given Modular System into Primary Systems including some Properties of Hilbert Numbers, *Math. Ann.* **74** (1913), 66–121;

- Macaulay F. S. , *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press (1916);

- Gröbner W., *Moderne Algebraische Geometrie*, Springer (1949);

- Möller H.M., Systems of Algebraic Equations Solved by Means of Endomorphisms, *L. N. Comp. Sci.* **673** (1993), 43–56, Springer;

- Marinari M.G., Möller H.M., On multiplicities in Polynomial System Solvin. *Trans. AMS*, **348** (1996), 3283–3321;

- Alonso M.E., Marinari M.G., The big Mother of all Dualities 2: Macaulay Bases, *J AAECC* To appear

Let $<$ be a semigroup ordering on $\mathcal{T}$ and $\mathsf{I} \subset \mathcal{P}$ an $\mathfrak{m}$-closed ideal.

$$\mathrm{Can}(t, \mathsf{I}, <) =: \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in k[[\mathbf{N}_<(\mathsf{I})]] \subset k[[X_1, \ldots, X_n]]$$

so that

$$t - \textstyle\sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in \mathsf{I},$$

$$t < \tau \implies \gamma(t, \tau, <) = 0.$$

Define, for each $\tau \in \mathbf{N}_<(\mathsf{I})$,

$$\ell(\tau) := M(\tau) + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <) M(t) \in k[[\mathbb{M}]].$$

Remark that $\ell(\tau) \in \mathfrak{M}(\mathsf{I})$ requires $\ell(\tau) \in k[\mathbb{M}]$ which holds iff $\{t : \gamma(t, \tau, <) \neq 0\}$ is finite and is granted if $\{t : t > \tau\}$ is finite.

To obtain this we must choose as $<$ a *standard ordering* i.e. such that

- $X_i < 1, \forall i$,
- for each infinite decreasing sequence in $\mathcal{T}$

$$\tau_1 > \tau_2 > \cdots \tau_\nu > \cdots$$

  and each $\tau \in \mathcal{T}$ there is $\nu : \tau > \tau_n$.

In this setting the generalization of the notion of Gröbner basis is called Hironoka/standard basis and deals with *series* instead of polynomials.

The choice of this setting is natural, since a Hironaka basis of an ideal I returns its $\mathfrak{m}$-closure.

Let $<$ be a standard ordering on $\mathcal{T}$ and let $\mathsf{I} \subset \mathcal{P}$ an m-closed ideal. Denote

$$\mathrm{Can}(t, \mathsf{I}, <) =: \sum_{\tau \in \mathbf{N}_<(\mathsf{I})} \gamma(t, \tau, <)\tau \in k[[\mathbf{N}_<(\mathsf{I})]]$$

and, for each $\tau \in \mathbf{N}_<(\mathsf{I})$,

$$\ell(\tau) := M(\tau) + \sum_{t \in \mathbf{T}_<(\mathsf{I})} \gamma(t, \tau, <) M(t) \in k[\mathbb{M}].$$

Then

$$\mathfrak{M}(\mathsf{I}) = \mathrm{Span}_k\{\ell(\tau), \tau \in \mathbf{N}_<(\mathsf{I})\}.$$

The set $\{\ell(\tau), \tau \in \mathbf{N}_<(\mathsf{I})\}$ is called the *Macaulay Basis* of I.

There is an algorithm which, given a finite basis (not necessarily Gröbner/standard) of an m-primary ideal I, computes its Macaulay Basis.

Such algorithm becomes an infinite procedure which, given a finite basis of an ideal $\mathsf{I} \subset \mathsf{m}$, returns the infinite Macaulay Basis of its m-closure.

III. Cerlienco–Mureddu Correspondence

- Cerlienco, L, Mureddu, M. Algoritmi combinatori per l'interpolazione polinomiale in dimensione ≥ 2. *Preprint* (1990)

- Cerlienco L., Mureddu M., From algebraic sets to monomial linear bases by means of combinatorial algorithms *Discrete Math.*, **139** (1995), 73–87.

- Cerlienco L., Mureddu M.,
  Multivariate Interpolation and Standard Bases for Macaulay Modules, *J. Algebra* **251** (2002), 686–726

Problem 1 *Given a finite set of points,*

$$\{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \dots, a_{in}),$$

to compute $\mathbf{N}_<(\mathsf{I})$ *w.r.t. the lexicographical ordering* $<$ *induced by* $X_1 < \cdots < X_n$ *where*

$$\mathsf{I} := \{f \in \mathcal{P} : f(\mathsf{a}_i) = 0, 1 \leq i \leq s\}.$$

Cerlienco–Mureddu Algorithm, to each *ordered* finite set of points

$$\mathsf{X} := \{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

associates

- an order ideal $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and
- a bijection $\Phi := \Phi(\mathsf{X}) : \mathsf{X} \mapsto \mathbf{N}$

which satisfies

Theorem 1 $\mathbf{N}(\mathsf{I}) = \mathbf{N}(\mathsf{X})$ *holds for each finite set of points* $\mathsf{X} \subset k^n$.

Since they do so by induction on $s = \#(\mathsf{X})$ let us consider the subset $\mathsf{X}' := \{\mathsf{a}_1, \ldots, \mathsf{a}_{s-1}\}$, and the corresponding order ideal $\mathbf{N}' := \mathbf{N}(\mathsf{X}')$ and bijection $\Phi' := \Phi(\mathsf{X}')$.

If $s = 1$ the only possible solution is $\mathbf{N} = \{1\}, \Phi(\mathsf{a}_1) = 1$.

$$\begin{aligned}\mathcal{T}[1,m] &:= \mathcal{T}\cap k[X_1,\ldots,X_m]\\ &= \{X_1^{a_1}\cdots X_m^{a_m} : (a_1,\ldots,a_m)\in\mathbb{N}^m\},\end{aligned}$$

$$\pi_m : k^n \mapsto k^m, \quad \pi_m(x_1,\ldots,x_n) = (x_1,\ldots,x_m),$$

$$\pi_m : \mathcal{T}\cong\mathbb{N}^n \mapsto \mathbb{N}^m \cong \mathcal{T}[1,m],$$

$$\pi_m(X_1^{a_1}\cdots X_n^{a_n}) = X_1^{a_1}\cdots X_m^{a_m}.$$

With this notation, let us set

$$m := \max\Big(j : \exists i < s : \pi_j(\mathsf{a}_i) = \pi_j(\mathsf{a}_s)\Big);$$

$$d := \#\{\mathsf{a}_i, i < s : \pi_m(\mathsf{a}_i) = \pi_m(\mathsf{a}_s)\};$$

$$\mathsf{W} := \{\mathsf{a}_i : \Phi'(\mathsf{a}_i) = \tau_i X_{m+1}^d, \tau_i \in \mathcal{T}[1,m]\} \cup \{\mathsf{a}_s\};$$

$$\mathsf{Z} := \pi_m(\mathsf{W});$$

$$\tau := \Phi(\mathsf{Z})(\pi_m(\mathsf{a}_s));$$

$$t_s := \tau X_{m+1}^d;$$

where $\mathbf{N}(\mathsf{Z})$ and $\Phi(\mathsf{Z})$ are the result of the application of the present algorithm to Z, which can be inductively applied since $\#(\mathsf{Z}) \le s-1$.

We then define

- $\mathbf{N} := \mathbf{N}' \cup \{t_s\}$,
- $\Phi(\mathsf{a}_i) := \begin{cases}\Phi'(\mathsf{a}_i) & i < s\\ t_s & i = s\end{cases}$

$$\mathsf{a}_1 := (0,0,1),$$

$$\Phi(\mathsf{a}_1) := t_1 := 1;$$

$$\mathsf{a}_2 := (0,1,-2), m = 1,$$

$$d = 1, \mathsf{W} = \{(0,1)\}, \tau = 1,$$

$$\Phi(\mathsf{a}_2) := t_2 := X_2,$$

$$\mathsf{a}_3 := (2,0,2), m = 0,$$

$$d = 1, \mathsf{W} = \{(2,0)\}, \tau = 1,$$

$$\Phi(\mathsf{a}_3) := t_3 := X_1,$$

$$\mathsf{a}_4 := (0,2,-2), m = 1,$$

$$d = 2, \mathsf{W} = \{(0,2)\}, \tau = 1,$$

$$\Phi(\mathsf{a}_4) := t_4 := X_2^2,$$

$$\mathsf{a}_5 := (1,0,3), m = 0,$$

$$d = 2, \mathsf{W} = \{(1,0)\}, \tau = 1,$$

$$\Phi(\mathsf{a}_5) := t_5 := X_1^2,$$

$$\mathsf{a}_6 := (1,1,3), m = 1,$$

$$d = 1, \mathsf{W} = \{(0,1),(1,1)\}, \tau = X_1,$$

$$\Phi(\mathsf{a}_6) := t_6 := X_1X_2.$$

$(0,2,-2)$		
$(0,1,-2)$	$(1,1,3)$	
$(0,0,1)$	$(2,0,2)$	$(1,0,3)$

$$\mathsf{a}_7 := (1,1,1), m = 2,$$

$$d = 1, \mathsf{W} = \{(1,1,1)\}, \tau = 1,$$

$$\Phi(\mathsf{a}_7) := t_7 := X_3.$$

$$\mathsf{a}_8 := (2,0,1), m = 2,$$

$$d = 1, \mathsf{W} = \{(1,1,1),(2,0,1)\}, \tau = X_1,$$

$$\Phi(\mathsf{a}_8) := t_8 := X_1 X_3,$$

$$\mathsf{a}_9 := (2,0,0), m = 2,$$

$$d = 2, \mathsf{W} = \{(2,0,0))\}, \tau = 1,$$

$$\Phi(\mathsf{a}_9) := t_9 := X_3^2,$$

$(0,2,-2)$		
$(0,1,-2)$	$(1,1,3)$	
$(0,0,1)$	$(2,0,2)$	$(1,0,3)$

- Gao S., Rodrigues V.M., Stroomer J., Gröbner basis structure of finite sets of points *Preprint* (2003)

A combinatorial reformulation which

– builds a tree on the basis of the point coordinates,

– cominatorially recombines the tree,

– reeds on this tree the monomial structure.

It returns $\mathbf{N}$ but not Φ; more important: it is *not* iterative.

- Marinari M.G., Cerlienco–Mureddu Correpondence and Lazard Structural Theorem. *Investigaciones Mathematicas* (2006). To appear.

Extends Cerlienco–Mureddu Algorithm to multiple points described via Macaulay Bases

IV. Macaulay's Algorithm

- Macaulay F. S., On the Resolution of a given Modular System into Primary Systems including some Properties of Hilbert Numbers, *Math. Ann.* **74** (1913), 66–121;

- Macaulay F. S. , *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press (1916);

- Gröbner W., *Moderne Algebraische Geometrie*, Springer (1949);

- Alonso M.E., Marinari M.G., The big Mother of all Dualities 2: Macaulay Bases, *J AAECC* To appear

$\mathsf{m} = (X_1, \ldots, X_n) \subset \mathcal{P} := k[X_1, \ldots, X_n]$,

$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\}$,

a standard-ordering $<$ on $\mathcal{T}$,

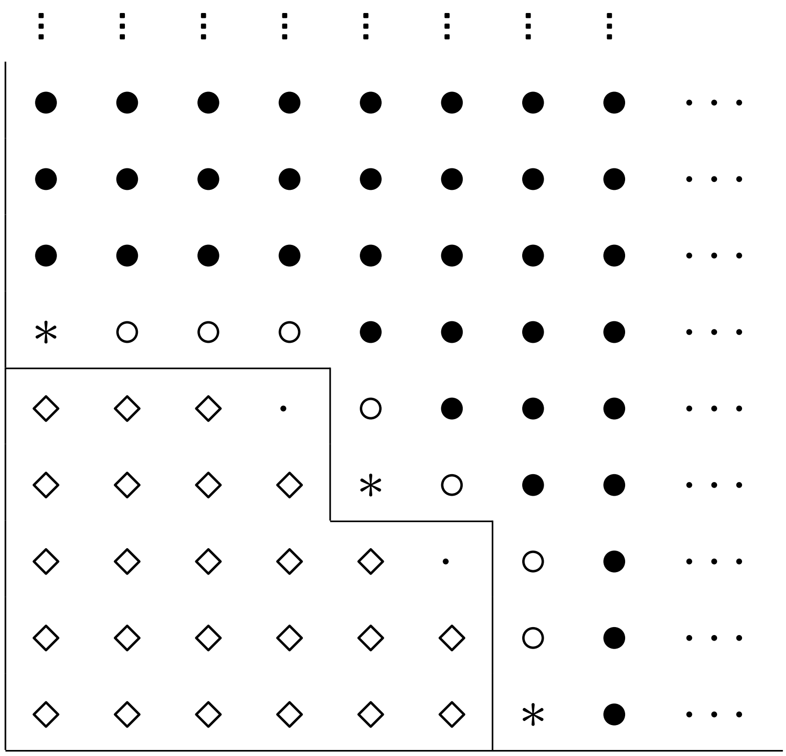
an m-closed ideal I,

the finite corner set $\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \ldots, \omega_s\}$,

the (not-necessarily finite) set $\mathbf{N}_<(\mathsf{I})$,

the Macaulay basis $\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\}$,

the k-vectorspace $\Lambda \subset \mathsf{Span}_k(\mathbb{M})$ generated by it.



$\mathsf{m} = (X_1, \ldots, X_n) \subset \mathcal{P} := k[X_1, \ldots, X_n]$,

$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\}$,

a standard-ordering $<$ on $\mathcal{T}$,

an m-closed ideal I,

the finite corner set $\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \ldots, \omega_s\}$,

the (not-necessarily finite) set $\mathbf{N}_<(\mathsf{I})$,

the Macaulay basis $\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\}$,

$\Lambda := \mathsf{Span}_k\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\} \subset \mathsf{Span}_k(\mathbb{M})$;

$\forall j, 1 \leq j \leq s, \Lambda_j := \mathsf{Span}_k\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\}$.

$\forall j, 1 \leq j \leq s, \mathfrak{q}_j := \mathfrak{I}(\Lambda_j)$.

Let $J \subset \{1, \ldots, s\}$ be the set such that $\{\mathfrak{q}_j : j \in J\}$ is the set of the minimal elements of $\{\mathfrak{q}_j : 1 \leq j \leq s\}$ and remark that $\mathfrak{q}_i \subset \mathfrak{q}_j \iff \Lambda_i \supset \Lambda_j$.

Lemma 1 (Macaulay) *With the notation above, for each j, denoting*

$$\Lambda'_j := \mathsf{Span}_K\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T} \cap \mathsf{m}\}$$

we have

$\dim_K(\Lambda'_j) = \dim_K(\Lambda_j) - 1$,

$\ell(\omega_j) \notin \Lambda'_j = \mathfrak{M}(\mathfrak{q}_j : \mathsf{m})$,

$\mathfrak{q}' \supset \mathfrak{q}_j \implies \mathfrak{M}(\mathfrak{q}') \subseteq \Lambda'_j$.

$\mathsf{m} = (X_1, \ldots, X_n) \subset \mathcal{P} := k[X_1, \ldots, X_n]$,

$\mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \ldots, a_n) \in \mathbb{N}^n\}$,

a standard-ordering $<$ on $\mathcal{T}$,

an m-closed ideal I,

the finite corner set $\mathbf{C}_<(\mathsf{I}) := \{\omega_1, \ldots, \omega_s\}$,

the (not-necessarily finite) set $\mathbf{N}_<(\mathsf{I})$,

the Macaulay basis $\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\}$,

$\Lambda := \mathsf{Span}_k\{\ell(\tau) : \tau \in \mathbf{N}_<(\mathsf{I})\} \subset \mathsf{Span}_k(\mathbb{M})$;

$\forall j, 1 \leq j \leq s, \Lambda_j := \mathsf{Span}_k\{\upsilon \cdot \ell(\omega_j) : \upsilon \in \mathcal{T}\}$.

$\forall j, 1 \leq j \leq s, \mathfrak{q}_j := \mathfrak{I}(\Lambda_j)$.

Let $J \subset \{1, \ldots, s\}$ be the set such that $\{\mathfrak{q}_j : j \in J\}$ is the set of the minimal elements of $\{\mathfrak{q}_j : 1 \leq j \leq s\}$ and remark that $\mathfrak{q}_i \subset \mathfrak{q}_j \iff \Lambda_i \supset \Lambda_j$.

Theorem 2 (Gröbner) *If I is m-primary, then:*

1. *each Λ_j is a finite-dim. stable vectorspace;*
2. *each $\mathfrak{q}_j$ is an m-primary ideal,*
3. *is <u>reduced</u>*
4. *and irreducible.*
5. *$\mathsf{I} := \cap_{j \in J} \mathfrak{q}_j$ is a <u>reduced representation</u> of I.*

V. Reduced Irreducible Decomposition

- Noether Noether E. Idealtheorie in Ringbereichen, *Math. Annalen*, **83** (1921), 25–66.

- Macaulay F. S., On the Resolution of a given Modular System into Primary Systems including some Properties of Hilbert Numbers, *Math. Ann.* **74** (1913), 66–121;

- Macaulay F. S. , *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press (1916);

- Gröbner W., *Moderne Algebraische Geometrie*, Springer (1949);

- Renschuch. B, *Elementare und praktische Idealtheorie*, Deutscher Verlag der Wissenschaften (1976);

- Alonso M.E., Marinari M.G., The big Mother of all Dualities 2: Macaulay Bases, *J AAECC* To appear

- (Lasker-Noether) In a noetherian ring R, every ideal $\mathfrak{a} \subset R$ is a finite intersection of irreducible ideals.

- (Noether) A representation $\mathfrak{a} = \cap_{j=1}^{r} \mathfrak{i}_j$ of an ideal $\mathfrak{a}$ in a noetherian ring R as intersection of finitely many irreducible ideals is called a *reduced representation* if
  - $\forall\, j \in \{1, \dots, r\}$, $\mathfrak{i}_j \not\supset \bigcap\limits_{\substack{h=1 \\ j \neq h}}^{r} \mathfrak{i}_h$ and
  - there is no irreducible ideal ${\mathfrak{i}_j}' \supset \mathfrak{i}_j$ such that $$\mathfrak{a} = \left(\bigcap_{\substack{h=1 \\ j \neq h}}^{r} \mathfrak{i}_h \right) \cap {\mathfrak{i}_j}'.$$

- (Noether) In a noetherian ring R, each ideal $\mathfrak{a} = \bigcap\limits_{i=1}^{r} \mathfrak{q}_i$ $\mathfrak{a} \subset R$ has a reduced representation as intersection of finitely many irreducible ideals.

- A primary component $\mathfrak{q}_j$ of an ideal $\mathfrak{a}$ contained in a noetherian ring R, is called *reduced* if there is no primary ideal ${\mathfrak{q}_j}' \supset \mathfrak{q}_j$ such that $$\mathfrak{a} = \left(\bigcap_{\substack{i=1 \\ j \neq i}}^{r} \mathfrak{q}_i \right) \cap {\mathfrak{q}_j}'.$$

- In an irredundant primary decomposition of an ideal of a noetherian ring, each primary component can be chosen to be reduced.

The decomposition

$$(X^2, XY) = (X) \cap (X^2, XY, Y^\lambda), \forall \lambda \in \mathbb{N}, \lambda \geq 1,$$

where $\sqrt{(X^2, XY, Y^\lambda)} = (X, Y) \supset (X)$, shows that embedded components are not unique; however,

$$(X^2, XY, Y) = (X^2, Y) \supseteq (X^2, XY, Y^\lambda), \forall \lambda > 1,$$

shows that (X^2, Y) is a reduced embedded irreducible component and that

$$(X^2, XY) = (X) \cap (X^2, Y)$$

is a reduced representation.

The decompositions

$$(X^2, XY) = (X) \cap (X^2, Y + aX), \forall a \in \mathbb{Q},$$

where $\sqrt{(X^2, Y + aX)} = (X, Y) \supset (X)$ and, clearly, each $(X^2, Y + aX)$ is reduced, show that also reduced representations are not unique; remark that, setting $a = 0$, we find again the previous one $(X^2, XY) = (X) \cap (X^2, Y)$.

If I is not m-primary, let

$\rho := \max\{\deg(\omega_j) + 1 : \omega_j \in \mathbf{C}(\mathsf{I})\}$ so that

$\mathfrak{q}' := \mathsf{I} + \mathsf{m}^\rho$ is an m-primary component of I;

$\mathsf{I} = \cap_{i=1}^r \mathfrak{q}_i$ an irredundant primary representation of I with $\sqrt{\mathfrak{q}_1} = \mathsf{m}$;

$\mathfrak{b} := \mathsf{I} : \mathfrak{m}^\infty = \cap_{i=2}^r \mathfrak{q}_i$;

$\mathfrak{b} = \cap_{i=1}^u \mathfrak{Q}_i$, a reduced representation of $\mathfrak{b}$;

$\mathfrak{q}_1 := \cap_{j=1}^s \mathfrak{q}_j$ a reduced representation of $\mathfrak{q}_1$ which is wlog ordered so that $\mathfrak{q}_i \supset \mathfrak{b} \iff i > t$;

$\mathfrak{q} := \cap_{j=1}^t \mathfrak{q}_j$.

Then

1. $\mathfrak{q}$ is a reduced m-primary component of I,
2. $\mathfrak{q} := \cap_{j=1}^t \mathfrak{q}_j$ is a reduced representation of $\mathfrak{q}$,
3. $\mathsf{I} = \cap_{i=1}^u \mathfrak{Q}_i \bigcap \cap_{j=1}^t \mathfrak{q}_j$ is a reduced representation of I.

$\mathsf{I} := (X^2, XY)$,

$\Lambda = \mathrm{Span}_k\{M(1), M(X)\} \cup \{M(Y^i), i \in \mathbb{N}\}$;

$\rho = 2$,

$\mathfrak{M}(\mathsf{I} + \mathsf{m}^2) = \{M(1), M(X), M(Y)\}$,

$\omega_1 := X, \Lambda_1 = \{M(1), M(X))\}, \mathfrak{q}_1 = (X^2, Y)$,

$\omega_2 := Y, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^2)$,

$\mathsf{I} : \mathsf{m}^\infty = (X) \subset (X, Y^2)$,

$(X^2, XY) = (X) \cap (X^2, Y)$.

$$\mathsf{I} := (X^2, XY),$$

$$\Lambda = \mathrm{Span}_k\{M(1), M(X)\} \cup \{M(Y^i), i \in \mathbb{N}\};$$

$$\rho = 2,$$

$$\mathfrak{M}(\mathsf{I} + \mathsf{m}^2) = \{M(1), M(X), M(Y)\},$$

$$\omega_1 := X, \Lambda_1 = \{M(1), M(X))\}, \mathfrak{q}_1 = (X^2, Y),$$

$$\omega_2 := Y, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^2),$$

$$\mathsf{I} : \mathsf{m}^\infty = (X) \subset (X, Y^2),$$

$$(X^2, XY) = (X) \cap (X^2, Y).$$

Both the reduced representation and the notion of Macaulay basis strongly depend on the choice of a frame of coordinates.
In fact, considering, for each $a \in \mathbb{Q}, a \neq 0$,

$$\Lambda = \mathrm{Span}_k\{M(1), M(X) - aM(Y)\} \cup \{M(Y^i), i \in \mathbb{N}\},$$

we obtain

$$\rho = 2,$$

$$\mathfrak{M}(\mathsf{I} + \mathsf{m}^2) = \{M(1), M(X) - aM(Y), M(Y)\},$$

$$\omega_1 := X, \Lambda_1 = \{M(1), M(X) - aM(Y)\}, \mathfrak{q}_1 = (X^2, Y + aX),$$

$$\omega_2 := Y, \Lambda_2 = \{M(1), M(Y)\}, \mathfrak{q}_2 = (X, Y^2),$$

$$\mathsf{I} : \mathsf{m}^\infty = (X) \subset (X, Y^2),$$

$$(X^2, XY) = (X) \cap (X^2, Y + aX).$$

VI. Lazard Structural Theorem

- Lazard D., Ideal Basis and Primary Decomposition: Case of two variables *J. Symb. Comp.* **1** (1985) 261–270

Theorem 3 *Let $\mathcal{P} := k[X_1, X_2]$ and let $<$ be the lex. ordering induced by $X_1 < X_2$.*

Let $\mathsf{I} \subset \mathcal{P}$ be an ideal and let $\{f_0, f_1, \ldots, f_k\}$ be a Gröbner basis of I ordered so that

$$\mathbf{T}(f_0) < \mathbf{T}(f_1) < \cdots < \mathbf{T}(f_k).$$

Then

- $f_0 = PG_1 \cdots G_{k+1}$,
- $f_j = PH_jG_{j+1} \cdots G_{k+1}, 1 \leq j < k$,
- $f_k = PH_kG_{k+1}$,

where

P *is the primitive part of* $f_0 \in k[X_1][X_2]$;

$G_i \in k[X_1], 1 \leq i \leq k+1$;

$H_i \in k[X_1][X_2]$ *is a monic polynomial of degree* $d(i)$, *for each* i;

$d(1) < d(2) < \cdots < d(k)$;

$H_{i+1} \in (G_1 \cdots G_i, \ldots, H_jG_{j+1} \cdots G_i, \ldots, H_{i-1}G_i, H_i), \forall i$.

VII. Axis-of-Evil Theorem

- Marinari M.G., Mora T., A remark on a remark by Macaulay or Enhancing Lazard Structural Theorem. *Bull. of the Iranian Math. Soc.*, **29** (2003), 103–145;

- Marinari M.G., Mora T.
  Some Comments on Cerlienco–Mureddu Algorithm and Enhanced Lazard Structural Theorem. Rejected by ISSAC-2004 (2004)

- Marinari M.G., Mora T.
  Cerlienco–Mureddu Correpondence and Lazard Structural Theorem.
  Investigaciones Mathematicas (2006). To appear.

Description of the combinatorial structure [Gröbner and border basis, linear and Gröbner representation] of a 0-dimensional ideal

$$\mathsf{I} = \cap \mathfrak{q}_i \subset \mathcal{P}, \sqrt{\mathfrak{q}_i} = (X_1 - a_{i1}, \cdots, X_n - a_{in})$$

in terms of a **Macaualy representation**, i.e. of its roots $(a_{i1}, \cdots, a_{in})$ and of the Macaulay basis of each $\mathfrak{q}_i$.

It is summarized into 22* statements.

The description is "algorithmical" in terms of elementary combinatorial tools and linear interpolation.

It extends Cerlienco–Mureddu Correspondence and Lazard's Structural Theorem.

The proof is essentially a direct application of Möller's Algorithm.

*in honour of Trythemius, the founder of cryptography (*Steganographia* [1500], *Polygraphia* [1508]) which introdiced in german the 22^{th} letter **W** in order to perform german gematria.

Let

$\mathsf{I} \subset \mathcal{P}$ be a zero-dimensional radical ideal;

$\mathsf{Z} := \{\mathbf{a}_1, \ldots, \mathbf{a}_s\} \subset k^n$ its roots;

$\mathbf{N} := \mathbf{N}(\mathsf{I})$;

$\mathbf{G}_<(\mathsf{I}) := \{\mathsf{t}_1, \ldots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \ldots < \mathsf{t}_r$, $\mathsf{t}_i := X_1^{d_1^{(i)}} \cdots X_n^{d_n^{(i)}}$ the minimal basis of its associated monomial ideal $\mathbf{T}_<(\mathsf{I})$;

$G := \{f_1, \ldots, f_r\}, \mathbf{T}(f_i) = \mathsf{t}_i \forall i$, the unique reduced lexicographical Gröbner basis of I.

There is a combinatorial algorithm which, given Z, returns sets of points

$$\mathsf{Z}_{m\delta i} \subset k^m, \forall m, \delta, i : 1 \leq i \leq r, 1 \leq m \leq n, 1 \leq \delta \leq d_m^{(i)},$$

thus allowing to compute

- by means of Cerlienco–Mureddu Algorithm the corresponding order ideal
$$F_{m\delta i} := \mathbf{N}(\mathsf{Z}_{m\delta i}) \subset \mathcal{T} \cap k[X_1, \ldots, X_{m-1}]$$
- and, by interpolation* unique polynomials
$$\gamma_{m\delta i} := X_m - \sum_{\omega \in F_{m\delta i}} c_\omega \omega$$
which satisfy the relation
$$f_i = \prod_m \prod_\delta \gamma_{m\delta i} \quad (\text{mod } (f_1, \ldots, f_{i-1}) \forall i.$$

Moreover, setting

ν the maximal value such that $d_\nu^{(i)} \neq 0, d_m^{(i)} = 0, m > \nu$ so that $f_i \in k[X_1, \ldots, X_\nu] \setminus k[X_1, \ldots, X_{\nu-1}]$,

$L_i := \prod_{m=1}^{\nu-1} \prod_\delta \gamma_{m\delta i}$ and

$P_i := \prod_\delta \gamma_{\nu\delta i}$

we have $f_i = L_i P_i$ where L_i is the *leading polynomial* of f_i.

$^* X_m(\mathsf{a}) = \sum_{\omega \in F_{m\delta i}} c_\omega \omega(\mathsf{a}), \mathsf{a} \in Z_{m\delta i}$.