Genericity of Parameters in Control Theory

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Special Semester on Gröbner Bases and Related Methods
Workshop D3 "Gröbner Bases in Control Theory and Signal Processing"

18.05.2006, Linz

Introduction

Many systems of equations, arising in the Control Theory, contain parameters.
In many cases the parameters are of algebraic nature, that is they do not depend on the variables and in some sense are autonomous.

Parameters Algebraically

Let $\mathbb{K}$ be the ground field. Under "parameter" we mean a symbolic quantity, which can take any values in $\mathbb{K}$ except 0.
Since *a priori* we do not know, whether the parameter may be algebraically dependent, we use them as $\mathbb{K}(p_1, \dots, p_n)$, the transcendental field extension.

We can find "bad" parameter constellation for a given module using symbolic algebraic algorithms!

Main Computational Tool: Gröbner bases

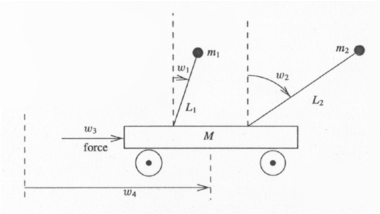
... the fundament for symbolic manipulations with modules.

Gröbner bases provide us with algorithms for

- a basis, unique up to a leading coefficient
- test for a membership of an element in the module
- test for equality of two modules (e.g. comparison with 0)
- dimension and a vector space dimension of a module
- elimination of variables
- elimination of module components
- intersection of submodules
- kernel of a module homomorphism
- kernel of a ring homomorphism
- algebraic relations between commuting polynomials

Example: Two pendula, mounted on a cart

[Polderman, Willems "Introduction to Mathematical Systems Theory"]



Denote $\partial := \frac{d}{dt}$. A linearized system becomes $\mathbf{S} \cdot \overline{\mathbf{w}} = \mathbf{0}$, where S is

$$\begin{array}{cccc} m_1 L_1 \partial^2 & m_2 L_2 \partial^2 & (m_1 + m_2 + M)\partial^2 & -1 \\ m_1 L_1^2 \partial^2 + d_1 \partial + k_1 - m_1 g L_1 & 0 & m_1 L_1 \partial^2 & 0 \\ 0 & m_2 L_2^2 \partial^2 + d_2 \partial + k_2 - m_2 g L_2 & m_2 L_2 \partial^2 & 0 \end{array}$$

Two pendula: Simplification and Computation

The associated algebra $A = \mathbb{K}(m_1, m_2, M, g, L_1, L_2)[\partial]$ is commutative. Let $T = S^t \subset A^3$ and $U = A^3/T$.

Simplification

We introduce new variables for $i = 1, 2$,

$$d_i' := \frac{d_i}{m_i L_i}, \text{ and } z_i := \frac{k_i}{m_i L_i} - g.$$

k_i and d_i can have only positive values (use d_i instead of d_i').

Moreover, we perform reductions with the last column of the basis. After these, the presentation matrix looks much easier:

$$T' = \begin{pmatrix} d_1 Dt + z_1 & 0 & Dt^2 \\ 0 & d_2 Dt + z_2 & Dt^2 \end{pmatrix}$$

Two pendula: computations

1. Generic Gröbner basis

G, the reduced Gröbner basis of T', is a 2×2 identity matrix, hence the module $U = A^3/T \cong A^2/G = 0$.
Control Theory: generic (strong) controllability and flatness of S follow.

2. Obstructions to Genericity

We compute the transformation matrix X between two bases T and G, i.e. $T \cdot X = G$ (with the algorithm `Lift`) and analyze X for suspicious denominators (with the procedure `genericity`). We get the list of polynomials in parameters.

Obstructions to Genericity: General methodology

Suppose we have a set of obstruction polynomials $\{f_1, \ldots, f_m\} \subset \mathbb{K}[p_1, \ldots, p_n]$. We have to solve $2^m - 1$ systems of equations and inequalities (constructible sets) of the form

$$\{f_{i_1} = 0, \ldots, f_{i_k} = 0; f_{i_{k+1}} \neq 0, \ldots, f_{i_m} \neq 0\}.$$

Solution of each such system $S_j, 1 \leq j \leq 2^m - 1$ corresponds to the algebraic variety $V_j \in \mathbb{A}^n_{\mathbb{K}}$, described by an ideal $I_j \in \mathbb{K}[p_1, \ldots, p_n]$. However, there is a need for check, whether the obtained values of parameters in each case really lead us to the torsion module. According to the result of such a check, we refine the list of components V_j. We can compute the exceptional variety $V = \cap V_j$, given by the ideal $I = \cup I(V_j)$. Outside of this variety, parameters of the system do not disturb the controllability.

We can avoid $2^m - 1$ computations

by e. g. ordering $\{f_i\}$ degree–wise and start building hierarchies from the lowest degrees, simplifying and cancelling some polynomials.

Two pendula: Obstructions to Genericity

Systems like before,

$$\{f_{i_1} = 0, \dots, f_{i_k} = 0; f_{i_{k+1}} \neq 0, \dots, f_{i_m} \neq 0\}.$$

can be solved with the factorizing Gröbner basis algorithm. In SINGULAR, one can use the function `facstd`$(F_0, F_{\neq 0})$.

Two Pendula Example

We obtain the following three obstruction polynomials:

1. $z_1 = 0$ (in original variables, $k_1 = m_1 L_1 g$),
2. $z_2 = 0$ (in original variables, $k_2 = m_2 L_2 g$),
3. $0 = P :=$
$z_2^2 L_1^2 - 2 z_1 z_2 L_1 L_2 + z_1^2 L_2^2 + z_2 L_2 d_1^2 - z_2 L_1 d_1 d_2 - z_1 L_2 d_1 d_2 + z_1 L_1 d_2^2$.

Ordering and check shows, that $P \in \langle z_1, z_2 \rangle$.

Two Pendula: Case Studies I

It is important to analyze P. Brute force use of solving with e. g. MAPLE does not give any enlightening information.
$P = P(z_1, z_2) = (L_2 z_1 - L_1 z_2)^2 + (L_2 d_1 - L_1 d_2) \cdot (d_1 z_2 - d_2 z_1)$.

1. The case $z_1 = z_2 = 0$

With this case we cover actually two cases: since $P \in \langle z_1, z_2 \rangle$, the case $z_1 = z_2 = 0$, $P \neq 0$ is impossible.
If $P = 0$, the system is not controllable.
The torsion submodule is annihilated by the ideal $\langle \partial \rangle$ and the kernel representation of the controllable part is given by

$$\begin{pmatrix} m_2 L_2^2 \partial + d_2 & 0 & m_1 L_1 \partial \\ 0 & m_1 L_1^2 \partial + d_1 & m_2 L_2 \partial \end{pmatrix}.$$

Two Pendula: Case Studies II

Preparations for computations with SINGULAR:

```
ring T = 0,(z1,z2,L1,L2,d1,d2),dp;
poly P = (z2*L2)*d1^2+(-z1*L2-z2*L1)*d1*d2+(z1*L1)*d2^2+
         (z1^2*L2^2-2*z1*z2*L1*L2+z2^2*L1^2);
```

2. The case $\{z_1 = 0, z_2 \neq 0, P(0, z_2) = 0\}$

```
ideal I2 = P,z1;
facstd(I2,z2);
==>  _[1]=z1
==>  _[2]=z2*L1^2+L2*d1^2-L1*d1*d2
```

From the 2nd equation we conclude, that $z_2 L_1^2 = d_1(L_1 d_2 - L_2 d_1) \neq 0$, what is true if $d_1 \neq 0$ and $L_1 d_2 \neq L_2 d_1$. We simplify our system to

$$\{z_1 = 0, d_2 = z_2 L_1 \frac{1}{d_1} + \frac{L_2}{L_1} d_1\} \cup \{z_2 \neq 0, d_1 \neq 0, L_1 d_2 \neq L_2 d_1\}.$$

Two Pendula: Case Studies II cont'd

2. The case $\{z_1 = 0, z_2 \neq 0, P(0, z_2) = 0\}$

is equivalent to

$$\{z_1 = 0, d_2 = z_2 L_1 \frac{1}{d_1} + \frac{L_2}{L_1} d_1\} \cup \{z_2 \neq 0, d_1 \neq 0, L_1 d_2 \neq L_2 d_1\}.$$

The torsion submodule is annihilated by the ideal $\langle \partial \rangle$ and the kernel representation of the controllable part is given by

$$\begin{pmatrix} 0 & X & m_1 m_2 L_1^2 L_2 d_1 \partial^2 \\ m_1 L_1^2 \partial + d_1 & 0 & m_1 L_1 \partial \end{pmatrix}.$$

where $X = m_1 m_2 L_1^2 L_2^2 d_1 \partial^2 - (m_1^2 m_2^2 g L_1^4 L_2^2 - m_1^2 m_2 L_1^4 L_2 k_2 - m_2 L_2^2 d_1^2)\partial - (m_1 m_2 g L_1^2 L_2 d_1 - m_1 L_1^2 k_2 d_1)$.

Two Pendula: Case Studies III

3. The case $\{z_1 \neq 0, z_2 = 0, P(z_1, 0) = 0\}$

is analogous to the case (2) above. The system is not controllable, the torsion submodule is annihilated by the ideal $\langle \partial \rangle$.

4. The case $\{z_1 \neq 0, z_2 \neq 0, P(z_1, z_2) = 0\}$

Since z_i, L_i are nonzero, we get the quadratic equation in d_1, d_2. We rewrite it naturally as $P = Q_1^2 + Q_2 Q_3 = 0$.

Observation: $\forall$ i, $Q_i = 0$ implies $Q_j = Q_k = 0$. With $d_1 \neq 0$, we get

$$\frac{L_2}{L_1} = \frac{z_2}{z_1} = \frac{d_2}{d_1}.$$

Going back to the original variables,

$$\frac{m_2 L_2^2}{m_1 L_1^2} = \frac{k_2 - m_2 L_2 g}{k_1 - m_1 L_1 g} = \frac{d_2}{d_1}.$$

Two Pendula: Case Studies IV

Subcase $L_2 z_1 = L_1 z_2$ ($Q_i = 0$) cont'd

Denote the fraction above by t. Then,

$$d_2 = t \cdot d_1, \; m_2 = t \cdot \frac{m_1 L_1^2}{L_2^2}, \; k_2 = t \cdot (k_1 - m_1 g \frac{L_1}{L_2}(L_2 - L_1)).$$

The system is not controllable, with the torsion submodule, annihilated by $\langle (m_1 L_1^2) \cdot \partial^2 + d_1 \cdot \partial + k_1 - m_1 L_1 g \rangle$.

Two Pendula: Case Studies V

Subcase $L_2 z_1 \neq L_1 z_2$ ($Q_i \neq 0$)

In variables d_1, d_2 the equation is quadratic.
$(z_2 L_2) d_1^2 - (z_1 L_2 + z_2 L_1) d_2) \cdot d_1 + ((z_1 L_1) d_2^2 + (z_1^2 L_2^2 - 2 z_1 z_2 L_1 L_2 + z_2^2 L_1^2) = 0.$
Its discriminant is $D = (z_1 L_2 - z_2 L_1)^2 \cdot (d_2^2 - 4 z_2 L_2)$.
$D = 0$, if $z_1 L_2 = z_2 L_1$ (already treated above) or
if $d_2 = 2\sqrt{z_2 L_2}$. In this case,

$$d_1 = \frac{z_1 L_2 + z_2 L_1}{2 z_2 L_2} \cdot d_2 = \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{L_1}{L_2}\right) \cdot 2\sqrt{z_2 L_2}.$$

Solving the quadratic equation for $D > 0$, we obtain general solutions

$$d_1 = \frac{1}{2}\left(\frac{z_1}{z_2} + \frac{L_1}{L_2}\right) \cdot d_2 \pm \frac{1}{2}\left|\frac{z_1}{z_2} - \frac{L_1}{L_2}\right| \sqrt{d_2^2 - 4 z_2 L_2}.$$

Implementation: control.lib

The library is designed to be scalable and modular.

Controllability and autonomy analysis: procedures `control(matrix M)` resp. `autonom(matrix M)`

Do not require algebraic knowledge, user–friendly: one has to input only an algebra A, a matrix S over it and just run one of the above procedures.

"Minimal knowlegde – maximal information" principle:

- the special heuristics is developed and implemented (it was never done before) for providing exactly the information which is needed,
- context dependency: based on the already computed information, performs additional case–specific computations, e.g. left inverse, annihilator of the torsion part and so on,
- provides textual comments to the algebraic output, hence the library can be used in courses on Control Theory and Applications of Computer Algebra.

Results

We compare the correctness and the performance with the MAPLE–package OREMODULES (F. Chyzak, A. Quadrat, D. Robertz).

Times below are in seconds (1,7 MHz proc. and 3GB).

Benchmarks (M. Becker, O. Motsak)

Example	OreModules Maple	control.lib Singular:Plural
Antenna	1.21	0.04
Bipendulum	0.30	0.08
ElecLine	5.06	0.09
FlexibleArm	1.38	0.10
FlexibleRod	0.44	0.06
Network	0.26	0.05
Pedalvit	0.19	0.05
RLC-circuit	0.32	0.07
Satellite	1.93	0.11
String	2.37	0.07

Three Main Cases

System with constant coefficients

An algebra A is a pure operator algebra. If operators commute pairwise, it is a commutative algebra, for example $\mathbb{K}[\partial_1, \ldots, d_n]$.
Used: SINGULAR and `control.lib`.

System with polynomial coefficients

An algebra A is a *GR*–algebra, for example $\mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1\rangle$.
Used: PLURAL and `ncontrol.lib` (under development).

System with rational coefficients

An algebra A is an Ore algebra, for example $\mathbb{K}(x)\langle \partial \mid \partial x = x\partial + 1\rangle$.
Used: future extension of PLURAL to rational algebras and `ratcontrol.lib` (development in progress).