Passivity & Coherence and their relation in Differential Algebra

Moritz Minzlaff

Fakultät für Informatik Universität Karlsruhe (TH)

Special Semester on Gröbner Bases (Workshop D2)

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Motivation Basic Notions Coherence and Passivity

Why Consider Coherence and Passivity?

Given: A finite set A of differential polynomials

Wanted:

- (P1) Decide solvability of A = 0
- (P2) Compute prime decompositions of the radical differential ideal of A
- (P3) Formulate existence and uniqueness statements about formal power series solutions

Tools: Coherence and Passivity

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Motivation Basic Notions Coherence and Passivity

Conventions

K, differential field

- 1 $CHAR(\mathbb{K}) = 0$
- 2 Δ , set of derivations
- **3** $\Theta \cong \mathcal{N}^m$, free commutative monoid generated by Δ
- K[[𝒴], differential polynomial ring in *n* differential indeterminates 𝒴 = { Y₁,..., Y_n}

• $A, H \subseteq \mathbb{K}[\mathcal{Y}]$ finite

- $1 \quad A \cap \mathbb{K} = \emptyset$
- 2 A triangular, partially autoreduced
- 3 H partially reduced w.r.t. A
- 4 H^{∞} contains all initials and separants of A

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Coherence

Definition (Pseudo-S-Polynomials)

Let $p, q \in \mathbb{K}[\mathcal{Y}]$ with 1 LEAD(p) = v = LEAD(q)2 DEG(p) = d, DEG(q) = e $S(p,q) := \frac{\text{INIT}(q)v^e p - \text{INIT}(p)v^d q}{\text{GCD}(d,e)}$

Definition

(A, H) is *coherent* if for all $p, q \in \Theta A$ with $extsf{LEAD}(p) = extsf{LEAD}(q)$

$$\mathcal{S}(\pmb{
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Motivation Basic Notions Coherence and Passivity

Passivity

Different authors: Different definitions for passivity

[Ritt, Wu, Chen& Gao]

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Common to all:

- Every $p \in [A]$: H^{∞} has a "nice representation"
- A is involutively completed
 - w.r.t. an involutive division and an algorithmic process

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Motivation Basic Notions Coherence and Passivity

Three Questions

Is passivity independent of the involutive completion?

- 2 Every (Wu-)passive system is coherent. [Li&Wang '99] Is the converse also true?
- 3 Is passivity independent of the involutive division?

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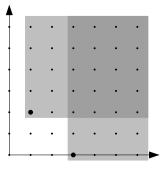
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Involutive Divisions Adding Structure to Differential Systems

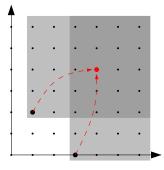
Involutive Divisions and \mathbb{N}^m



Consider: Span of finite $\mathcal{N} \subseteq \mathbb{N}^m$ Note: No unique divisor in \mathcal{N} Wanted: Uniqueness Idea: Replace division by an *involutive division* \mathcal{L}

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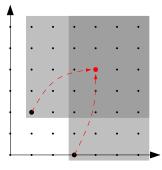


Consider: Span of finite $\mathcal{N} \subseteq \mathbb{N}^m$ Note: No unique divisor in \mathcal{N} Wanted: Uniqueness Idea: Replace division by an involutive division \mathcal{L} Need: Involutively complete \mathcal{N}

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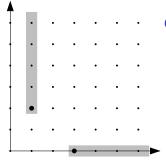


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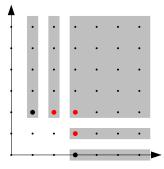
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Involutive Divisions Adding Structure to Differential Systems

Transfer from \mathbb{N}^m to $\mathbb{K}[\mathcal{Y}]$

Idea: Derivative operators in leaders play role of *m*-tuples

 $\mathcal{N}_{A,Y} := \{ \alpha \in \mathbb{N}^m \mid \alpha \, Y \in \mathsf{LEAD}(A) \}$

Multiplicative Prolongations: Elements βp with $p \in A$, LEAD $(p) = \alpha Y$, and $\alpha \mathcal{L}$ -divides $\beta \alpha$ w.r.t. $\mathcal{N}_{A,Y}$

Definition (Involutive Span)

Ideal generated by all multiplicative prolongations of all $p \in A$

 $[A]_{\mathcal{L}} := \langle \beta p \mid \beta p \text{ a mult. prolongation of } p \in A \rangle$

For $u \in \Theta \mathcal{Y}$:

 $[A]_{\mathcal{L}}^{u} := \langle \beta p \mid \beta p \leq u, \beta p \text{ a mult. prolongation of } p \in A \rangle$

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Involutive Divisions Adding Structure to Differential Systems

Involutive Completions

Definition

 $A^{\mathcal{L}}\subseteq \mathbb{K}[\![\mathcal{Y}]\!]$ is an involutive completion of A if $A^{\mathcal{L}}\supseteq A$ is minimal with

1 $\mathcal{N}_{A^{\mathcal{L}},Y}$ are involutive completions of $\mathcal{N}_{A,Y}$

2 $p \in A^{\mathcal{L}}$ are "simple" differential consequences of A

$$\blacksquare p \notin A \Rightarrow \mathsf{DEG}(p) = 1$$

■ INIT(
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), SEP(p) \in (INIT(A) \cup SEP(A)) ^{\sim}

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Note:

Desired properties are defined without use of algorithms

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Example

Assume: Involutive completions exist

Claim: There is an involutive completion $A^{\mathcal{L}} \subseteq \Theta A$

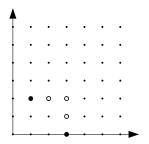


Adding Structure to Differential Systems

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Reason:

1 Let $\beta \in \mathcal{N}_{A^{\mathcal{L}},Y} \setminus \mathcal{N}_{A,Y}$

Adding Structure to Differential Systems

2 Choose $p \in A$ with LEAD $(p) = \alpha Y$, α divides β

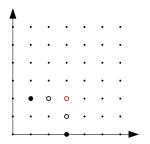
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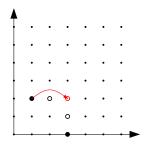
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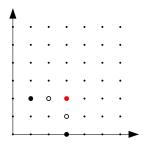
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Involutive Divisions Adding Structure to Differential Systems

Passive Systems

Theorem

 $p \in [A^{\mathcal{L}}]^{u}_{\mathcal{L}} : H^{\infty} \Rightarrow$ exists $h \in H^{\infty}$ with $hp = \lambda(\gamma r) + \sum_{i=1}^{k} \lambda_{i} m_{i}$

1 γr , unique multiplicative prolongation with LEAD $(\gamma r) = u$

2 m_i, product of multiplicative prolongations < u</p>

3 LEAD $(p) \le u \Rightarrow \lambda_i$ partially reduced w.r.t. A

Wanted: The above property for all $p \in [A]$: H^{∞}

Definition

```
(A, H) is passive w.r.t. A^{\mathcal{L}} if
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 $\langle \Theta A_{\leq u} \rangle : H^{\infty} = [A^{\mathcal{L}}]^{u}_{\mathcal{L}} : H^{\infty} \quad \forall u \in \Theta \mathcal{Y}$

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$$\langle \Theta A_{\leq u} \rangle : H^{\infty} = [A^{\mathcal{L}}]^{u}_{\mathcal{L}} : H^{\infty} \quad \forall u \in \Theta \mathcal{Y}$$

Consider: Involutive completions $A_1^{\mathcal{L}}$ and $A_2^{\mathcal{L}}$ of A

Theorem

(A,H) is passive w.r.t. $A_1^{\mathcal{L}} \Leftrightarrow (A,H)$ is passive w.r.t. $A_2^{\mathcal{L}}$

Proof.

Wanted: $\alpha p \in [A_2^L]_{\mathcal{L}}^u : H^{\infty}$ whenever $p \in A$, LEAD $(\alpha p) = u$ Have: $h(\alpha p) = \lambda_1(\gamma_1 r_1) + s$, where $s \in [A_1^L]_{\mathcal{L}}^v$ some v < uInduction: $[A_1^L]_{\mathcal{L}}^v : H^{\infty} = [A_2^L]_{\mathcal{L}}^v : H^{\infty}$ Pseudo-Division: INIT $(\gamma_2 r_2)\gamma_1 r_1 = INIT(\gamma_1 r_1)\gamma_2 r_2 + s'$

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Consequence:

Consider: Involutive division \mathcal{L} such that involutive completions of A exist

Theorem

(A, H) is passive w.r.t. $\mathcal{L} \Leftrightarrow (A, H)$ is coherent

Proof.

⇒ as done by Li & Wang '99

 \Leftarrow **Choose**: Involutive completion $A^{L} \subseteq \Theta A$

Note: $(A^{\mathcal{L}}, H)$ is coherent

Deduce: $p \in A$ implies

 $S(\alpha p, \gamma r) = \mathsf{INIT}(\gamma r)\alpha p - \mathsf{INIT}(\alpha p)\gamma r \in \langle \Theta A_{\mathsf{CLEAD}(\alpha p)} \rangle : H^{\infty}$ Induction: $\langle \Theta A_{\le u} \rangle : H^{\infty} = [A^{\mathcal{L}}]^{u}_{c} : H^{\infty}, \forall u < \mathsf{LEAD}(\alpha p)$ [

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Independence of Involutive Divisions

Consider: Involutive divisions \mathcal{L}_1 and \mathcal{L}_2 such that involutive completions of A exist

Corollary

(A, H) is passive w.r.t. $\mathcal{L}_1 \Leftrightarrow (A, H)$ is passive w.r.t. \mathcal{L}_2

Proof.

Follows from equivalence with coherence.

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Moritz Minzlaff A New Definition for Passivity

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Conclusion

- **1** Is passivity independent of the involutive completion? Yes
- Every (Wu-)passive system is coherent. [Li&Wang '99]
 Is the converse also true? Yes
- 3 Is passivity independent of the involutive division? Yes

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