Passivity & Coherence and their relation in Differential Algebra

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Special Semester on Gröbner Bases (Workshop D2)

Why Consider Coherence and Passivity?

Given: A finite set A of differential polynomials

Wanted:

(P1) Decide solvability of $A = 0$

(P2) Compute prime decompositions of the radical differential ideal of A

(P3) Formulate existence and uniqueness statements about formal power series solutions

Tools: *Coherence* and *Passivity*

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Conventions

- $\mathbb{K}$, differential field
  1. $\mathrm{CHAR}(\mathbb{K}) = 0$
  2. Δ, set of derivations
  3. $\Theta \cong \mathcal{N}^m$, free commutative monoid generated by Δ
- $\mathbb{K}[\![\mathcal{Y}]\!]$, differential polynomial ring in n differential indeterminates $\mathcal{Y} = \{Y_1, \ldots, Y_n\}$
- $A, H \subseteq \mathbb{K}[\![\mathcal{Y}]\!]$ finite
  1. $A \cap \mathbb{K} = \emptyset$
  2. A triangular, partially autoreduced
  3. H partially reduced w.r.t. A
  4. H^∞ contains all initials and separants of A
- $\langle A \rangle$, $[A]$, ideal resp. differential ideal generated by A

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Coherence

Definition (Pseudo-S-Polynomials)

Let $p, q \in \mathbb{K}[\![\mathcal{Y}]\!]$ with

1. $\text{LEAD}(p) = v = \text{LEAD}(q)$
2. $\text{DEG}(p) = d, \text{DEG}(q) = e$

$$\mathcal{S}(p,q) := \frac{\text{INIT}(q)v^e p - \text{INIT}(p)v^d q}{\text{GCD}(d,e)}$$

Definition

(A, H) is *coherent* if for all $p, q \in \Theta A$ with $\text{LEAD}(p) = \text{LEAD}(q)$

$$\mathcal{S}(p,q) \in \left\langle \Theta A_{<\text{LEAD}(p)} \right\rangle : H^\infty$$

Passivity

Different authors: Different definitions for passivity

[Ritt, Wu, Chen& Gao]

Common to all:

1. Every $p \in [A] : H^\infty$ has a "nice representation"
2. A is involutively completed
   w.r.t. an involutive division and an algorithmic process

However: Different involutive divisions and different algorithmic processes are used!

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Three Questions

1. Is passivity independent of the involutive completion?
2. Every (Wu-)passive system is coherent. [Li&Wang '99]
   Is the converse also true?
3. Is passivity independent of the involutive division?

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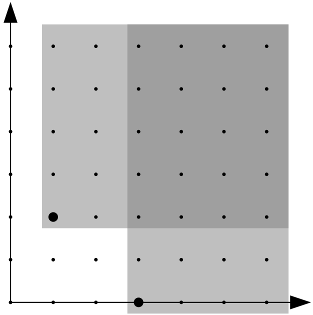
Involutive Divisions and $\mathbb{N}^m$



Consider: Span of finite $\mathcal{N} \subseteq \mathbb{N}^m$

Note: No unique divisor in $\mathcal{N}$

Wanted: Uniqueness

Idea: Replace division by an *involutive division* $\mathcal{L}$

Need: *Involutively complete* $\mathcal{N}$

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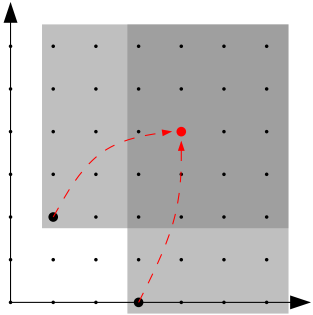

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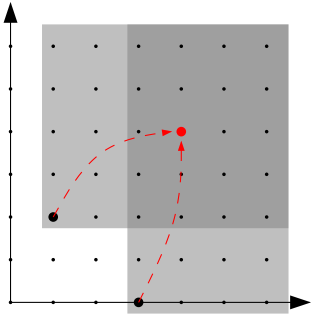

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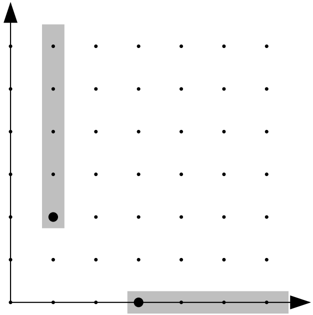

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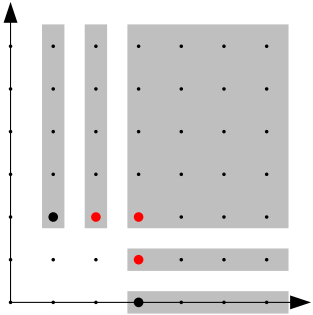

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Transfer from $\mathbb{N}^m$ to $\mathbb{K}[\![\mathcal{Y}]\!]$

Idea: Derivative operators in leaders play role of m-tuples

$$\mathcal{N}_{A,Y} := \{\alpha \in \mathbb{N}^m \mid \alpha Y \in \text{LEAD}(A)\}$$

Multiplicative Prolongations: Elements βp with $p \in A$, $\text{LEAD}(p) = \alpha Y$, and α $\mathcal{L}$-divides $\beta\alpha$ w.r.t. $\mathcal{N}_{A,Y}$

Definition (Involutive Span)

Ideal generated by all multiplicative prolongations of all $p \in A$

$$[A]_{\mathcal{L}} := \langle \beta p \mid \beta p \text{ a mult. prolongation of } p \in A \rangle$$

For $u \in \Theta\mathcal{Y}$:

$$[A]^u_{\mathcal{L}} := \langle \beta p \mid \beta p \leq u, \beta p \text{ a mult. prolongation of } p \in A \rangle$$

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Involutive Completions

Definition

$A^{\mathcal{L}} \subseteq \mathbb{K}[\![\mathcal{Y}]\!]$ is an *involutive completion* of A if $A^{\mathcal{L}} \supseteq A$ is minimal with

1. $\mathcal{N}_{A^{\mathcal{L}},Y}$ are involutive completions of $\mathcal{N}_{A,Y}$
2. $p \in A^{\mathcal{L}}$ are "simple" differential consequences of A
   - $p \notin A \;\Rightarrow\; \textsc{deg}(p) = 1$
   - $\textsc{init}(p), \textsc{sep}(p) \in (\textsc{init}(A) \cup \textsc{sep}(A))^{\infty}$
   - $\textsc{rank}(p) = u \;\Rightarrow\; p \in \langle \Theta A_{\leq u} \rangle$

Note:
Desired properties are defined without use of algorithms

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Example

Assume: Involutive completions exist

Claim: There is an involutive completion $A^{\mathcal{L}} \subseteq \Theta A$

Reason:

1. Let $\beta \in \mathcal{N}_{A^{\mathcal{L}},Y} \setminus \mathcal{N}_{A,Y}$
2. Choose $p \in A$ with $\mathrm{LEAD}(p) = \alpha Y$, α divides β
3. Add $(\beta - \alpha)p$ to $A^{\mathcal{L}}$

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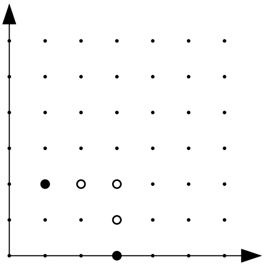

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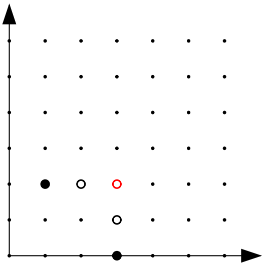

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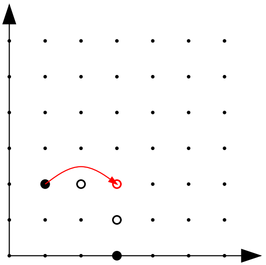

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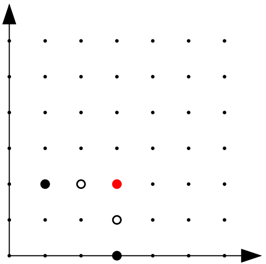

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Passive Systems

Theorem

$p \in [A^{\mathcal{L}}]^u_{\mathcal{L}} : H^\infty \Rightarrow$ *exists* $h \in H^\infty$ *with* $hp = \lambda(\gamma r) + \sum_{i=1}^k \lambda_i m_i$

1. γr, *unique multiplicative prolongation with* $\text{LEAD}(\gamma r) = u$
2. m_i, *product of multiplicative prolongations* $< u$
3. $\text{LEAD}(p) \leq u \Rightarrow \lambda_i$ *partially reduced w.r.t.* A

Wanted: The above property for all $p \in [A] : H^\infty$

Definition

(A, H) is passive w.r.t. $A^{\mathcal{L}}$ if

$$\langle \Theta A_{\leq u} \rangle : H^\infty = [A^{\mathcal{L}}]^u_{\mathcal{L}} : H^\infty \quad \forall u \in \Theta \mathcal{Y}$$

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Independence of Involutive Completions

Consider: Involutive completions $A_1^{\mathcal{L}}$ and $A_2^{\mathcal{L}}$ of A

Theorem

(A, H) is passive w.r.t. $A_1^{\mathcal{L}} \Leftrightarrow (A, H)$ is passive w.r.t. $A_2^{\mathcal{L}}$

Proof.

$\Rightarrow$

Wanted: $\alpha p \in [A_2^{\mathcal{L}}]_{\mathcal{L}}^{u} : H^{\infty}$ whenever $p \in A$, $\text{LEAD}(\alpha p) = u$

Have: $h(\alpha p) = \lambda_1(\gamma_1 r_1) + s$, where $s \in [A_1^{\mathcal{L}}]_{\mathcal{L}}^{v}$ some $v < u$

Induction: $[A_1^{\mathcal{L}}]_{\mathcal{L}}^{v} : H^{\infty} = [A_2^{\mathcal{L}}]_{\mathcal{L}}^{v} : H^{\infty}$

Pseudo-Division: $\text{INIT}(\gamma_2 r_2)\gamma_1 r_1 = \text{INIT}(\gamma_1 r_1)\gamma_2 r_2 + s'$ $\square$

Consequence:

The new definition specializes to Wu's notion of passivity

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Consequence:

The new definition specializes to Wu's notion of passivity

Independence of Involutive Completions

Consider: Involutive completions $A_1^{\mathcal{L}}$ and $A_2^{\mathcal{L}}$ of A

Theorem

(A, H) is passive w.r.t. $A_1^{\mathcal{L}} \Leftrightarrow (A, H)$ is passive w.r.t. $A_2^{\mathcal{L}}$

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Equivalence of Passivity and Coherence

Consider: Involutive division $\mathcal{L}$
such that involutive completions of A exist

Theorem

(A, H) is passive w.r.t. $\mathcal{L} \Leftrightarrow (A, H)$ is coherent

Proof.

$\Rightarrow$ as done by Li & Wang '99

$\Leftarrow$ **Choose**: Involutive completion $A^{\mathcal{L}} \subseteq \Theta A$

Note: $(A^{\mathcal{L}}, H)$ is coherent

Deduce: $p \in A$ implies

$S(\alpha p, \gamma r) = \mathrm{INIT}(\gamma r)\alpha p - \mathrm{INIT}(\alpha p)\gamma r \in \langle \Theta A_{<\mathrm{LEAD}(\alpha p)} \rangle : H^\infty$

Induction: $\langle \Theta A_{\leq u} \rangle : H^\infty = [A^{\mathcal{L}}]^u_{\mathcal{L}} : H^\infty, \forall u < \mathrm{LEAD}(\alpha p)$ □

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Independence of Involutive Divisions

Consider: Involutive divisions $\mathcal{L}_1$ and $\mathcal{L}_2$ such that involutive completions of A exist

Corollary

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Proof.

Follows from equivalence with coherence. □

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Proof.

Follows from equivalence with coherence. □

Conclusion

1. Is passivity independent of the involutive completion? Yes
2. Every (Wu-)passive system is coherent. [Li&Wang '99]
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