Coherent Configurations and Association Schemes. Part I

Definitions, examples, simple facts

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Contents

1. Graphs, relations and matrices

- *A (directed) graph* $\Gamma = (X, R)$, where X is a finite set of vertices of cardinality n, and $R \subseteq X^2$ is a set of arcs (a binary relation). Usually, $X = \{1, 2, \ldots, n\} = [1, n]$ or $X = [0, n-1]$.

- *A (complete) colour graph* $\left(X, \{R_l\}_{1 \le l \le r}\right)$ is defined by a partition of X^2 into r binary relations. Usually each index, i, is regarded as a *colour*.

- *Adjacency matrix* $A_l = A(\Gamma_l)$ for $\Gamma_l = (X, R_l)$ is a $(0, 1)$-matrix $A = (a_{ij})$ such that

$$a_{ij} = 1 \iff (i, j) \in R_l$$

  In this case, R_l is called *support* of A_l.

- Let H be a group, $S \subseteq H$.
  A graph $\Gamma = Cay(H, S)$ is defined as follows:
  $\Gamma = (H, R)$, where

  $$R = \{(x, x + h) | x \in H, h \in S\}$$

  Γ is called *Cayley graph* over H with a *connection set* S.
  No loops in $\Gamma \iff 0 \notin S$.
  Γ is undirected $\iff S = -S$, where
  $-S = \{-x | x \in S\}$.

- A partition $\{S_0, S_1, \ldots, S_d\}$ of H such that $S_0 = \{0\}$ defines a complete colour Cayley graph over H.

- Let $R, S \subseteq X^2$, we can define $R \circ S$ (*product* of two relations) as a multi-subset of X^2:
  A pair (a, b) appears in $R \circ S$ with the multiplicity $m(a, b)$ where

$$m(a,b) = |\{c \in X | (a,c) \in R \wedge (c,b) \in S\}|$$

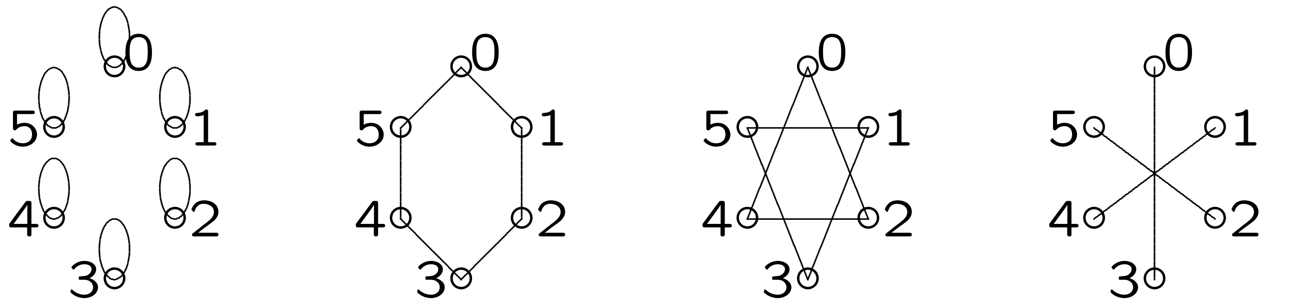
- This multi-subset, $R \circ S$, may be regarded as a colour graph over X.

- Clearly in this case we get for $\Gamma_1 = (X, R)$, $\Gamma_2 = (X, S)$,

$$A(\Gamma_1) \cdot A(\Gamma_2) = B$$

  where B is the linear combination of the adjacency matrices of "partial" relations in $R \circ S$.

Example 1. *a) Graphs:*



R_0 R_1 R_2 R_4

b) Connection sets of Cayley graphs over $\mathbb{Z}_6$*:*

$$S_0 = \{0\}, \quad S_1 = \{1, 5\}, \quad S_2 = \{2, 4\}, \quad S_3 = \{3\}$$

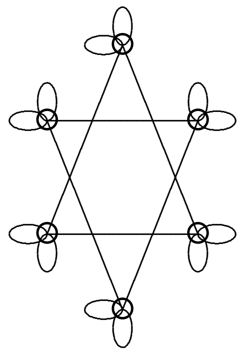
*c) Matrices (*0 *is substituted by* 6*)*

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Multiplication:

(i)



$R_1 \circ R_1 \qquad A_1^2 = 2A_0 + A_2$

$$\begin{aligned} \underline{S_1} \circ \underline{S_1} &= \underline{1,5} \circ \underline{1,5} = \\ &= \underline{1+1, 1+5, 5+1, 5+5} = \\ &= \underline{2,0,0,4} = 2 \cdot \underline{0} + \underline{2,4} = \\ &= 2\underline{S_0} + \underline{S_2} \end{aligned}$$

(ii) $A_1 \cdot A_2 = A_1 + 2A_3$

$$\begin{aligned} \underline{S_1} \circ \underline{S_2} &= \underline{1,5} \circ \underline{2,4} = \\ &= \underline{3,5,1,3} = \underline{S_1} + 2\underline{S_3} \end{aligned}$$

Here $\underline{x_1, \ldots, x_k}$ *is element of group algebra over group* H *for* $x_1, \ldots, x_k \in H$*. (More about this in part II).*

2. 2-orbits of permutation groups

- *Permutation* g of a finite set X is a bijection on set X.
  For $x \in X$ and permutation g, x^g denotes image of x under g.
  For permutations g_1, g_2 on X, their *composition* (product) $g = g_1 g_2$ is $x^g = x^{g_1 g_2} = (x^{g_1})^{g_2}$ for $x \in X$.
  $S(X)$ is *Symmetric group* of X, if $|X| = n$, we denote it by S_n.

- A *permutation group* (G, X) of degree n is a subgroup G of the symmetric group $S(X)$; G *acts* on set X.

- A subset $G \subseteq S(X)$ is a permutation group $\iff$ G is closed under composition.

- For $x \in X$ and permutation group (G, X), the *orbit* $Orb(x)$ is $Orb(x) = x^G = \{x^g | g \in G\}$.
  $Orb(G, X)$ is the set of all orbits of (G, X).
  Transitive group $(G, X) \iff Orb(G, X) = \{X\}$.

- For (G, X) we consider induced action (G, X^2):
  For $(a, b) \in X^2$, $(a, b)^g = (a^g, b^g)$.
  Orbits of this induced action are called (following H. Wielandt) *2-orbits* of (G, X).
  $2 - Orb(G, X)$ is the set of all 2-orbits.

- Pair $(X, 2 - Orb(G, X))$ is a complete colour graph with the vertex set X.

Example 1. *(Revisited) Let D_6 be the dihedral group of degree 6 and order 12,*

$$D_6 = < (0,1,2,3,4,5),(1,5)(2,4) >$$

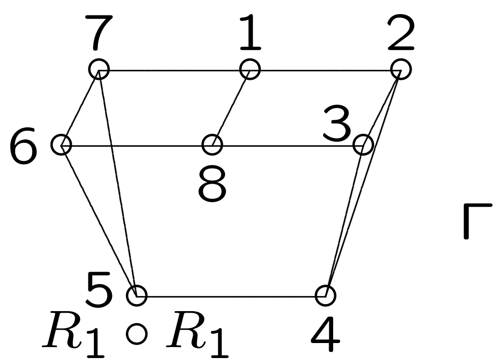
Clearly, D_6 is transitive. Then,

$$2 - orb(D_6, [0,5]) = \{R_0, R_1, R_2, R_3\}$$

*Here all graphs defined by the 2-orbits (*orbital *graphs in alternative terminology) are regular undirected graphs. The rank r of a permutation group is $|2 - Orb(G, X)|$, thus*

$$r(D_6, [0,5]) = 4$$

Example 2.

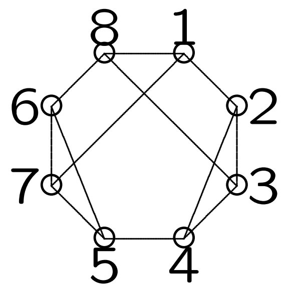


*"Chemical" graph cuneane=wedge=*КЛИН *(determines a synthesized compound).*

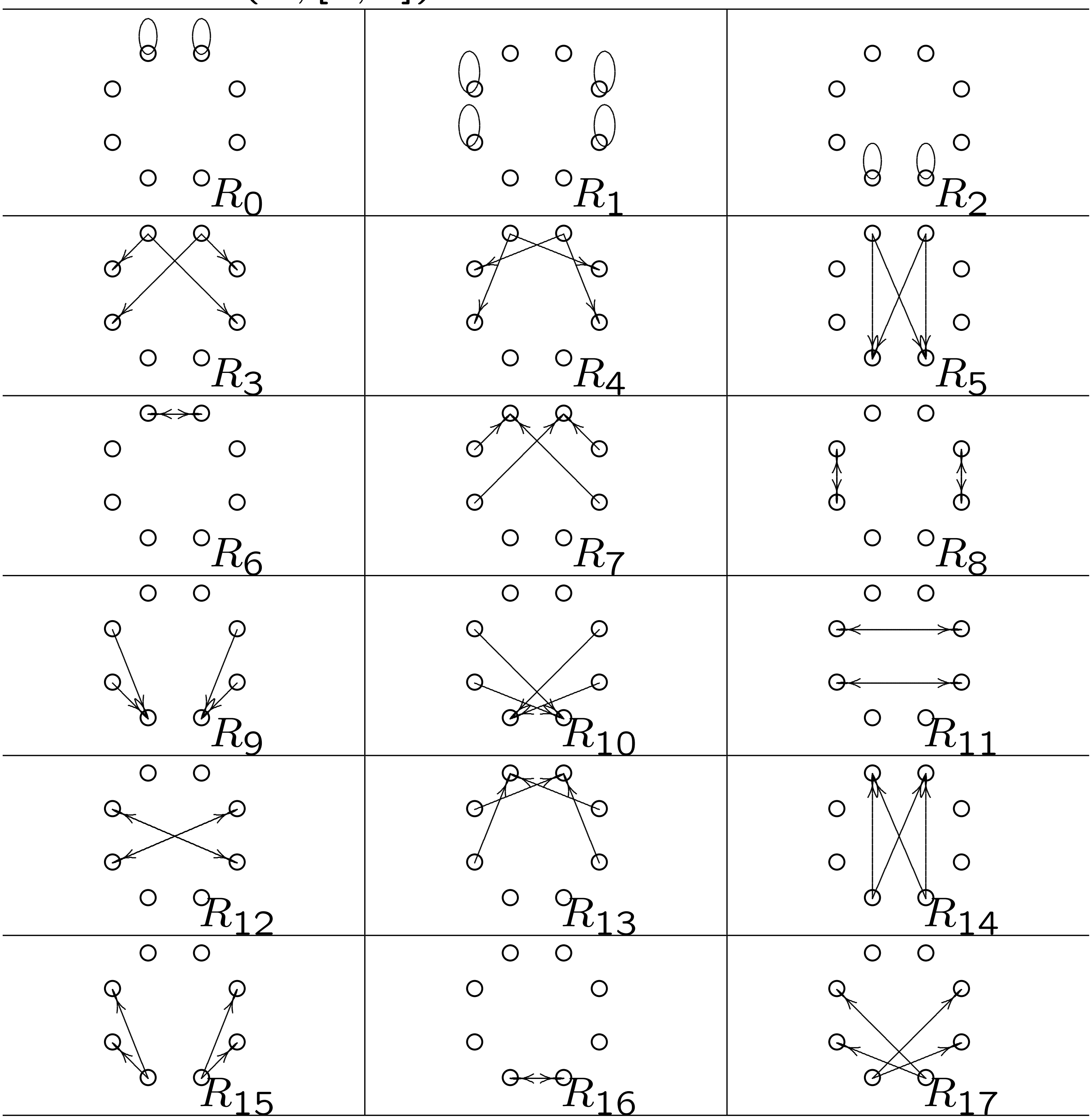
$G = Aut(\Gamma) = \{e, g_2, g_3, g_4\}$, *where* e *is the identity permutation,* $g_2 = (1,8)(2,3)(6,7)$, $g_3 = (2,7)(3,6)(4,5)$, $g_4 = (1,8)(2,6)(3,7)(4,5)$. $Orb(G,[1,8]) = \{\{1,8\},\{2,3,6,7\},\{4,5\}\}$. *Using the orbit counting lemma we get*

$$r(G,[1,8]) = \frac{1}{4}(8^2 + 2 \cdot 2^2 + 0^2) = 18$$

Cuneane again:



2-orbits of $(G, [1, 8])$:



- Let $g \in S(X)$ denote by $M(g)$ a permutation matrix of order $n = |X|$,

$$M(g) = (m_{ij})_{1 \le i,j \le n}$$

where $m_{ij} = \begin{cases} 1 & j = i^g \\ 0 & \text{otherwise} \end{cases}$.
Clearly, $M(g^{-1}) = (M(g))^{-1} = (M(g))^t$, where M^t is the matrix transposed to M.

- Let K be a ring or a field. Let $M_n(K)$ be the ring (algebra) of all square matrices of order n over K. Define for a permutation group (G, X),

$$V_K(G, X) = \{A \in M_n(K) | \forall g \in G \; AM(g) = M(g)A\}$$

- $V_K(G, X)$ is a ring (algebra) which is called the *centralizer ring* (algebra) of (G, X). (Most important cases for us are $K = \mathbb{Z}$, $K = \mathbb{C}$. Usually sign K is omitted.)

- **Theorem** Let (G, X) be a permutation group of degree n and rank r. Then

  a) $V_K(G, X)$ as a vector space over K has dimension r;

  b) $V_K(G, X)$ has a special basis which consists of $(0, 1)$-matrices;

  c) The members of this *standard basis* are in bijective correspondence with the set $2 - Orb(G, X)$.

- To prove this theorem we have to detect that if

$$2 - Orb(G, X) = (R_1, \ldots, R_r)$$

  then $V(G, X) =< A_1, \ldots, A_r >$ where $A_i = A(\Gamma_i)$, $\Gamma_i = (X, R_i)$, $1 \leq i \leq r$.

Example 1. *(Revisited)*

$$V(D_6, [0,5]) = < A_0, A_1, A_2, A_3 >$$

Multiplication table: $R_i \cdot R_j = \sum_{k=0}^{3} p_{ij}^k R_k$

$\backslash R_j$ $R_i \backslash$	R_0	R_1	R_2	R_3	control sum
	1	0	0	0	1
R_0	0	1	0	0	1
	0	0	1	0	1
	0	0	0	1	1
	0	2	0	0	2
R_1	1	0	1	0	2
	0	1	0	1	2
	0	0	2	0	2
	0	0	2	0	2
R_2	0	1	0	1	2
	1	0	1	0	2
	0	2	0	0	2
	0	0	0	1	1
R_3	0	0	1	0	1
	0	1	0	0	1
	1	0	0	0	1

Example 2. *Here $V(G,X)$ has rank 18, thus it has 18 basis matrices. All these matrices are presented in a compact form as a colour matrix*

$$A(V(G,X)) = \sum_{i=0}^{17} iA_i$$

where $A_0, A_1, \ldots, A_{17}$ is the standard basis of $V(G,X)$.

$$\begin{pmatrix} 0 & 3 & 4 & 5 & 5 & 4 & 3 & 6 \\ 7 & 1 & 8 & 9 & 10 & 11 & 12 & 13 \\ 13 & 8 & 1 & 9 & 10 & 12 & 11 & 7 \\ 14 & 15 & 15 & 2 & 16 & 17 & 17 & 14 \\ 14 & 17 & 17 & 16 & 2 & 15 & 15 & 14 \\ 13 & 11 & 12 & 10 & 9 & 1 & 8 & 7 \\ 7 & 12 & 11 & 10 & 9 & 8 & 1 & 13 \\ 6 & 4 & 3 & 5 & 5 & 3 & 4 & 0 \end{pmatrix}$$

3. Coherent configurations

- We are now giving an axiomatic analogue of $2 - Orb(G, X)$ and $V(G, X)$ for an arbitrary permutation group.

- Let $X = [1, n]$, $\mathfrak{R} = \{R_1, \ldots, R_r\}$ a collection of binary relations on X such that:

CC1 $R_i \cap R_j = \emptyset$ for $1 \leq i \neq j \leq r$;

CC2 $\bigcup_{i=1}^{r} R_i = X^2$;

CC3 $\forall i \in [1, r] \exists i' \in [1, r] R_i^t = R_{i'}$, where $R_i^t = \{(y, x) | (x, y) \in R_i\}$;

CC4 $\exists I' \subseteq [1, r] \bigcup_{i \in I'} R_i = \Delta$, where $\Delta = \{(x, x) | x \in X\}$;

CC5 $\forall i, j, k \in [1, r] \forall (x, y) \in R_k | \{z \in X | (x, z) \in R_i \wedge (z, y) \in R_j\}| = p_{ij}^k$

$\mathfrak{M} = (X, R)$ is called a *coherent configuration* (D. G. Higman, 1970).

- The numbers p_{ij}^k are *intersection numbers*.

- Switching to matrix language: Let $W \subseteq M_n(\mathbb{C})$ be a matrix algebra over $\mathbb{C}$ such that

CA1 W as a linear space over $\mathbb{C}$ has some basis, $A_1, A_2, \ldots, A_r$, consisting of $(0,1)$-matrices;

CA2 $\sum_{i=1}^{r} A_i = J_n$, where J_n is the matrix of order n all entries of which are equal to 1;

CA3 $\forall i \in [1,r] \exists i' \in [1,r] A_i^t = A_{i'}$;

Then W is called a *coherent algebra* of rank r and order n with the *standard basis* $A_1, A_2, \ldots, A_r$. We write $W = < A_1, \cdots, A_r >$.

Remark: similar notion of a *cellular algebra* (B. Weisfeiler, A. A. Leman, 1968).

- To each coherent configuration $\mathfrak{M} = (X, \mathfrak{R})$ we associate its *adjacency algebra* $W =< A_i = A(X, R_i) | R_i \in \mathfrak{R} >$ which is a coherent algebra.

- **Proposition**: For each permutation group (G, X) its centralizer algebra $V(G, X)$ is a coherent algebra.

- Remark: Converse proposition is not true: there are many examples of coherent algebras which are not centralizer algebras of a suitable permutation group.

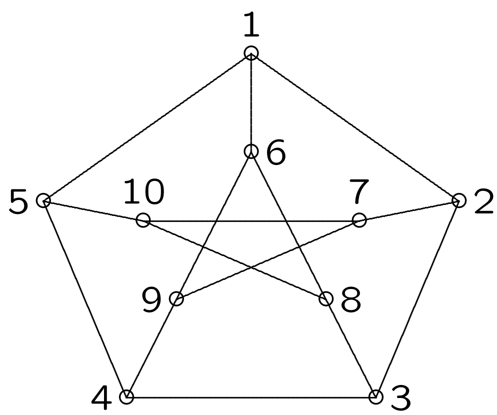
- **Theorem** A matrix algebra W is a coherent algebra $\iff$ W is closed with respect to Schur-Hadamard (entry-wise) multiplication, transposition, and contains matrices I, J, where I is the identity matrix.

- Remark An equivalent formulation uses closure with respect to the Hermitean adjoint instead of transposition.

- **Corollary** If W_1 and W_2 are two coherent algebras of the same order n, then $W = W_1 \cap W_2$ is also a coherent algebra of order n.

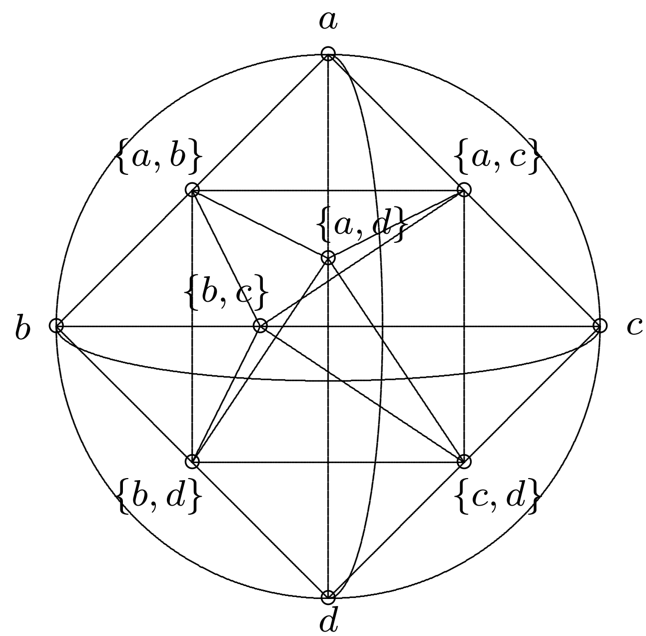
(More details in part II.)

Example 3. Part a)



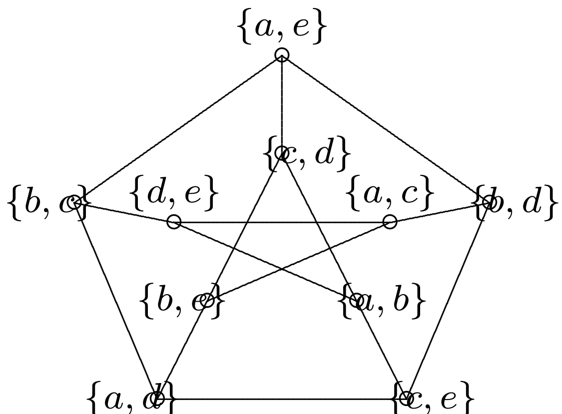
Γ'

$D_5 \leq Aut(\Gamma')$

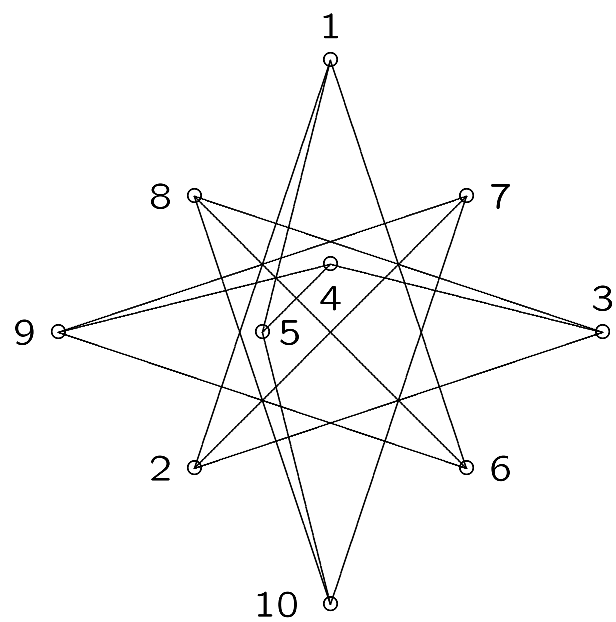


$\overline{\Gamma''}$

$S_4 \leq Aut(\overline{\Gamma''}) = Aut(\Gamma'')$



Γ'



Γ''

$\Gamma' = \Gamma'' = \Gamma =$ *Petersen graph.* $|Aut(\Gamma)| \geq$ gcd$(10, 24) = 120$. *In fact,* $Aut(\Gamma) \simeq S_5$.

Part b)

$$\begin{pmatrix}
0 & 2 & 3 & 3 & 2 & 4 & 5 & 6 & 6 & 5 \\
2 & 0 & 2 & 3 & 3 & 5 & 4 & 5 & 6 & 6 \\
3 & 2 & 0 & 2 & 3 & 6 & 5 & 4 & 5 & 6 \\
3 & 3 & 2 & 0 & 2 & 6 & 6 & 5 & 4 & 5 \\
2 & 3 & 3 & 2 & 0 & 5 & 6 & 6 & 5 & 4 \\
7 & 8 & 9 & 9 & 8 & 1 & 10 & 11 & 11 & 10 \\
8 & 7 & 8 & 9 & 9 & 10 & 1 & 10 & 11 & 11 \\
9 & 8 & 7 & 8 & 9 & 11 & 10 & 1 & 10 & 11 \\
9 & 9 & 8 & 7 & 8 & 11 & 11 & 10 & 1 & 10 \\
8 & 9 & 9 & 8 & 7 & 10 & 11 & 11 & 10 & 1
\end{pmatrix}$$

$W' = V(D_5, [1..10])$, $rank(W') = \frac{1}{10}(10^2 + 5 \cdot 2^2) = 12$

$$\begin{pmatrix}
0 & 2 & 3 & 4 & 2 & 2 & 4 & 4 & 3 & 3 \\
5 & 1 & 5 & 6 & 6 & 6 & 7 & 6 & 8 & 8 \\
3 & 2 & 0 & 2 & 4 & 4 & 4 & 2 & 3 & 3 \\
8 & 6 & 5 & 1 & 7 & 6 & 6 & 6 & 5 & 8 \\
5 & 6 & 8 & 7 & 1 & 6 & 6 & 6 & 8 & 5 \\
5 & 6 & 8 & 6 & 6 & 1 & 6 & 7 & 5 & 8 \\
8 & 7 & 8 & 6 & 6 & 6 & 1 & 6 & 5 & 5 \\
8 & 6 & 5 & 6 & 6 & 7 & 6 & 1 & 8 & 5 \\
3 & 4 & 3 & 2 & 4 & 2 & 2 & 4 & 0 & 3 \\
3 & 4 & 3 & 4 & 2 & 4 & 2 & 2 & 3 & 0
\end{pmatrix}$$

$W'' = V(S_4, [1, 10])$, $rank(W'') = \frac{1}{24}(10^2 + 6 \cdot 4^2 + 3 \cdot 2^2 + 8 \cdot 1^2) = 9$

$$\begin{pmatrix}
0 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 & 2 \\
2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 2 \\
1 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 1 & 1 \\
2 & 2 & 1 & 2 & 2 & 1 & 2 & 0 & 2 & 1 \\
2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0 & 2 \\
2 & 2 & 2 & 2 & 1 & 2 & 1 & 1 & 2 & 0
\end{pmatrix}$$

$W = W' \cap W'' = V(Aut(\Gamma)) = V(S_5, [1, 10])$ $rank(W) = 3$.

There exists a nice simple purely combinatorial algorithm which allows to construct W from W' and W''.

- If $W' \subseteq W$ are coherent algebras of the same order then W' is called a *coherent subalgebra* of W.

- If $\mathfrak{M}'$ is a coherent configuration with adjacency algebra W' and $\mathfrak{M}$ corresponds to W then in this case $\mathfrak{M}'$ is called *fusion* coherent configuration of $\mathfrak{M}$ or *merging* of relations (classes) of $\mathfrak{M}$.

- Enumeration of coherent subalgebras of a given coherent algebra is an important task (see further talks).

- *Diagonal relation* $\Delta = \{(x, x)|x \in X\}$ is splitted in a coherent configuration $\mathfrak{M}$ into (one or more) basis relations. Each such basis relation uniquely defines a *fiber* of $\mathfrak{M}$. (See more precise definition in abstract).

- Fiber is a combinatorial analogue of an orbit of a permutation group. Coherent configurations with one fiber are analogues of transitive permutation groups.

- Coherent configurations with one fiber is a *homogeneous coherent configuration* or an *association scheme* (not obligatory commutative).

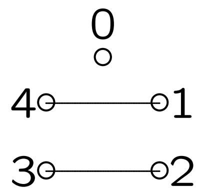
4. Association schemes

- Again: homogeneous coherent configuration = association scheme = coherent configuration with one fiber = diagonal relation Δ is one of basis relations = all basis graphs are regular.

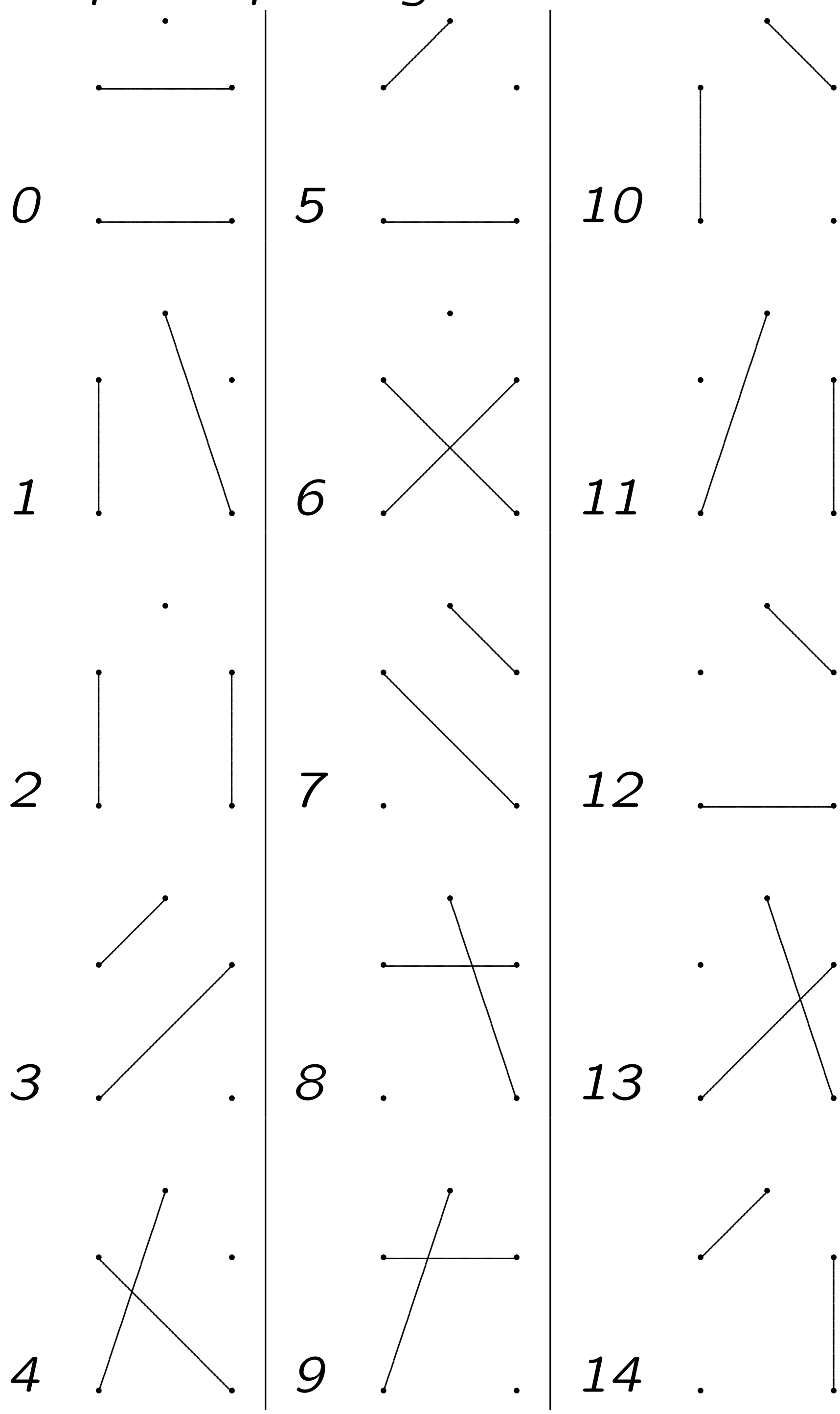
- In this context association scheme is a very natural combinatorial analogue of a transitive permutation group.

- *Schurian* association scheme $\mathfrak{M}$ corresponds to pair $(X, 2-Orb(G,X))$ for a suitable transitive permutation group (G,X); otherwise, $\mathfrak{M}$ is *non-Schurian*.

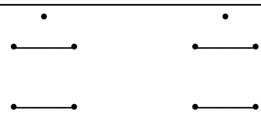
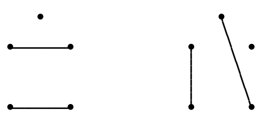
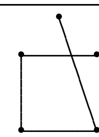
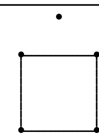
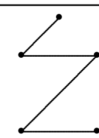
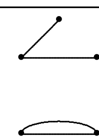
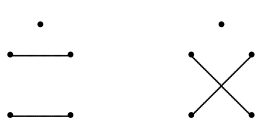
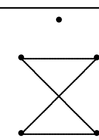
Example 4. *Let us start with* $(A_5, [0, 4])$*, the* alternating group *of degree* 5 *and order* 60.
Let X *be set of all labeled graphs of form*



$|X| = 15$. A_5 *acts transitively on* X. *Labeling produced by computer package COCO:*

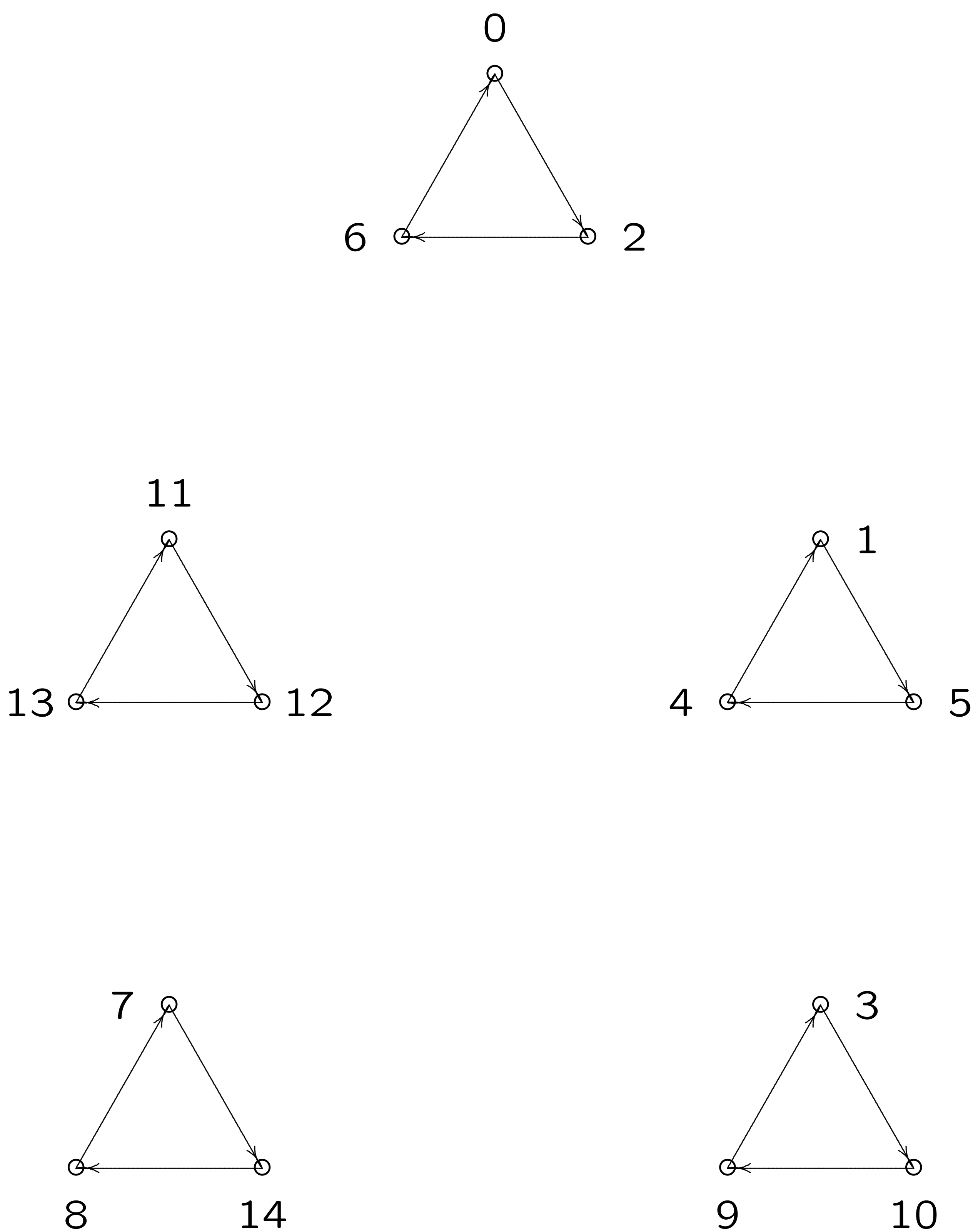


"Typical" representatives of 2-orbits:

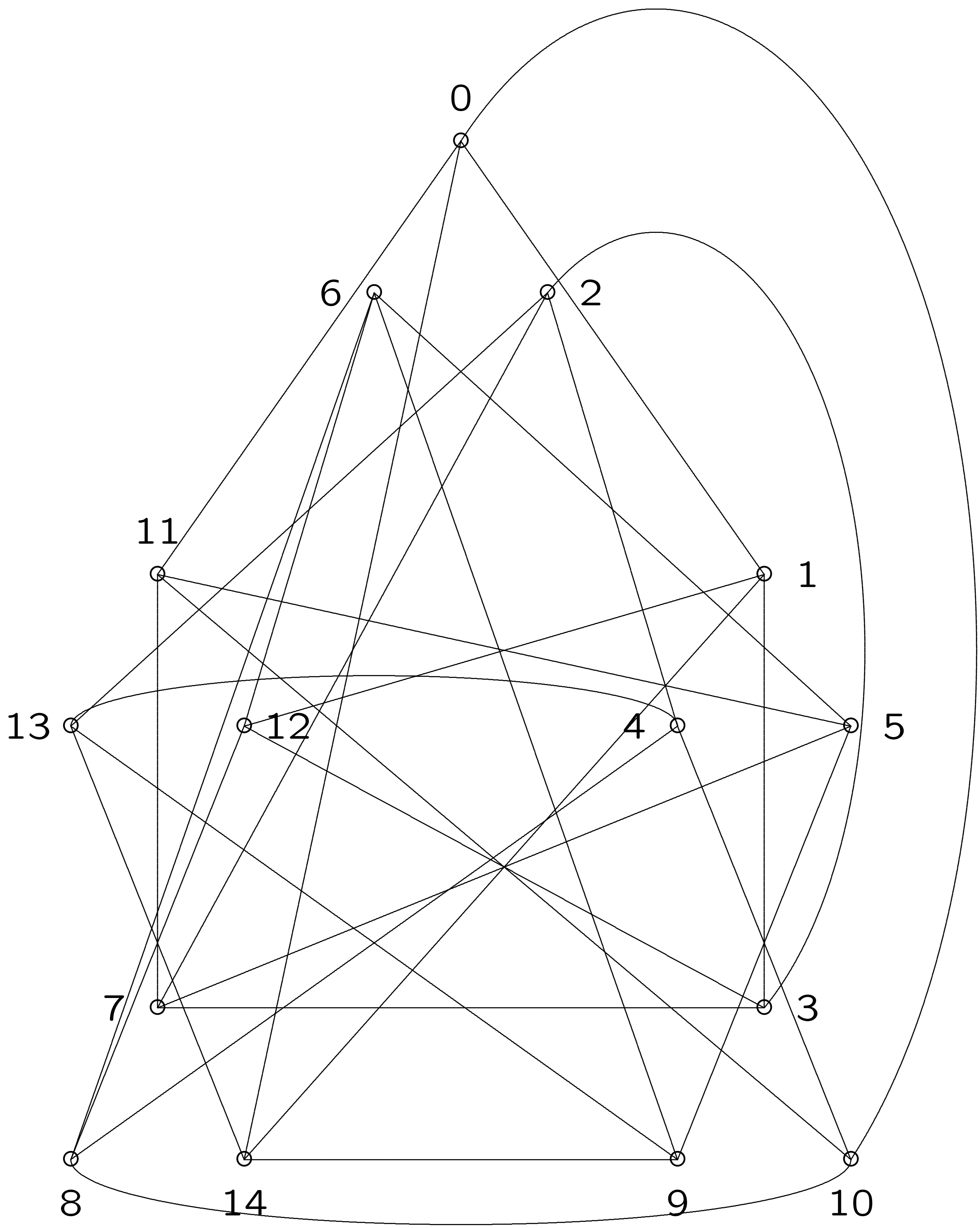
R_i	Pair	Picture	Valency	Comments
R_0	$(0,0)$		1	*diagonal*
R_1	$(0,1)$		4	
R_2	$(0,2)$		1	
R_3	$(0,3)$		4	
R_4	$(0,5)$		4	
R_5	$(0,6)$		1	
		Total:	*15*	

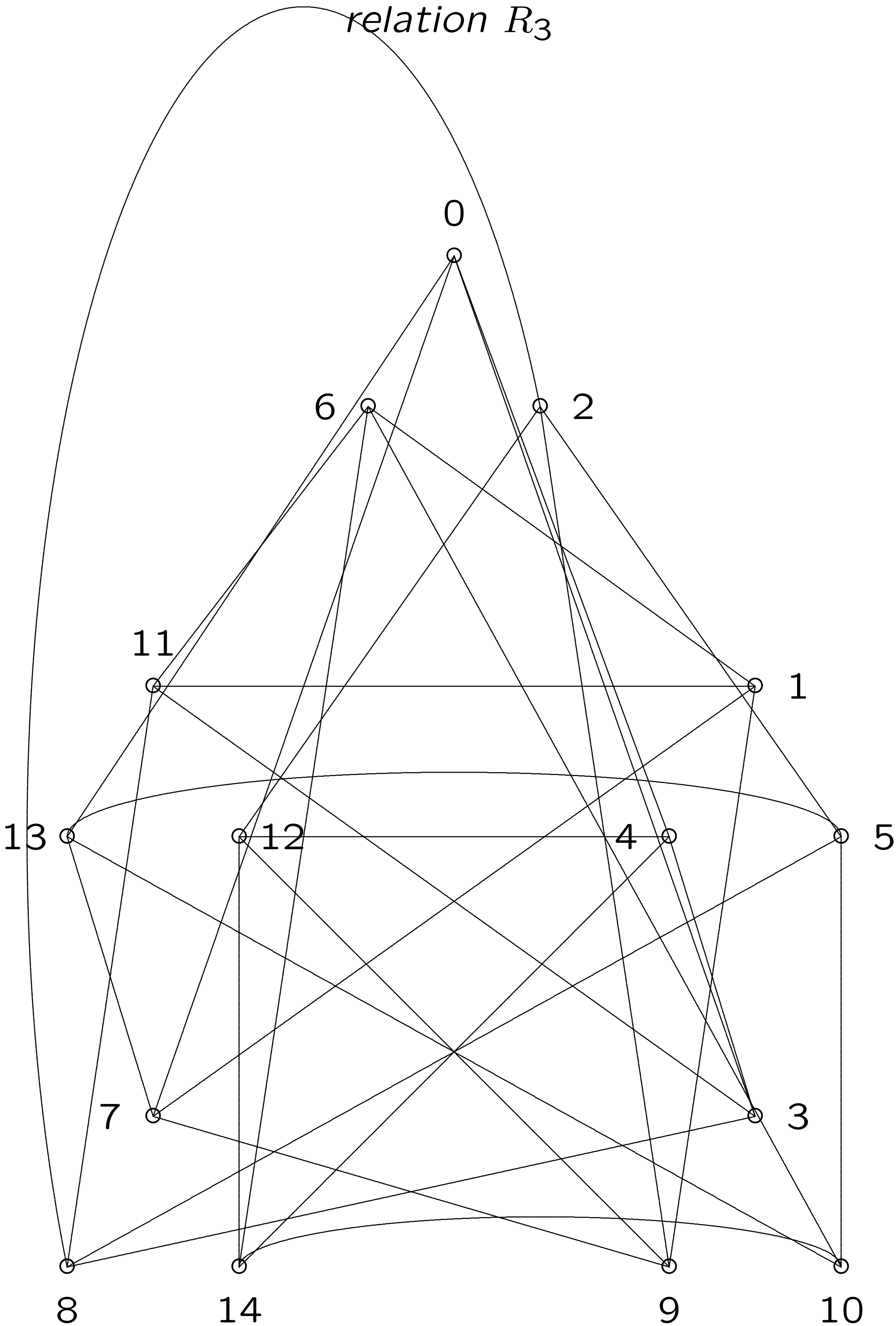
Remark: distinction between R_1 and R_3, R_2 and R_5 requires "chirality vision".

relation R_2 ($R_5 = R_2^t$)

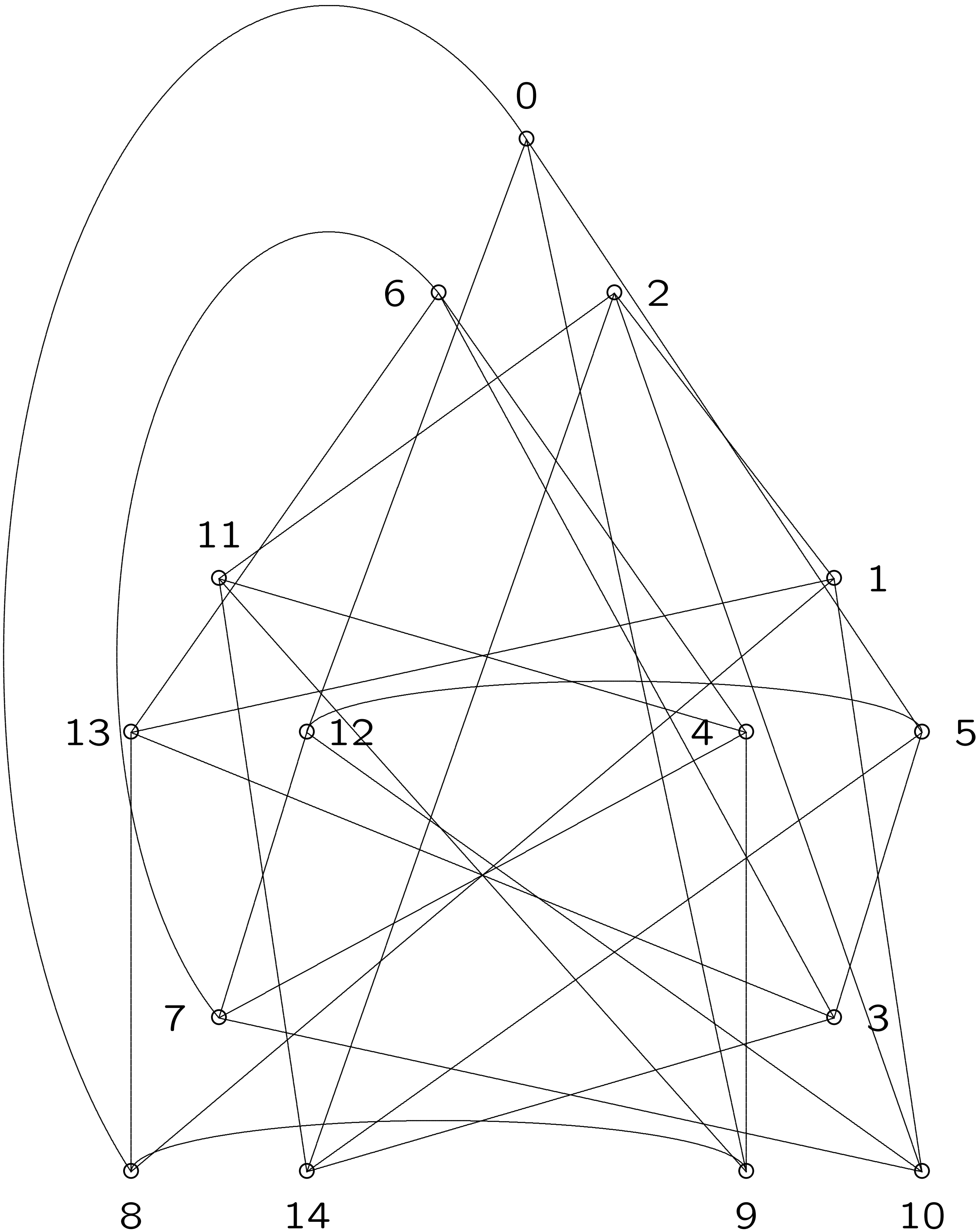


relation R_1

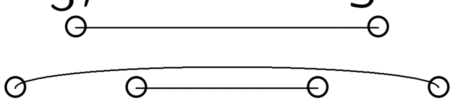



relation R_3
0
6
2
11
1
13
12
4
5
7
3
8
14
9
10

relation R_4



Comments

1. We have 3 times the same situation: 3-fold covering of K_5 (triangles in R_2 are vertices of K_5; each edge of K_5 is "blown-up" to 1-factor ).

2. The scheme is not commutative.

3. This is the first member of an exceptional infinite series of so-called *Siamese* association schemes on $(q+1)(q^2+1)$ vertices, q is a prime power (here, $q=2$).

5. Adjacency and intersection algebras

- Recall that structure constants form a 3-dimensional tensor.

  Let $\mathfrak{M} = (X, \{R_0, R_1, \ldots, R_d\})$ be an association scheme with d *classes* (diagonal relation R_0 does not have status of a class). Let us arrange all structure constants P_{ij}^k into $d+1$ matrices $B_0, B_1, \ldots, B_d$

$$B_i = \begin{pmatrix} p_{i0}^0 & p_{i0}^1 & \cdots & p_{i0}^d \\ p_{i1}^0 & p_{i1}^1 & \cdots & p_{i1}^d \\ \cdots & \cdots & \cdots & \cdots \\ p_{id}^0 & p_{id}^1 & \cdots & p_{id}^d \end{pmatrix} \quad 0 \le i \le d$$

  Let $\mathfrak{B} = \ll B_0, B_1, \ldots, B_d \gg$ be the subalgebra of $M_{d+1}(\mathbb{C})$ generated by $B_0, B_1, \ldots, B_d$.

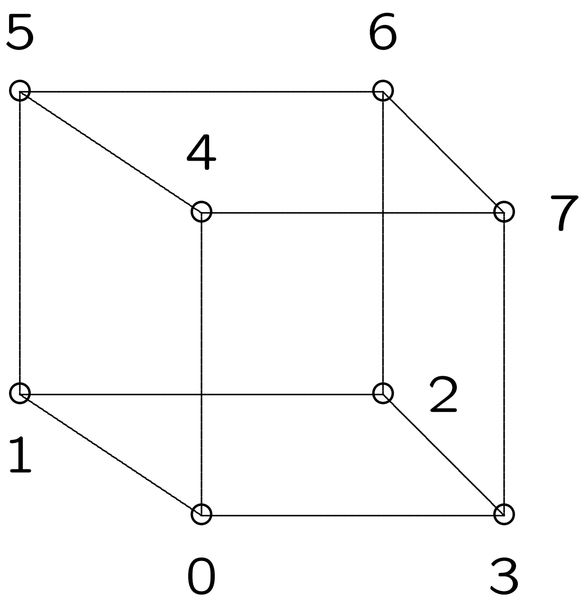
- **Theorem** The adjacency algebra of $\mathfrak{M}$ and algebra $\mathfrak{B}$ are anti-isomorphic; the isomorphism is established by correspondence $A_i \to B_i$. As a result, in particular, matrices A_i and B_i have the same minimal polynomial.

- **Corollary** In commutative case, both algebras are isomorphic.

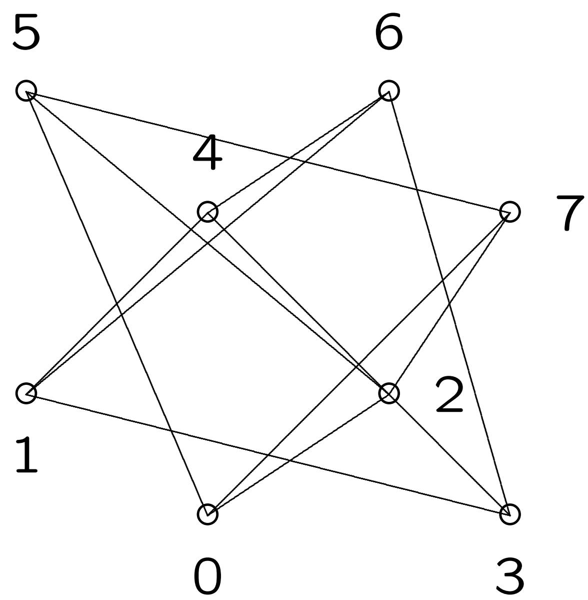
- The algebra $\mathfrak{B}$ is called *the intersection algebra* of association scheme $\mathfrak{M}$.

- Remark Similar theorem is valid for coherent configurations.

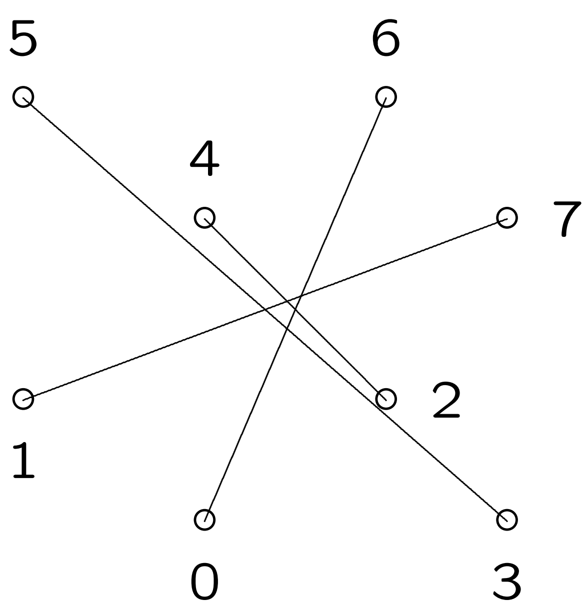
Example 5. *Metric association scheme with 3 classes, generated by 3-dimensional cube* Q_3.



R_1



R_2



R_3

Table of multiplication

$\backslash R_j$ $R_i \backslash$	R_0	R_1	R_2	R_3
R_0	1	0	0	0
	0	1	0	0
	0	0	1	0
	0	0	0	1
R_1	0	3	0	0
	1	0	2	0
	0	2	0	1
	0	0	3	0
R_2	0	0	3	0
	0	2	0	1
	1	0	2	0
	0	3	0	0
R_3	0	0	0	1
	0	0	1	0
	0	1	0	0
	1	0	0	0

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 3 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For example, $A_1^2 = 3A_0 + 2A_2$ (easy combinatorial inspection)

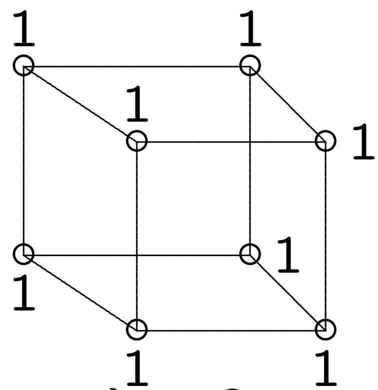
$$B_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 7 & 0 & 6 \\ 6 & 0 & 7 & 0 \\ 0 & 2 & 0 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 6 \\ 6 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} = 3B_0 + 2B_2$$

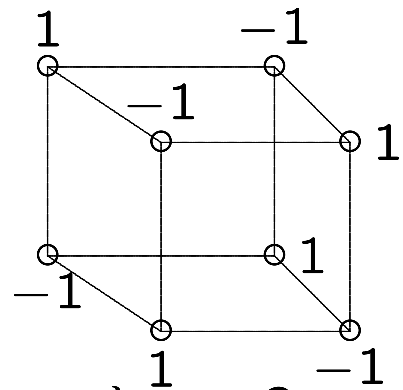
Why this knowledge is helpful?

We want to get $Spec(Q_3)$.

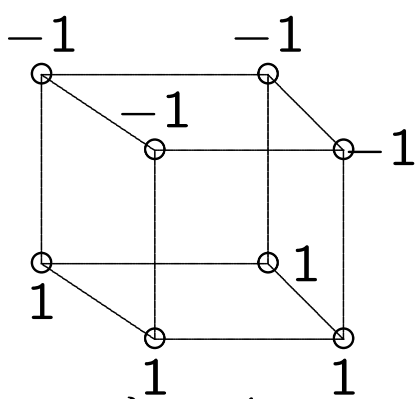
Way 1 Tricks a la C. Godsil.



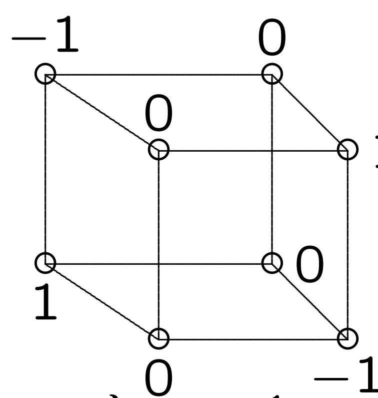
$\lambda = 3$



$\lambda = -3$



$\lambda = 1$



$\lambda = -1$

Way 2

$$\begin{vmatrix} \lambda & -1 & 0 & 0 \\ -3 & \lambda & -2 & 0 \\ 0 & -2 & \lambda & -3 \\ 0 & 0 & -1 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -2 & 0 \\ -2 & \lambda & -3 \\ 0 & -1 & \lambda \end{vmatrix} + \lambda \begin{vmatrix} -3 & -2 & 0 \\ 0 & \lambda & -3 \\ 0 & -1 & \lambda \end{vmatrix} =$$

$$= \lambda(\lambda(\lambda^2 - 3) + 2(-2\lambda)) - 3(\lambda^2 + 3) = \lambda^4 - 10\lambda^2 - 9 =$$

$$= (\lambda^2 - 9)(\lambda^2 - 1)$$

Eigenvalues are $-1, 1, 3, -3$.

(More about spectrum in part II).

6. Strongly regular graphs

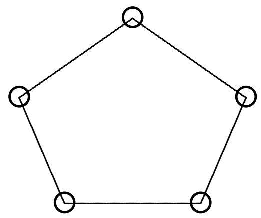
- A graph $\Gamma = (V, E)$ is *strongly regular* if it is regular of valency k and each pair of adjacent (unadjacent) vertices has exactly λ (μ) common neighbours.
  In this case we write $SRG(v, k, \lambda, \mu)$, where $v = |V|$.

- Γ is an srg $\iff$

$$\begin{cases} A^2 = kI + \lambda A + \mu(J - A - I) \\ AJ = JA = kJ \end{cases}$$

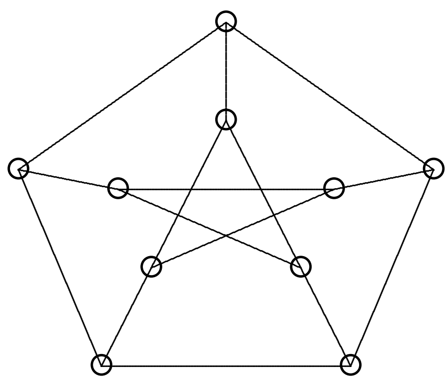
- Γ is an srg $\iff$ $\mathfrak{M}(\Gamma) = (V, < \Delta, E, \overline{E} >)$ is a symmetric association scheme with 2 classes, here $\overline{\Gamma} = (V, \overline{E})$ is graph *complementary* to Γ.

Example 6. *a) Pentagon*



$SRG(5, 2, 0, 1)$

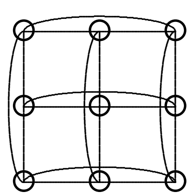
b) Petersen graph



$SRG(10, 3, 0, 1)$

c) $l \circ K_m$*: disjoint union of* l *copies of complete graph with* m *vertices.* $SRG(lm, m-1, m-2, 0)$.

d) Lattice square graph $L_2(3)$.



$SRG(9, 4, 1, 2)$

- SRG Γ is called *primitive* if both Γ and $\overline{\Gamma}$ are connected graphs.

- Using association scheme $\mathfrak{M}(\Gamma)$ with 2 classes (that is of dimension 3) it is easy to prove that an undirected regular Γ is an SRG $\iff$ Γ has exactly 3 distinct eigenvalues.

- There are various combinatorial and spectral equations and restrictions for sequence (v, k, λ, μ) of parameters of an SRG $\Rightarrow$ feasible set of parameters.

- a central question: existence of an SRG with a given feasible set of parameters.

- Rank 3 graph $\Gamma \iff Aut(\Gamma)$ is acting transitively on vertices, ordered pairs of adjacent and non-adjacent vertices.

- Rank 3 graph $\Gamma \iff$ for $G = Aut(\Gamma)$, $2-Orb(G,V) = \{\Delta, E, \overline{E}\}$.

- Rank 3 graph $\Gamma \Rightarrow \Gamma$ is an SRG.

- All rank 3 graphs are described via the use of classification of finite simple groups.

Two most known infinite series of SRG's:

- Triangular graph $T(m)$ = line graph of K_m is $SRG(\frac{m(m-1)}{2}, 2(m-2), 1, 4)$. For $m \neq 8$ is uniquely determined by parameters.

- Lattice square graph $L_2(m)$ = line graph of the complete bipartite graph $K_{m,m}$ is

$$SRG(m^2, 2(m-1), 0, 2)$$

For $m \neq 4$ is uniquely determined by parameters.

- Many other infinite series and sporadic examples are known.

- Origins in design of statistical experiments and partial geometries (R. C. Bose, 1963).

- Important and beautiful links with coding theory and finite geometries. In particular, poster by Axel Kohnert.

- Diverse behaviour:

  - for some parameter sets there exists "prolific" constructions;

  - for some parameter sets uniqueness is proved.

Example 7. *A non-rank-3 SRG on 280 vertices = non-Schurian association scheme with 2 classes (A.A. Ivanov, Klin, Faradžev, 1984), (Mathon-Rosa, 1985).*

- *Consider the symmetric group* S_9 *acting on the partitions of 9-set into 3 subsets of size 3;*

- *Degree is* $\frac{9!}{(3!)^4} = 280$*;*

- *Rank is 5 with subdegrees 1, 27, 36, 54, 162 (any two partitions have 3, 5, 9, 6, 7 non-empty intersections).*

- *The relation defined by 7 intersections gives*

$$\Gamma = SRG(280, 162, 96, 90)$$

$$\left(\overline{\Gamma} = SRG(280, 117, 44, 52)\right)$$

Problem Classify all SRG's Γ such that $Aut(\Gamma)$ is a primitive permutation group representation of the symmetric group S_n.
Known examples:

- K_n (trivial);
- $T(m)$ $(m \neq 4)$;
- a few sporadic examples, including the above one are known (Klin et al).

(Certain progress was achieved by Muzychuk and his followers.)

7. Distance regular graphs and metric schemes

- $\Gamma = (V, E)$ is (undirected) connected regular graph of valency k;
  for $x, y \in V$, $d(x, y)$ is a *distance* between vertices x and y, that is, the length of shortest path in Γ with the ends x, y;
  $d = d(\Gamma)$ is *diameter* of Γ.

- for $0 \leq i \leq d$ the *distance graph* $\Gamma_i = (V, R_i)$ has the same vertex set V; x, y are adjacent in $\Gamma_i \iff d(x, y) = i$;
  $\Gamma_0 = \Delta$, $\Gamma_1 = \Gamma$.

- Consider colour graph $(V, \{R_0, R_1, \ldots, R_d\})$: in general it does not form an association scheme.

- Graph Γ is called a *distance regular graph* (DRG) if $\mathfrak{M}(\Gamma) = (V, \{R_0, R_1, \ldots, R_d\})$ is an association scheme with d classes. In this case, $\mathfrak{M}(\Gamma)$ is called *metric* association scheme.

- Let W be an adjacency algebra of a metric association scheme. In this case W is generated by one matrix $A = A(\Gamma)$, moreover the generating process is in a sense canonical (cf talk by S. Reichard).

- We may associate to DRG Γ certain sequence of parameters (a part of intersection numbers of $\mathfrak{M}(\Gamma)$), which is called *intersection array*.

- A distance regular graph of diameter 2 = strongly regular graph.

- DRG Γ is called *primitive* if all distance graphs Γ_i, $1 \leq i \leq d$ are connected, otherwise Γ is *imprimitive*.

- DRG Γ is called *antipodal* if $\Gamma_d = m \circ K_l$ (a disjoint union of complete graphs, $v = |V| = ml$).

- **Theorem** Each imprimitive DRG is bipartite or antipodal.

- Graph Γ is called *distance transitive* (DTG) if for $G = Aut(\Gamma)$ the centralizer algebra $V(Aut(\Gamma))$ has rank $d+1$ (in other words, G acts transitively on ordered pairs of equidistant vertices).

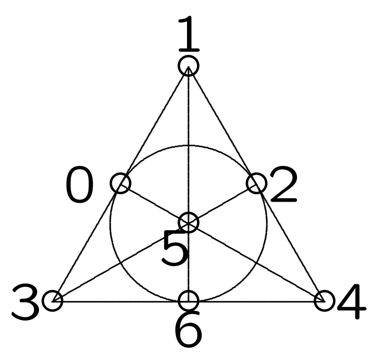
- Each DTG is DRG. There are many examples of DRG's which are not DTG's.

- Classification of DTG is one of the most monumental projects in algebraic graph theory (using CFSG): many infinite series and a lot of striking sporadic examples. This project is close to completion.

- Simplest non-trivial examples of imprimitive DRG's are coming from 3-dimensional euclidean geometry:
  - cube Q_3 ($k = 3$ $d = 3$, antipodal and bipartite),
  - icosahedron ($k = 5$, $d = 3$, antipodal),
  - dodecahedron ($k = 3$, $d = 6$, antipodal).

Example 8. *(Primitive graph on 21 vertices)*

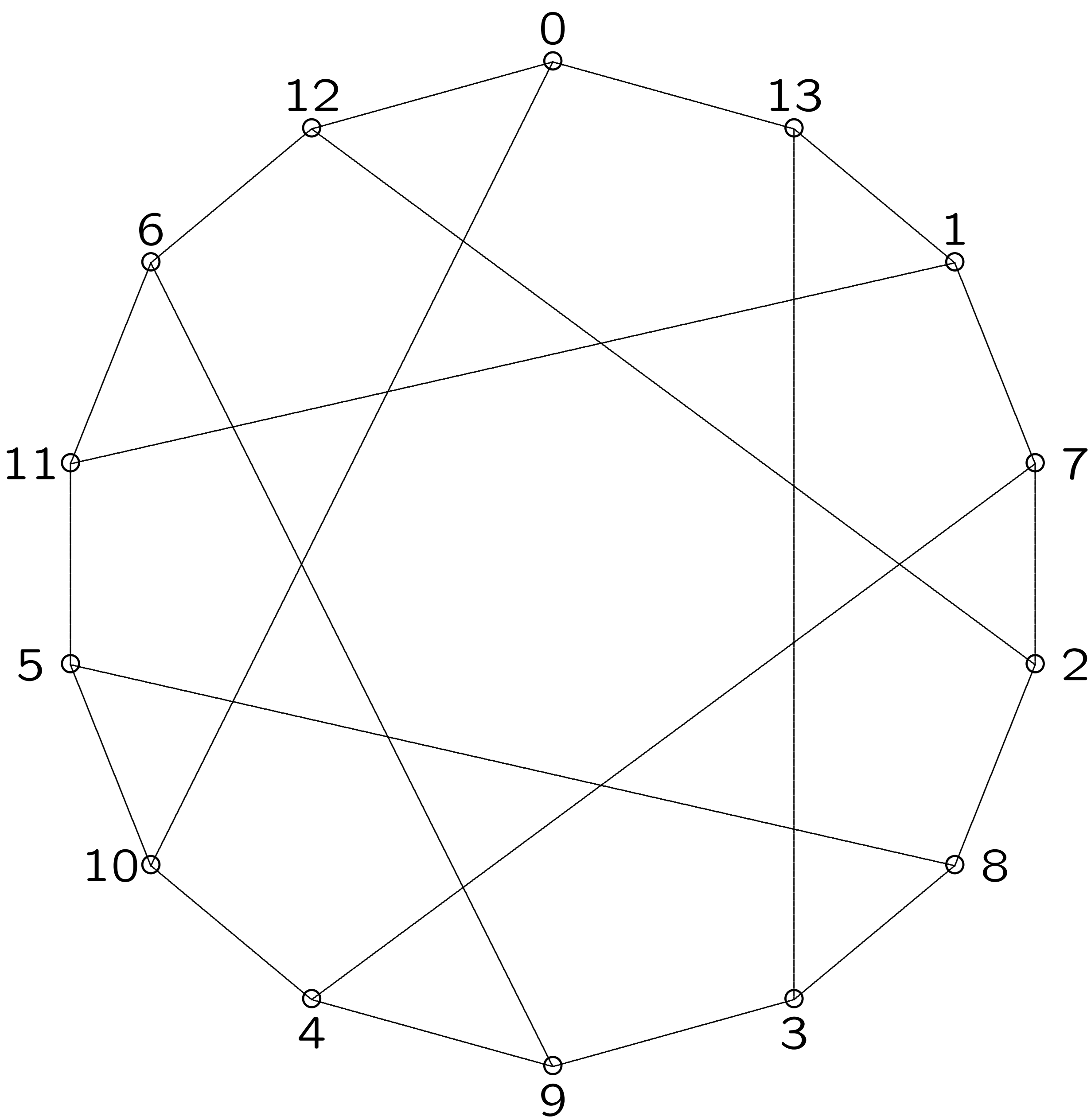
a) begin with Fano plane = $PG(2, 2)$*:*



$\{1, 2, 4\} = 7$
$\{2, 3, 5\} = 8$
$\{3, 4, 6\} = 9$
$\{4, 5, 0\} = 10$
$\{5, 6, 1\} = 11$
$\{6, 0, 2\} = 12$
$\{0, 1, 3\} = 13$

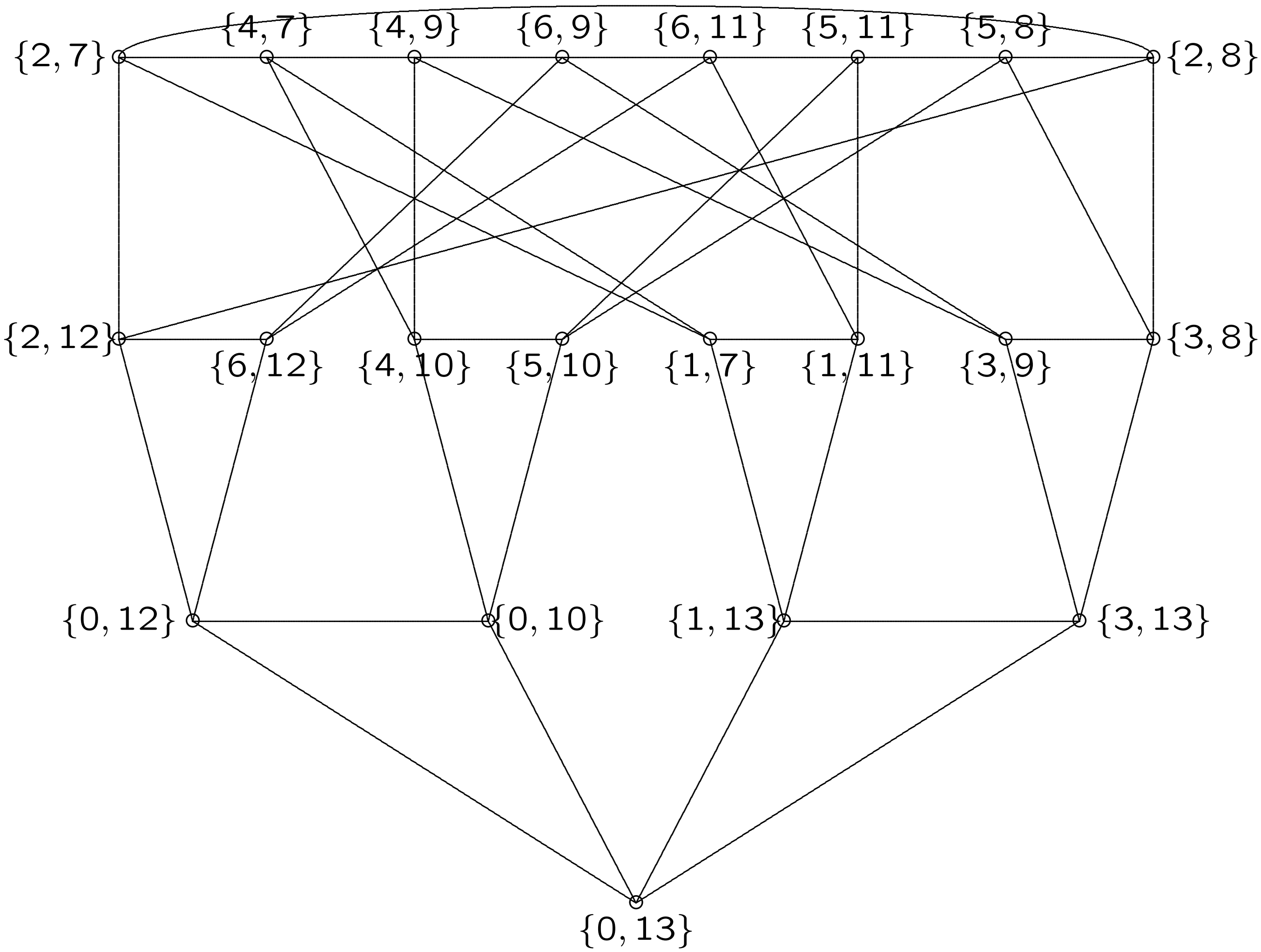
b) Construct its Levi graph: vertices are points and lines, adjacency=incidence.

Get Heawood graph H on 14 vertices:

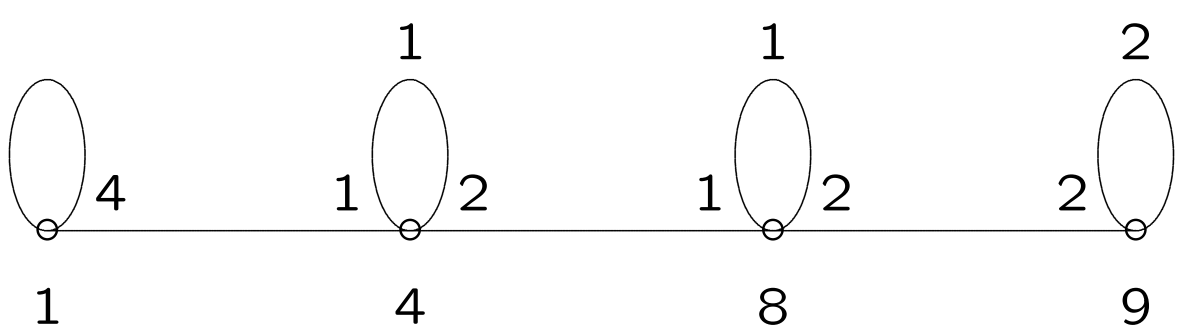


H itself is antipodal DRG ($k = 3$, $d = 3$).

c) Construct the line graph $L(H)$ - vertices are edges of H, adjacency = common ends of edges:



Intersection Diagram:



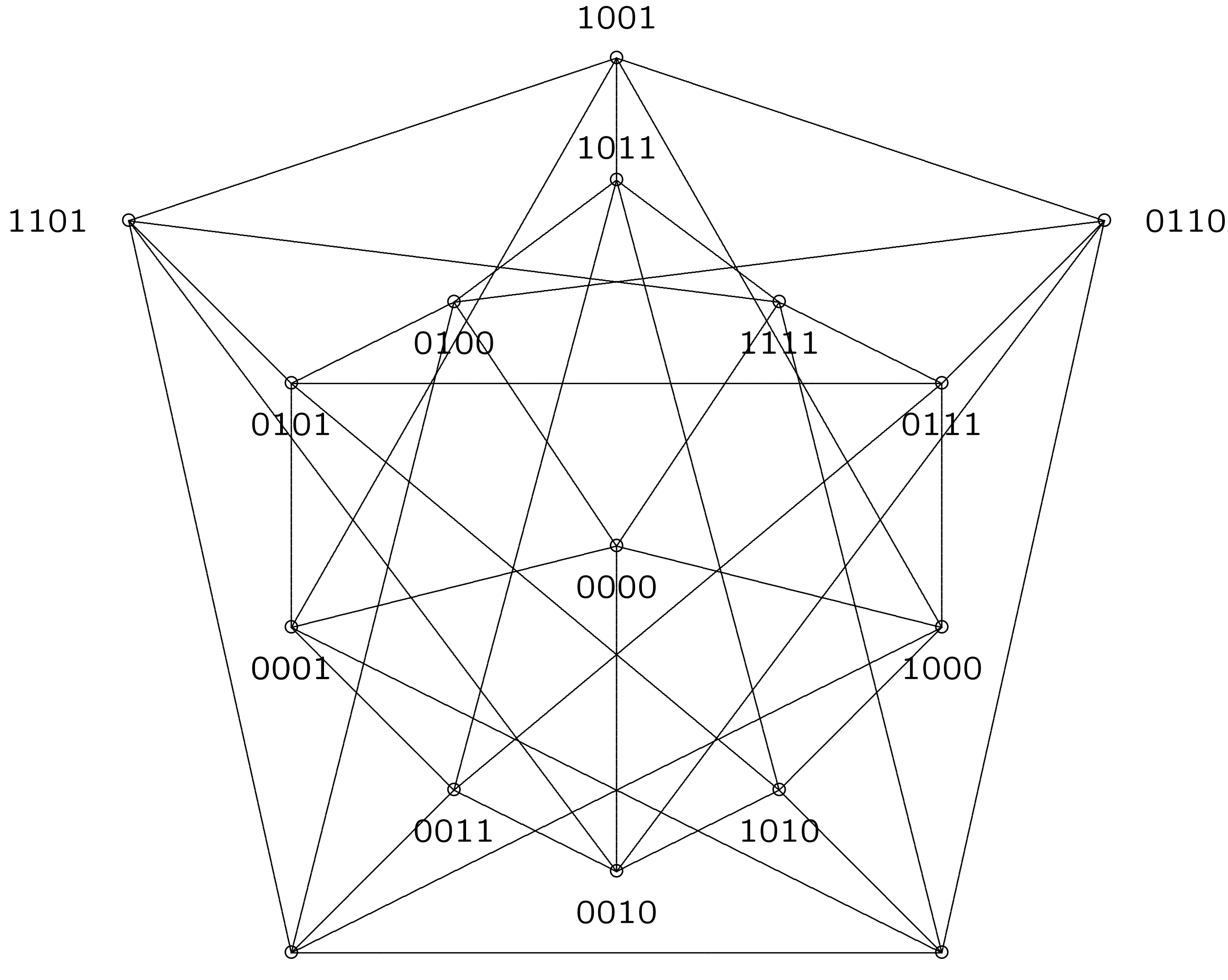
does not depend on a selection of vertex.

- A very famous family of DRG's is *n-dimensional cubes* Q_n with 2^n vertices which define *Hamming* (metric) association scheme $H(n, 2)$ with n classes.

- merging of classes in various metric schemes will be a subject of PART II.

- Now we consider just one example of merging in $H(4, 2)$.

Example 9. *Clebsch graph $\square_5$ is defined as merging of classes Γ_1 and Γ_4 in Q_4:*



Alternative way:

$$\square_5 = CAY(E_{2^4}, \{0001, 0010, 0100, 1000, 1111\})$$

is an $SRG(16, 5, 0, 1)$.

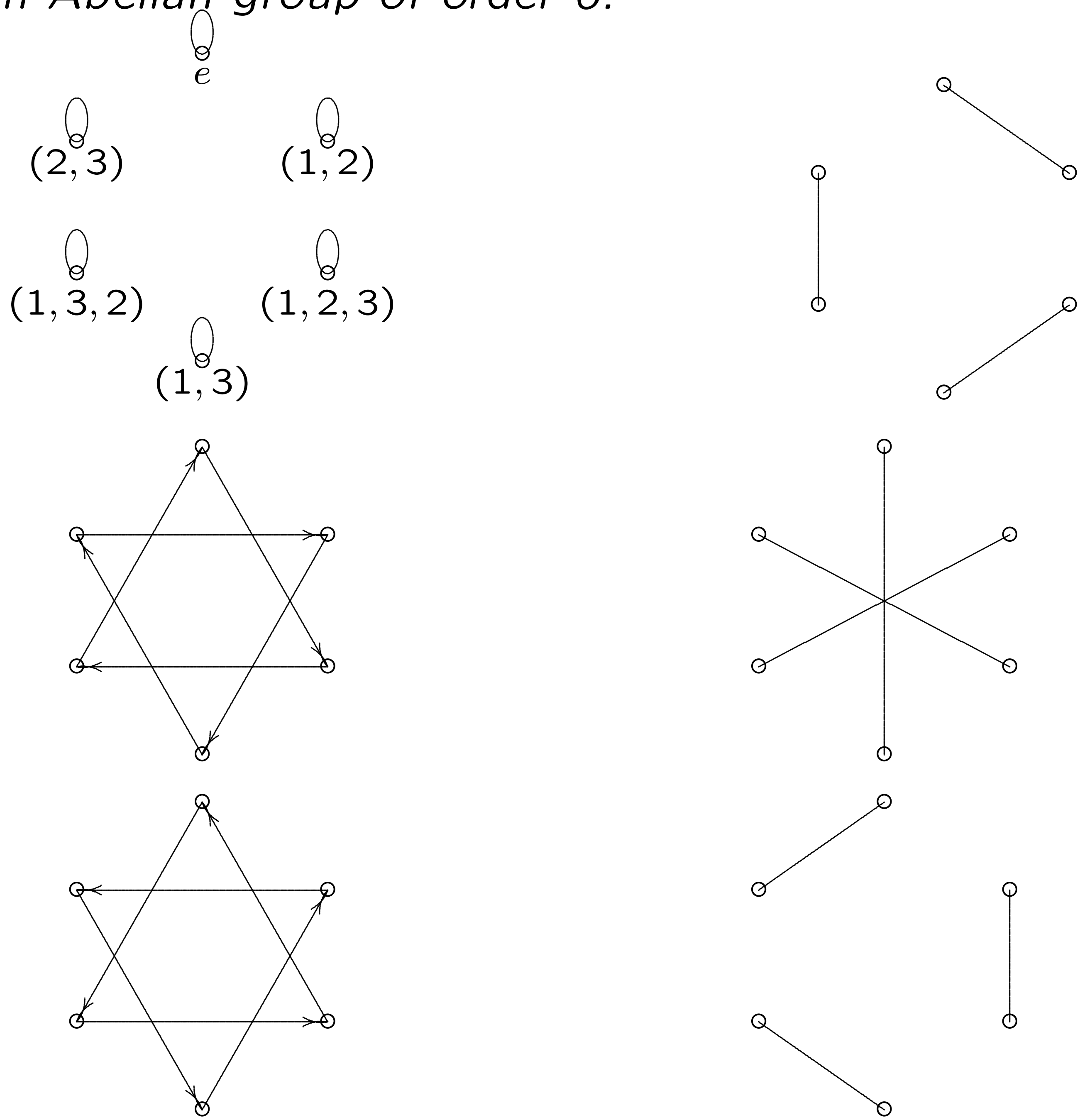
8. Dihedral association schemes

- Metric association schemes have in a sense one natural generator - a corresponding DRG.

- We now introduce a family of association schemes with two generators (*dihedral* schemes according to Zieschang).

- The name goes back to a simple example of a so-called *thin* scheme (all valencies of basis graphs are equal to 1).

Example 10. *Regular action of group $S_3 = D_3$, a non-Abelian group of order 6:*



Cayley graphs with respect to elements of S_3. Any two involutions generate the scheme.

Example 11.

- *Consider action of S_n on ordered pairs of distinct elements from* $[1, n]$, $n \geq 4$.
- *Get permutation group* (S_n, X) *of degree* $n(n-1)$.
- *Describe* $2-Orb(S_n, X)$ *via typical pairs of points:*
  R_0: $((a,b),(a,b))$
  R_1: $((a,b),(a,c))$
  R_2: $((a,b),(c,b))$
  R_3: $((a,b),(b,c))$
  R_4: $((a,b),(c,a))$
  R_5: $((a,b),(c,d))$
  R_6: $((a,b),(b,a))$
  Here R_1 *and* R_2 *generate association scheme* $(X, 2-Orb(S_n, X))$.
- *More examples in part 3.*
  (challenge for GB methods!)

9. Amorphic association schemes

- Association scheme is called *amorphic* if any partition of its classes leads to a fusion scheme.

- A necessary condition: each basic graph is an SRG.

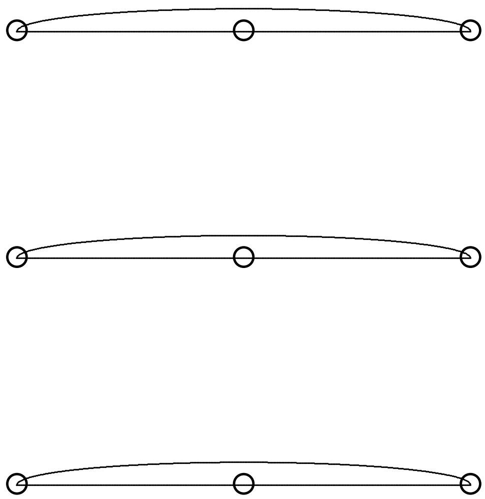
- **Theorem** (D.G. Higman, A.V. Ivanov)

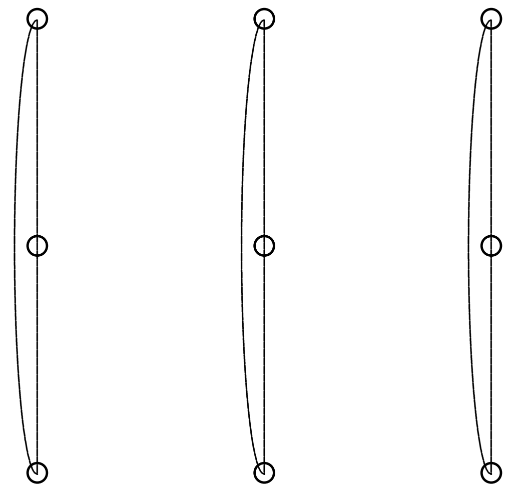
  a) If number of classes is at least 3, then number of points is n^2, $n \in \mathbb{N}$;

  b) Each basis graph is of a positive (negative) Latin square type graph.

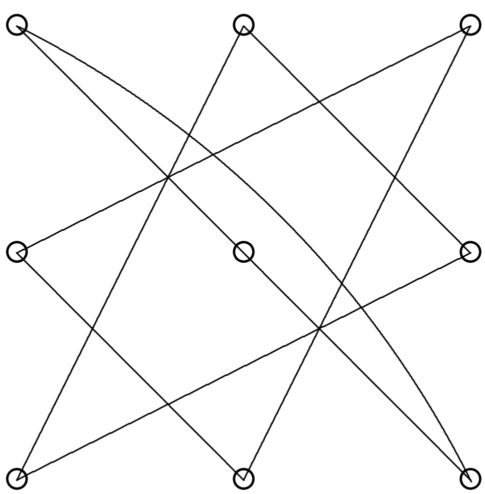
- "Generic" examples: complete affine scheme of order n^2 with $n+1$ classes.

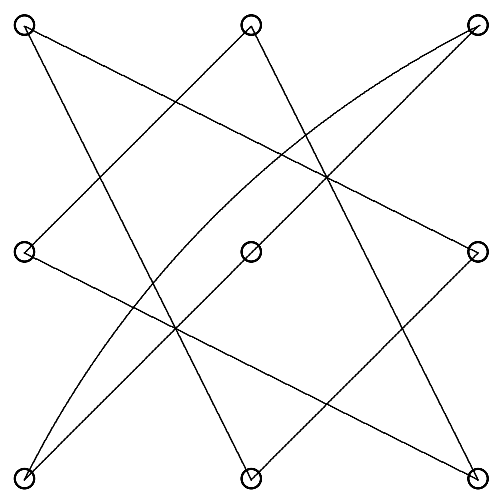
Example 12. $n = 3$





Γ_1 Γ_2





Γ_3 Γ_4

Here scheme is generated by arbitrary 3 basis relations (In general we need n *generators).*

Acknowledgments

- Boris Weisfeiler is indeed alive in our memory.

- Fruitful long-standing collaboration with:
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