Computing characteristic cycles of local cohomology

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Running example

Let $R = k[x_1, x_2, x_3, x_4, x_5, x_6]$ and

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

Let I be the ideal generated by the three 2×2-minors of A:
$f_1 = x_1x_5 - x_2x_4$, $f_2 = x_1x_6 - x_3x_4$, $f_3 = x_2x_6 - x_3x_5$.

Are there g_1, g_2 such that $V(I) = V(g_1, g_2)$?

Arithmetic rank $\geq$ cohomological dimension

The answer to the above question is no if $H^3_I(R) \neq 0$.

- In char $k > 0$ the module $H^3_I(R)$ does vanish!
- If char $k = 0$ then $H^3_I(R) \neq 0$ (Hochster)

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The answer to the above question is no if $H_I^3(R) \neq 0$.

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Let k be a field of characteristic 0.

Definition

The algebra $D = A_n(k) = k\langle x, \partial\rangle = k\langle x_1, \partial_1, \ldots, x_n, \partial_n\rangle$ with relations $[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1$ (and all other pairs commuting) is called *the n-th Weyl algebra*.
(algebra of differential operators with polynomial coefficients)

Convention:

We would use only left ideals in D as well as left D-modules.

Example (one variable)

For $D = A_1 = k\langle x, \partial\rangle$ the module $R = k[x]$ and its localization R_x are left D-modules:

$$\partial \cdot \frac{1}{x^m} = \frac{-m}{x^{m+1}}$$

Moreover, both have cyclic presentations:

$$R = D/D\partial, \quad R_x \cong D/D(x\partial + 2)$$

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Definition (Characteristic ideal)

For an ideal $I \subset A_n$, the ideal $\mathrm{in}_{(0,e)}(I) \subset k[x,\xi]$ is called the *characteristic ideal* of I.
Here $w = (0,e)$ is the weight that assigns $w(x_i) = 0$ and $w(\partial_i) = 1$ for all i.

Theorem (Fundamental theorem of algebraic analysis)

Let I be a nonzero left A_n-ideal, then $n \leq \dim(\mathrm{in}_{(0,e)}(I)) \leq 2n$*,*

Definition (Holonomic)

An ideal $I \subset D = A_n$ is called *holonomic* if its characteristic ideal has dimension n.
The D-module $M = D/I$ is called *holonomic* if I is holonomic.

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Localization

$R_f = k[x, f^{-1}]$ possesses the following natural structure of a D-module:

$$x_i \cdot \frac{g}{f^d} = \frac{x_i g}{f^d}, \quad \partial_i \cdot \frac{g}{f^d} = \frac{\partial g / \partial x_i}{f^d} - \frac{dg(\partial f/\partial x_i)}{f^{d+1}},$$

for all $1 \leq i \leq n,\ f, g \in R,\ d \in \mathbb{Z}_{>0}$.

Theorem

The D-module R_f is holonomic.

Why view $R_f = R[f^{-1}]$ as a D-module?

- R_f can not be finitely generated as an R-module,
- but is generated by f^{-a} for some positive integer a as a D-module.

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Algorithm for localization?

Let $M = D/I$ be a holonomic D-module. Can we compute its localization $R_f \otimes M$, i.e. find $J \subset D$ such that $R_f \otimes M \cong D/J$?

If M is f-saturated (i.e. $f \cdot m = 0 \Leftrightarrow m = 0$ for all $m \in M$)...

... there is an algorithm (Oaku), the main steps of which are:

1. Find $J^I(f^s)$, annihilator of $f^s \otimes \bar{1} \in R_f[s]f^s \otimes M$ in $D[s]$, where
   - $\bar{1}$ is the cyclic generator of $M = D/I$;
   - f^s – the generator of $R_f[s]f^s$.
2. Compute the b-polynomial $b^I_f(s)$ (relative to the the ideal I); Take its smallest integer root a and "plug in" $s = a$ in the generators of $J^I(f^s)$.

Alternative algorithm

Oaku, Takayama, Walther: A localization algorithm for D-modules. J. Symbolic Computation 29 (2000), 721-728.

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Čech complex for computing local cohomology

Let $R = k[x_1, ..., x_n]$ and $I = (f_1, ..., f_d)$. To calculate $H_I^k(R)$ consider the Čech complex:

$$0 \to C^0 \to C^1 \to ... \to C^d \to 0,$$

$$C^k = \bigoplus_{1 \le i_1 < ... < i_k \le d} R_{f_{i_1} \cdots f_{i_k}}$$

and the map $C^k \to C^{k+1}$ is the alternating sum of maps

$$R_{f_{i_1} \cdots f_{i_k}} \to R_{f_{j_1} \cdots f_{j_{k+1}}}.$$

The complex $C^\bullet$ makes it possible to compute the local cohomology algorithmically viewing C^k as holonomic D-modules.

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Example (running)

$I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5) \subset R = \mathbb{Q}[x_1, ..., x_6]$
Does $H^3_I(R)$ vanish if $\operatorname{char} k = 0$?

Walther: computation of LC via D-modules

This was the first computational approach.

Joint with Tsai: software

D-modules for Macaulay 2.

Motivation for the rest of slides

Is there a way to answer the above question computationally without using Gröbner bases in noncommutative setting?

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Let $X = \mathbb{C}^n$ be the complex affine space with the coordinate ring $R = \mathbb{C}[x_1, \ldots, x_n]$. By $\mathcal{D}$ denote either $A_n := \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ or $D_n := \mathbb{C}\{x_1, \ldots, x_n\}\langle\partial_1, \ldots, \partial_n\rangle$.

Support of a $\mathcal{D}$-module

Let $C(M)$ be the characteristic variety and let
$\pi : \mathrm{Spec}(R[a_1, \ldots, a_n]) \longrightarrow \mathrm{Spec}(R)$, $\pi(x, a) = x$.
Then $\mathrm{Supp}_R(M) = \pi(C(M))$.

Definition (Characteristic cycle of M)

$$CC(M) = \sum m_i \Lambda_i$$

The sum is taken over all irreducible components Λ_i of $C(M)$ and m_i is the multiplicity of the module M along Λ_i.

A very useful property

CC is additive.

Let $X = \mathbb{C}^n$ be the complex affine space with the coordinate ring $R = \mathbb{C}[x_1, \ldots, x_n]$. By $\mathcal{D}$ denote either $A_n := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ or $D_n := \mathbb{C}\{x_1, \ldots, x_n\}\langle \partial_1, \ldots, \partial_n \rangle$.

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Let $X = \mathbb{C}^n$ be the complex affine space with the coordinate ring $R = \mathbb{C}[x_1, \ldots, x_n]$. By $\mathcal{D}$ denote either $A_n := \mathbb{C}[x_1, \ldots, x_n]\langle\partial_1, \ldots, \partial_n\rangle$ or $D_n := \mathbb{C}\{x_1, \ldots, x_n\}\langle\partial_1, \ldots, \partial_n\rangle$.

Support of a $\mathcal{D}$-module

Let $C(M)$ be the characteristic variety and let
$\pi : \mathrm{Spec}(R[a_1, \ldots, a_n]) \longrightarrow \mathrm{Spec}(R)$, $\pi(x, a) = x$.
Then $\mathrm{Supp}_R(M) = \pi(C(M))$.

Definition (Characteristic cycle of M)

$$CC(M) = \sum m_i \Lambda_i$$

The sum is taken over all irreducible components Λ_i of $C(M)$ and m_i is the multiplicity of the module M along Λ_i.

A very useful property

CC is additive.

Analytic vs. algebraic

Given an A_n-module M we consider $M^{an} := \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} M$

- M regular holonomic A_n-module $\Rightarrow$ M^{an} regular holonomic D_n-module.
- $\{M_i\}_{i \geq 0}$ good filtration on M $\Rightarrow$ $\{M_i^{an}\}_{i \geq 0}$ good filtration on M^{an}
- $gr(M^{an}) \simeq \mathbb{C}\{x\} \otimes_{\mathbb{C}[x]} gr(M)$

The analytic characteristic variety $C(M^{an})$ is the analytic extension of the algebraic characteristic variety $C(M)$, i.e. $C(M^{an}) = C(M)^{an}$.

Caveat: $CC(M) \neq CC(M^{an})$

Algebraically irreducible components can be analytically reducible.

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Conormal bundles

Let X_i° be the smooth part of $X_i \subseteq X$. Set:

$$Z = \{(x,a) \in T^*X \mid x \in X_i^\circ \text{ and } a \text{ kills } T_x X_i^\circ\}.$$

The conormal bundle $T^*_{X_i}X$ is the closure of Z in $T^*X|_{X_i}$.

For M with $CC(M) = \sum_{i\in\Im} m_i \Lambda_i$

... there exists a Whitney stratification $\{X_i\}_{i\in\Im}$ of X such that

$$CC(M) = \sum_{i\in\Im} m_i \; T^*_{X_i}X.$$

In particular, $\mathrm{Supp}_R(M) = \bigcup X_i$.

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Direct computation of CC of a localization

To compute $CC(M_f)$ for a holonomic M and a polynomial f directly, one needs to:

1. construct a representation of M_f;
2. find the characteristic ideal $J(M_f)$;
3. compute primary decomposition of $J(M_f)$.

Example ($R = \mathbb{C}\{x, y, z\}$, $f = x$)

$$CC(R_x) = T^*_X X + T^*_{\{x=0\}} X$$

Example ($M = H^1_{(x)}(R)$, $g = y$)

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Indirect computation (joint with Josep Àlvarez)

Definition ($T^*_{f|_Y}$ = conormal bundle relative to f)

Let Y° be the smooth part of $Y \subseteq X$ where $f|_Y$ is a submersion.

$$W = \{(x,a) \in T^*X \mid x \in Y^\circ \text{ and } a \text{ annihilates } T_x(f|_Y)^{-1}(f(x))\}.$$

$T^*_{f|_Y}$ is the closure of W in $T^*X|_Y$.

Theorem (Ginsburg, Briançon-Maisonobe-Merle (BMM))

Let M be a regular holonomic D_n-module with $CC(M) = \sum_i m_i \; T^*_{X_i} X$ and let $f \in R$ be a polynomial. Then

$$CC(M_f) = \sum_{f(X_i) \neq 0} m_i(\Gamma_i + T^*_{X_i} X)$$

with $\Gamma_i = \sum_j m_{ij} \Gamma_{ij}$, where Γ_{ij} are the irreducible components of multiplicity m_{ij} of the divisor defined by f in $T^*_{f|_{X_i}}$.

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Therefore, $CC(R_x) = T^*_X X + T^*_{\{x=0\}}X$

Example ($M = H^1_{(x)}(R)$, $g = y$, (from above) $CC(M) = T^*_{\{x=0\}}X$)

$T^*_{g|_{\{x=0\}}} = \{(x,y,z,a,b,c) \in T^*X \mid c = 0, x = 0\}$, then the divisor defined by g in $T^*_{g|_{\{x=0\}}}$ is

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How is this better?

We will consider $R = \mathbb{C}[x_1, \dots, x_n]$. Given a polynomial $f \in \mathbb{Q}[x_1, \dots, x_n]$, we would like to compute $CC(R_f)$. The [BMM] formula reduces to

$$CC(R_f) = T^*_X X + \sum m_i \ T^*_{X_i} X,$$

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Advantages of the indirect approach

- Do not have to compute the D-module presentations of localizations; in particular, no Bernstein-Sato polynomials.
- All computations take place in a commutative ring.

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Primary decomposition over $\mathbb{Q}$ is used in the implementation.

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Outline of the algorithm

Compute the smooth part Y° of Y where $f|_Y$ is a submersion

(0a) Compute $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$

(0b) Compute the ideal $I^\circ \subset R$ such that $Y^\circ = \{x \in Y \mid \nabla f(x) \notin T_x Y\}$ is described as $Y^\circ = Y \setminus V(I^\circ)$.

Compute the conormal relative to f

(1a) Compute $K = \ker \varphi$, where the $\varphi : R^n \longrightarrow R^{d+1}/I$ sends

$$s \mapsto (\nabla f, \nabla g_1, \ldots, \nabla g_d) \cdot s \in R^{d+1}/I.$$

(1b) Let $J \subset \operatorname{gr} A_n = R[a_1, \ldots, a_n]$ be the ideal generated by $\{(a_1, \ldots, a_n) \cdot b \mid b \in K\}$.

(1c) Compute $J_{sat} = J : ((\operatorname{gr} A_n) I^\circ)^\infty$; then $I(T^*_{f|_Y}) = \sqrt{J_{sat}}$.

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Compute the divisor defined by f in $T^*_{f|_Y}$

(2a) Compute $K_f = \ker \varphi_f$, where $\varphi_f : R^n \longrightarrow R^{d+1}/(I+(f))$:

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(2c) $C = J_{sat} + (f) + J_f \subset \operatorname{gr} A_n$.

For every $Y = X_i$ in $CC(M) = \sum m_i \; T^*_{X_i} X$ compute C_i such that $T^*_{f|_Y} = V(C_i)$.

Compute the components of C_i

(3a) Compute the associated primes C_{ij} of C_i.

(3b) Get $I_{ij} = C_{ij} \cap R$ (to know the defining ideal of $X_{ij} = \pi(\Gamma_{ij})$).

(3c) Calculate the multiplicity m_{ij} as the multiplicity of a generic point along each component C_{ij} of C_i via Hilbert functions.

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Example (running)

Let $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$, $I = \langle f_1, f_2, f_3 \rangle$, where
$f_1 = x_2x_6 - x_3x_5,\ f_2 = x_1x_6 - x_3x_4, f_3 = x_1x_5 - x_2x_4$.

Looking for $H_I^\bullet(R)$ we use Čech complex $C^\bullet(f_1, f_2, f_3; R)$:

$$\begin{array}{ccccccccc} & C_0 & & C_1 & & C_2 & & C_3 & \\ & \| & & \| & & \| & & \| & \\ 0 \to & R & \to & \begin{bmatrix} R_{f_1} \\ \oplus \\ R_{f_2} \\ \oplus \\ R_{f_3} \end{bmatrix} & \to & \begin{bmatrix} R_{f_1 f_2} \\ \oplus \\ R_{f_1 f_3} \\ \oplus \\ R_{f_2 f_3} \end{bmatrix} & \to & R_{f_1 f_2 f_3} & \to 0 \end{array}$$

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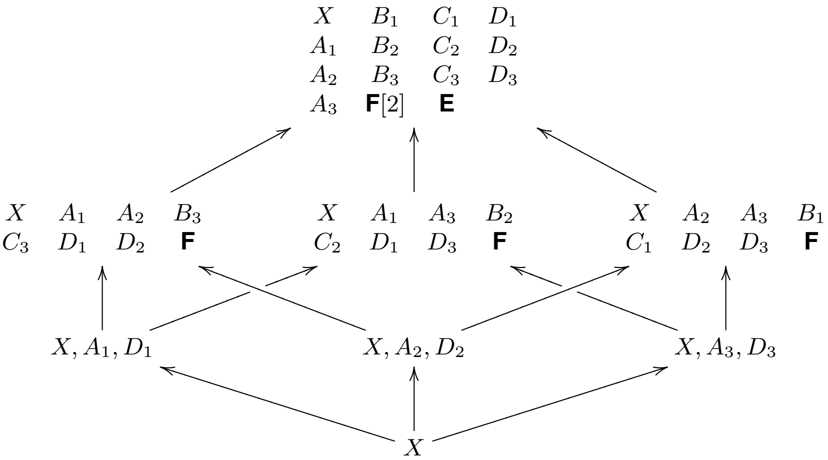
$$



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Utilize additivity

If a module N is $f_1 \dots f_n$-saturated

- observe that $C^\bullet(f_i; R)$,

$$0 \longrightarrow N \longrightarrow N_{f_i} \longrightarrow 0,$$

  means that $CC(H^1_{(f_i)}) = CC(N_{f_i}) - CC(N)$.

- After computing the CCs of chains of

$$C^\bullet(f_1, \dots, f_m; R) = C^\bullet(f_1; R) \otimes_R \cdots \otimes_R C^\bullet(f_m; R)$$

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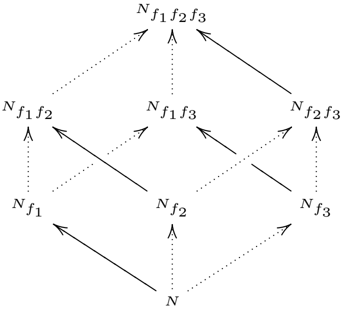
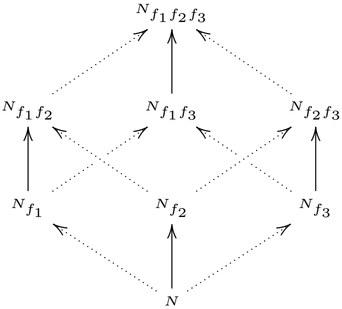
Prune pairs connected via solid edges

“Prune” = cancel out the components shared by the corresponding CCs (taking multiplicity into account).





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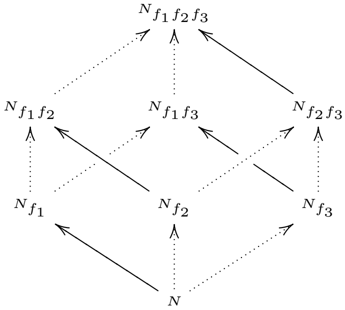

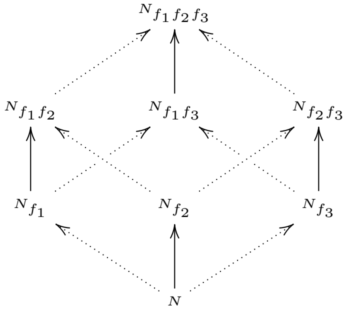

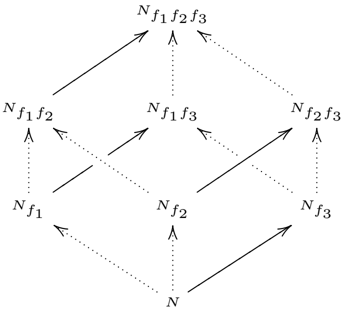

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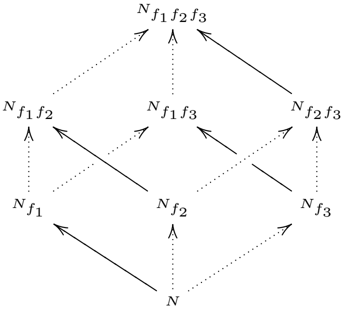

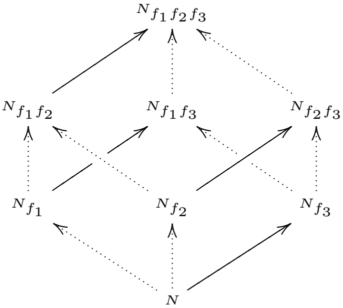

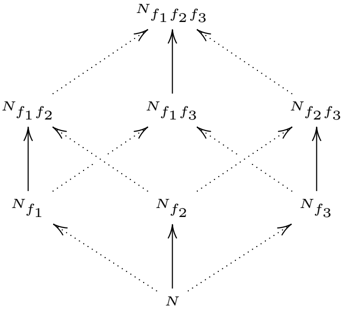

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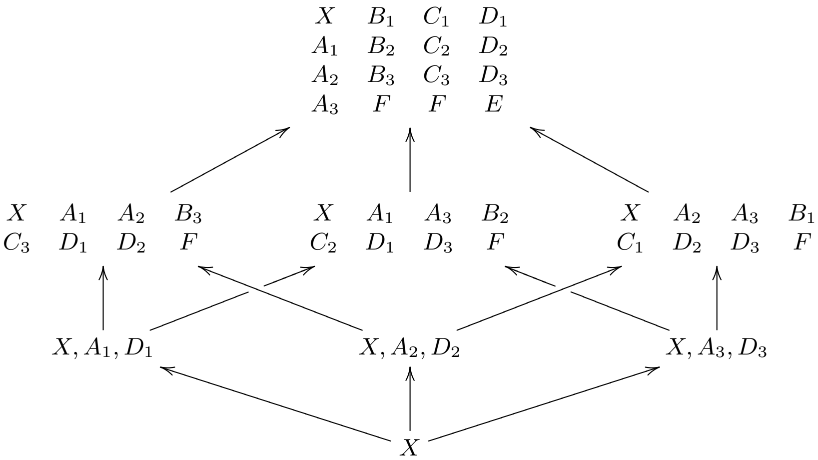

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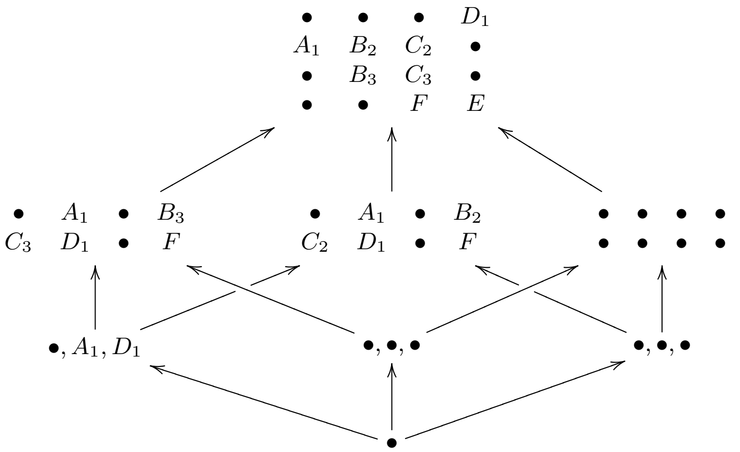

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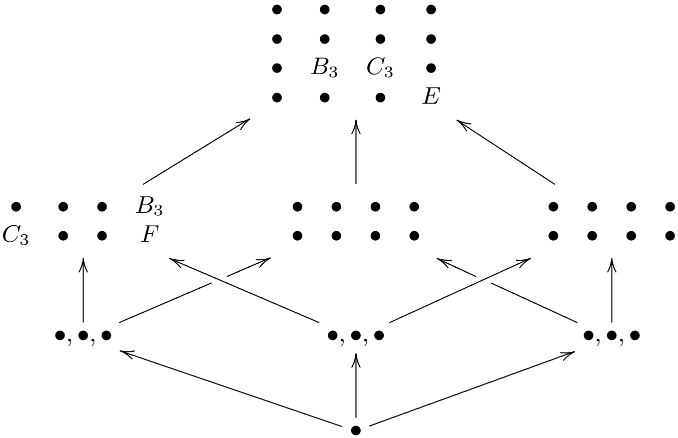

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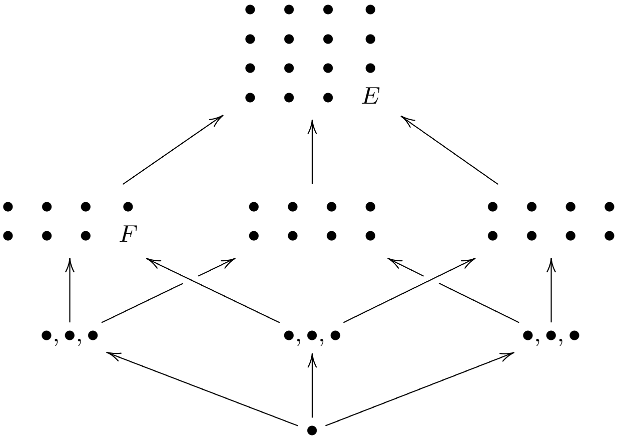

CC of cohomology modules

$CC(H_I^2(R)) = F,\ CC(H_I^3(R)) = E$

Lyubeznik numbers

Let $R = k[x_1, ..., x_n]$ be the polynomial ring over a field k of characteristic zero. Let $I \subseteq R$ be an ideal and .

Definition (Lyubeznik (1993))

Let $\mathfrak{m} = (x_1, ..., x_n) \subset R = k[x_1, ..., x_n]$,

$$\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)) := \dim_k \operatorname{Ext}_R^p(k, H_I^{n-i}(R)).$$

Let $E = V(\mathfrak{m})$, then

$$CC(H_{\mathfrak{m}}^p(H_I^{n-i}(R))) = \lambda_{p,i} \, T_E^* X$$

Example (running)

$I = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5) \subset R = \mathbb{Q}[x_1, ..., x_6]$.
What is the characteristic cycle of the local cohomology modules $H_{\mathfrak{m}}^p(H_I^i(R))$ for $i = 2, 3$ and $\forall p$?

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Čech complex for M either $H^2_I(R)$ or $H^3_I(R)$

$$0 \to M \to \bigoplus_{i=1}^{6} M_{x_i} \to \cdots \to M_{x_1 \cdots x_6} \to 0,$$

$\lambda_{0,3}(R/I) = 1$

For $M = H^3_I(R)$ the CC is $T^*_E X$, so applying [BMM] the Čech complex reduces to the first term. The nonvanishing entry is

$$CC(H^0_{\mathfrak{m}}(H^{6-3}_I(R))) = T^*_E X.$$

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Component E in CC(Čech) for $M = H^2_I(R)$

$$
\begin{array}{ccccccccccccc}
\emptyset & \to & \emptyset & \to & E[12] & \to & E[34] & \to & E[39] & \to & E[18] & \to & E[3] & \to & \emptyset \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & + & & & & & & & & & & \\
 & & & & 11 & & 11 & & & & & & & & \\
 & & & & & & + & & & & & & & & \\
 & & & & & & 23 & & 23 & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & 15 & & 15 & & & & \\
 & & & & & & & & & & + & & & & \\
 & & & & & & & & & & 3 & & 3 & &
\end{array}
$$

$\lambda_{2,4}(R/I) = \lambda_{4,4}(R/I) = 1$

$$CC(H^2_{\mathfrak{m}}(H^{(6-4)}_I(R))) = T^*_E X, \qquad CC(H^4_{\mathfrak{m}}(H^{(6-4)}_I(R))) = T^*_E X$$

Component E in CC(Čech) for $M = H_I^2(R)$

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\begin{array}{ccccccccccccc}
\emptyset & \to & \emptyset & \to & E[12] & \to & E[34] & \to & E[39] & \to & E[18] & \to & E[3] & \to & \emptyset \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & 1 & & & & & & & & & & \\
 & & & & + & & & & & & & & & & \\
 & & & & 11 & & 11 & & & & & & & & \\
 & & & & & & + & & & & & & & & \\
 & & & & & & 23 & & 23 & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & 1 & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & 15 & & 15 & & & & \\
 & & & & & & & & & & + & & & & \\
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$$\begin{array}{ccccccccccccc}
\emptyset & \to & \emptyset & \to & E[12] & \to & E[34] & \to & E[39] & \to & E[18] & \to & E[3] & \to & \emptyset \\
 & & & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
 & & & & \color{red}{1} & & & & & & & & & & \\
 & & & & + & & & & & & & & & & \\
 & & & & 11 & & 11 & & & & & & & & \\
 & & & & & & + & & & & & & & & \\
 & & & & & & 23 & & 23 & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & \color{red}{1} & & & & & & \\
 & & & & & & & & + & & & & & & \\
 & & & & & & & & 15 & & 15 & & & & \\
 & & & & & & & & & & + & & & & \\
 & & & & & & & & & & 3 & & 3 & &
\end{array}$$

$\lambda_{2,4}(R/I) = \lambda_{4,4}(R/I) = 1$

$$CC(H^2_{\mathfrak{m}}(H^{(6-4)}_I(R))) = T^*_E X, \qquad CC(H^4_{\mathfrak{m}}(H^{(6-4)}_I(R))) = T^*_E X$$

Conclusion

Noncommutative GB $\longrightarrow$ commutative GB + primary decomposition

- D-presentation of localization and local cohomology modules: done via GB in the Weyl algebra;
- Support of a D-module: think "characteristic cycle";
- Compute CCs: need GB in a (commutative) polynomial ring and primary decomposition.

Numerical algebraic geometry

Subvarieties of $\mathbb{C}^n$ can me described numerically by approximations of the points in so-called *witness sets*;

To run the algorithm for CC of localization numerically we need

1. numerical representation of the cotangent bundle;
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