Applications of Gröbner Bases in Non-commutative *GR*–algebras

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Implementation in PLURAL

What is PLURAL?

- PLURAL is the kernel extension of SINGULAR
- PLURAL is distributed with SINGULAR (from version 3-0-0 on)
- freely distributable under GNU Public License
- available for most hardware and software platforms

PLURAL as a Gröbner engine

- implementation of all the Gröbner basics available
- `slimgb` is available for Plural (and it is fast!)
- `janet` is available for two–sided input
- non–commutative Gröbner basics:
  - as kernel functions (`twostd, opposite` etc)
  - as libraries (NCDECOMP.LIB, NCTOOLS.LIB, NCPREIMAGE.LIB etc)

Algebras in PLURAL: Preliminaries

Let $\mathbb{K}$ be a field and R be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$.

$$\text{Mon}(R) \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

Definition

1. a total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if
   - $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of F, in particular $\forall\ a \in \mathbb{N}^n,\ 0 \prec a$
2. an ordering $\prec$ is called a **monomial ordering on** R, if
   - $\forall \alpha, \beta \in \mathbb{N}^n\ \alpha \prec \beta \ \Rightarrow\ x^\alpha \prec x^\beta$
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.
3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in \mathbb{K}^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c' x^{\alpha'}$ of f'. We define

   $\text{lm}(f) = x^\alpha$, the **leading monomial** of f

   $\text{lc}(f) = c$, the **leading coefficient** of f

Towards G–algebras

We start with the following collection of data:

1. a field $\mathbb{K}$ and a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset \mathbb{K}^*$, $1 \leq i < j \leq n$
3. a set $D = \{d_{ij}\} \subset R, \quad 1 \leq i < j \leq n$

Assume, that there exists a monomial well–ordering $\prec$ on R such that

$$\forall 1 \leq i < j \leq n, \;\; \mathrm{lm}(d_{ij}) \prec x_i x_j.$$

The Construction

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}\} \; \forall 1 \leq i < j \leq n \rangle$$

PBW Bases and G–algebras

Define the (i, j, k)–*nondegeneracy condition* to be the polynomial

$$NDC_{ijk} := c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik}x_j + d_{jk}x_i - c_{ij}c_{ik} \cdot x_i d_{jk}.$$

Theorem (V. L.)

$A = A(R, C, D, \prec)$ has a PBW basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}\}$ if and only if

$$\forall\ 1 \le i < j < k \le n,\ \ NDC_{ijk} \text{ reduces to } 0 \text{ w.r.t. the relations.}$$

Easy Constructive Check $NDC_{ijk} = x_k(x_j x_i) - (x_k x_j)x_i$.

Definition

An algebra $A = A(R, C, D, \prec)$, where nondegeneracy conditions vanish, is called **a G–algebra** (in n variables).

Gel'fand–Kirillov dimension

Let R be an associative $\mathbb{K}$–algebra with generators $x_1, \ldots, x_m$.

A degree filtration

Consider the vector space $V = \mathbb{K}x_1 \oplus \ldots \oplus \mathbb{K}x_m$.
Set $V_0 = \mathbb{K}$, $V_1 = \mathbb{K} \oplus V$ and $V_{n+1} = V_n \oplus V^{n+1}$.
For any fin. gen. left R–module M, there exists a fin.–dim. subspace $M_0 \subset M$ such that $RM_0 = M$.
An ascending filtration on M is defined by $\{H_n := V_n M_0, n \geq 0\}$.

Definition

The **Gel'fand–Kirillov dimension** of M is defined to be

$$\mathrm{GKdim}(M) \;=\; \lim_{n\to\infty} \sup \log_n(\dim_{\mathbb{K}} H_n)$$

Implementation: GKDIM.LIB, function `GKdim`. Uses Gröbner basis.

Nice Properties of G–algebras

We collect the properties in the following Theorem.

Theorem (Properties of G–algebras)

Let A be a G–algebra in n variables. Then

- *A is left and right Noetherian,*
- *A is an integral domain,*
- *the Gel'fand–Kirillov dimension* $\operatorname{GKdim}(A) = n$,
- *the global homological dimension* $\operatorname{gl.dim}(A) \leq n$,
- *the Krull dimension* $Kr.dim(A) \leq n$,
- *A is Auslander-regular and a Cohen-Macaulay algebra.*

We say that a GR**–algebra** $\mathcal{A} = A/T_A$ is a factor of a G–algebra in n variables A by a proper two–sided ideal T_A.

Examples of GR–algebras

- algebras of solvable type, skew polynomial rings
- univ. enveloping algebras of fin. dim. Lie algebras
- quasi–commutative algebras, rings of quantum polynomials
- positive (resp. negative) parts of quantized enveloping algebras
- some iterated Ore extensions, some nonstandard quantum deformations, some quantum groups
- Weyl, Clifford, exterior algebras
- Witten's deformation of $U(\mathfrak{sl}_2)$, Smith algebras
- algebras, associated to (q–)differential, (q–)shift, (q–)difference and other linear operators
- ...

Wide Scope: q–Calculus and Quantum Algebras

Let $\mathbb{K}$ be a field of char 0.

q–dilation operator

$D_q : C \to C, \quad D_q(f(x)) = f(qx)$:

$$\mathbb{K}(q)\langle x, D_q \mid D_q \cdot x = q \cdot x \cdot D_q \rangle.$$

Continuous q–difference Operator

$\Delta_q : C \to C, \quad \Delta_q(f(x)) = f(qx) - f(x)$:

$$\mathbb{K}(q)\langle x, \Delta_q \mid \Delta_q \cdot x = q \cdot x \cdot \Delta_q + (q-1) \cdot x \rangle.$$

q–differential Operator

$\partial_q : C \to C, \; \partial_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$:

$$\mathbb{K}(q)\langle x, \partial_q \mid \partial_q \cdot x = q \cdot x \cdot \partial_q + 1 \rangle.$$

Gröbner basics

Gröbner Basics are ...

...the most important and fundamental applications of Gröbner Bases.

- Ideal (resp. module) membership problem (NF, REDUCE)
- Intersection with subrings (elimination of variables) (ELIMINATE)
- Intersection of ideals (resp. submodules) (INTERSECT)
- Quotient and saturation of ideals (QUOT)
- Kernel of a module homomorphism (MODULO)
- Kernel of a ring homomorphism (NCPREIMAGE.LIB)
- Algebraic relations between pairwise commuting polynomials
- Hilbert polynomial of graded ideals and modules

Anomalies With Elimination

Contrast to Commutative Case

In terminology, we rather use "intersection with subalgebras" instead of "elimination of variables", since the latter may have no sense.

Let $A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}\}_{1 \leq i < j \leq n}\rangle$ be a G–algebra.
Consider a subalgebra A_r, generated by $\{x_{r+1}, \ldots, x_n\}$.
We say that such A_r is an *admissible subalgebra*, if d_{ij} are polynomials in $x_{r+1}, \ldots, x_n$ for $r+1 \leq i < j \leq n$ and $A_r \subsetneq A$ is a G–algebra.

Definition (Elimination ordering)

Let A and A_r be as before and $B := \mathbb{K}\langle x_1, \ldots, x_r \mid \ldots \rangle \subset A$
An ordering $\prec$ on A is an **elimination ordering for** $x_1, \ldots, x_r$
if for any $f \in A$, $\;\mathrm{lm}(f) \in B$ implies $f \in B$.

Anomalies With Elimination: Conclusion

"Elimination of variables $x_1, \ldots, x_r$ from an ideal I"

means the intersection $I \cap A_r$ with an admissible subalgebra A_r.
In contrast to the commutative case:

- not every subset of variables determines an admissible subalgebra
- there can be no admissible elimination ordering $\prec_{A_r}$

Example

Consider the algebra $A = \mathbb{K}\langle a, b \mid ba = ab + b^2 \rangle$. It is a G–algebra with respect to any well–ordering, such that $b^2 \prec ab$, that is $b \prec a$. Any elimination ordering for b must satisfy $b \succ a$, hence A is not a G–algebra w.r.t. any elimination ordering for b.
The Gröbner basis of a two–sided ideal, generated by $b^2 - ba + ab$ in $\mathbb{K}\langle a, b \rangle$ is infinite and equals to $\{ba^{n-1}b - \frac{1}{n}(ba^n - a^n b) \mid n \geq 1\}$.

Non–commutative Gröbner basics

For the non–commutative PBW world, we need even more:

- Gel'fand–Kirillov dimension of a module (GKDIM.LIB)
- Two–sided Gröbner basis of a bimodule (`twostd`)
- Central Character Decomposition of a module (NCDECOMP.LIB)
- Preimage of a module under algebra morphism
- Ext and Tor modules for centralizing bimodules (NCHOMOLOG.LIB)
- Maximal two–sided ideal in a left ideal (NCANN.LIB in work)
- Check whether a module is simple
- Center of an algebra and centralizers of polynomials
- Operations with opposite and enveloping algebras

A Very Recent Development

Announcement

The newest addition to SINGULAR:PLURAL is the library DMOD.LIB, containing algorithms of algebraic *D*–Module Theory.
A joint work of V. L. and J. M. Morales (Zaragoza).

Functionality: an algorithm ANNFS

- Oaku–Takayama approach (ANNFSOT command)
- Briançon–Maisonobe approach (ANNFSBM command)
- Bernstein polynomial is computed within both approaches

Constructively: two bigger rings are constructed and two eliminations are applied in a sequence.
Complexity of such computations is high!

D–modules

What's Behind

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $f \in R$. We are interested in $R[f^{-s}] = \mathbb{K}[x_1, \ldots, x_n, \frac{1}{f^s}]$ as an R–module for $s \in \mathbb{N}$.
On the one hand, $R[f^{-s}] \cong R[y]/\langle yf^s - 1\rangle$.
On the other hand, $R[f^{-s}]$ is a D–module, where D is the n–th Weyl algebra $\mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_j x_i = x_i \partial_j + \delta_{ij}\}\rangle$.
The algorithm AnnFs computes a D–module structure on $R[f^{-s}]$, that is a left ideal $I \subset D$, such that $R[f^{-s}] \cong D/I$.

Especially interesting are cases when f is irreducible singular (among other, a `reiffen` curve), reducibly singular or when f is a hyperplane arrangement (`arrange`).

Morphisms of general GR–algebras

Setup

Let $\mathcal{A} = A/T_A$ and $\mathcal{B} = B/T_B$ be two GR–algebras and $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a map (respectively, a map $\phi : A \longrightarrow B$). Define $f_i := \text{NF}(\Phi(x_i), T_B)$ resp. $f_i := \phi(x_i)$.

Let $E^o := A \otimes_{\mathbb{K}} B^{\text{opp}}$ (a G–algebra), $T_E^o := T_A + T_B^{\text{opp}}$ a two–sided ideal and $\mathcal{E}^o := \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}^{\text{opp}} = E^o / \langle T_E^o \rangle$ a GR–algebra.

Asymmetric construction

Define the set $S^o := \{x_i - \phi(x_i)^{\text{opp}} \mid 1 \leq i \leq n\} \subset E^o$. We view the (A, B)–bimodule ${}_A\langle S \rangle_B$ as the left ideal $I_\phi^o := {}_{A \otimes_{\mathbb{K}} B^{\text{opp}}}\langle S^o \rangle$.

Morphisms of GR–algebras. Asymmetric method

Lemma

For ϕ, Φ and I^o_ϕ as above, the following holds:

- $\phi \in \text{Mor}(A, B)$ *if and only if* $I^o_\phi \cap B^{opp} = \langle 0 \rangle$,
- $\Phi \in \text{Mor}(\mathcal{A}, \mathcal{B})$ *if and only if* $\text{NF}(I^o_\phi \cap B^{opp} \mid T^{opp}_B) = \langle 0 \rangle$.

Theorem (Asymmetric construction)

Let $\mathcal{A}, \mathcal{B}$ be GR–algebras (resp. A, B be G–algebras). Then the following assertions hold:

- *for any* $\phi \in \text{Mor}(A, B)$, $\quad \ker \phi = I^o_\phi \cap A$,
- *for any* $\Phi \in \text{Mor}(\mathcal{A}, \mathcal{B})$,

$$\ker \Phi = I^o_\Phi \cap \mathcal{A} = \text{NF}(T_A + (T^{opp}_B + I^o_\phi) \cap A \mid T_A).$$

Example ($U(\mathfrak{sl}_2) \to A_1$)

Let $A_1 = \mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1\rangle$ be the first Weyl algebra.

Consider the map $U(\mathfrak{sl}_2) \xrightarrow{\phi} A_1$, defined by

$$e \mapsto x,\ f \mapsto -x^2\partial,\ h \mapsto 2x\partial$$

Performing the elimination $I_\phi^o \cap A_1^{opp}$, we obtain zero ideal, hence

$$\phi \in \mathrm{Mor}(U(\mathfrak{sl}_2), W_1).$$

Computing another elimination $I_\phi^o \cap U(\mathfrak{sl}_2)$, we get

$$\ker \phi = \langle 4ef + h^2 - 2h\rangle.$$

So, there is an embedding

$$0 \to U(\mathfrak{sl}_2)/\langle 4ef + h^2 - 2h\rangle \longrightarrow A_1$$

Limitations of the Asymmetric method

With this method, we can check whether a map is a morphism and compute the kernel of a morphism, or the preimage of a two–sided ideal.

Problem

We cannot compute the preimage of a left ideal.

Lemma (No Module Structure)

Consider the set $X := \{f - \phi(f) \mid f \in A\} \subseteq A \otimes_{\mathbb{K}} B$*. It is spanned by* $\{x^\alpha - \phi(x^\alpha) \mid \alpha \in \mathbb{N}^n\}$*. Let* $S = \{x_i - \phi(x_i) \mid 1 \leq i \leq n\} \subseteq A \otimes_{\mathbb{K}} B$*. There are the following inclusions of* $\mathbb{K}$*–vector-spaces:*

$$X \subset {}_A\langle S\rangle_{\phi(A)} \subseteq {}_A\langle S\rangle_B.$$

Symmetric Deformation: Motivation

Let $\phi : A \to B$ be a map of K–algebras. There are the natural actions of A on B, induced by ϕ:

$$a \circ_L b := \phi(a)b \text{ and } b \cdot a := b \circ_R a := b\phi(a).$$

Observation

These actions provide a well–defined left and right A–module structures on B if and only if ϕ is a morphism.

Hence, B is an (A, A)–bimodule. We extend both actions to A by $a_1 \circ_L a_2 := a_1 \cdot a_2$ and thus turn $A \otimes_{\mathbb{K}} B$ into an (A, A)–bimodule.

Lemma

Consider the set $G = \{g - \phi(g) \mid g \in A\} \subset A \otimes_{\mathbb{K}} B$. Then

$$G = {}_A\langle \{x_i - \phi(x_i) \mid 1 \le i \le n\} \rangle_A \subset A \otimes_{\mathbb{K}} B.$$

Symmetric Deformation: Method

For $1 \leq i \leq n$, $1 \leq j \leq m$, define $q_{ij} \in \mathbb{K} \setminus \{0\}$ to be $q_{ij} := \frac{\mathrm{lc}(y_j f_i)}{\mathrm{lc}(f_i y_j)}$ and $r_{ij} \in B \subset A \otimes_{\mathbb{K}} B$ to be $r_{ij} := y_j f_i - q_{ij} f_i y_j$. Then, for all indices in the same range as above $y_j x_i = q_{ij} \cdot x_i y_j + r_{ij}$ or $[y_j, x_i]_{q_{ij}} = [y_j, f_i]_{q_{ij}}$.

Observation

If all $q_{ij} = 1$, we have $r_{ij} = y_j f_i - f_i y_j = [y_j, f_i]$ and relation becomes just $[y_j, x_i] = [y_j, f_i]$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.

Notation

$(A, B, \phi) \to A \otimes_{\mathbb{K}}^{\phi} B$

Given GR–algebras $\mathcal{A}, \mathcal{B}$, we construct $\mathcal{A} \otimes_{\mathbb{K}}^{\Phi} \mathcal{B}$ as a factor–algebra of $A \otimes_{\mathbb{K}}^{\phi} B$ by the two–sided ideal $T = T_A + T_B$.

Symmetric Deformation: Theorem

Theorem

Let $\mathcal{A}, \mathcal{B}$ be GR–algebras and $\Phi \in \mathrm{Mor}(\mathcal{A}, \mathcal{B})$.
Let I_Φ be the $(\mathcal{A}, \mathcal{A})$–bimodule ${}_{\mathcal{A}}\langle\{x_i - \Phi(x_i) \mid 1 \leq i \leq n\}\rangle_{\mathcal{A}} \subset \mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}$ and $f_i := \Phi(x_i)$. Suppose there exists an elimination ordering for B on $A \otimes_{\mathbb{K}} B$, such that

$$1 \leq i \leq n, 1 \leq j \leq m, \quad \mathrm{lm}(\mathrm{lc}(f_i y_j) y_j f_i - \mathrm{lc}(y_j f_i) f_i y_j) \prec x_i y_j.$$

Then
1) *$A \otimes_{\mathbb{K}}^{\phi} B$ is a G–algebra (resp. $\mathcal{A} \otimes_{\mathbb{K}}^{\Phi} \mathcal{B}$ is a GR–algebra).*
2) *Let $\mathcal{J} \subset \mathcal{B}$ be a left ideal, then*

$$\Phi^{-1}(\mathcal{J}) = (I_\Phi + \mathcal{J}) \cap \mathcal{A}.$$

Symmetric Deformation: Example

Example ($U(\mathfrak{sl}_2) \to A_1$)

Let $A_1 = \mathbb{K}\langle x, \partial \mid \partial x = x\partial + 1\rangle$ be the first Weyl algebra.

Consider the map $U(\mathfrak{sl}_2) \xrightarrow{\phi} A_1$, defined by
$e \mapsto x,\ f \mapsto -x^2\partial,\ h \mapsto 2x\partial$.
We already showed that $\phi \in \mathrm{Mor}(U(\mathfrak{sl}_2), A_1)$.
Define $E' = U(\mathfrak{sl}_2) \otimes_{\mathbb{K}}^{\phi} A_1$, by introducing new relations
$\{[d, e] = 1, [x, f] = 2xd, [d, f] = -d^2, [x, h] = -2x, [d, h] = 2d\}$.
The ordering restrictions on E' $fx \succ xd$ and $fd \succ d^2$ hold iff $f \succ d$. But then the elimination condition $\{x, d\} \gg \{e, f, h\}$ cannot be satisfied on E' and preimage cannot be computed.

Still,

For many cases, preimage can be efficiently computed.

Central Character Decomposition

Let $\mathbb{K}$ be algebraically closed and $C \subset A$ be a fin. gen. commutative subalgebra of A. Denote by C^* the set of maximal ideals of C.
Let M be a fin. gen. A-module and $\chi \in C^*$.
Define $M^\chi = \{v \in M \mid \exists n \in \mathbb{N}, \forall c \in C, \ (c - \chi(c))^n v = 0\}$.
We call $\operatorname{Supp}_C M = \{\chi \in C^* | M^\chi \neq 0\}$ a **support of M** w.r.t. C.

Lemma

Let $M \cong A^N / I_M$ for a left submodule $I_M \subset A^N$. We define a module

$$J_M = \operatorname{preAnn}(M) = \bigcap_{j=1}^{N} \operatorname{Ann}_A^M e_j.$$

Then $Z \cap J_M = Z \cap \operatorname{Ann}_A M$ and
the Zariski closure of $\operatorname{Supp}_Z M$ equals $V(J_M \cap Z(A))$.

Central Character Decomposition

Definition

Let $I \subset A^N$ be a left submodule and $Z = Z(A)$ be a center of A.

1. For $z \in Z$, $(I : z) := \{v \in A^N \mid zv \in I\}$
2. For an ideal $J \subset Z$, $(I : J) := \{v \in A^N \mid \ zv \in I \text{ for all } z \in J\}$.
3. The submodule $I : z^\infty = \varinjlim_{n \in \mathbb{N}} I : z^n$.
4. The submodule $I : J^\infty = \varinjlim_{n \in \mathbb{N}} I : J^n$ (a **central saturation** of I by J).

Theorem (Khomenko, V. L.)

Suppose that $|\operatorname{Supp}_Z M| = s < \infty$. *Then* $M = \bigoplus_{\chi \in Z^*} M^\chi$,

$$M^\chi \cong A^N / I_M : J_\chi^\infty, \ \textit{where} \ \ J_\chi = \bigcap_{\substack{\psi \in \operatorname{Supp}_Z M \\ \psi \neq \chi}} \ker \psi.$$

Central Character Decomposition: Example

Let $S = \{e^3, f^3, h^3 - 4h\} \subset U(\mathfrak{sl}_2)$ and I_L be a left ideal and I_T be a two–sided ideal, generated by S. Easy computation shows $I_L \supset I_T$. For $M_T = U(\mathfrak{sl}_2)/I_T$, $\dim_{\mathbb{K}} M_T = 10$ and $\operatorname{Supp}_Z M_T = \{z, z-8\}$.

Decomposition of M_T:

$$M_T = M_T^{(z)} \oplus M_T^{(z-8)} = U(\mathfrak{sl}_2)/\mathfrak{m} \ \oplus \ U(\mathfrak{sl}_2)/I_9$$

For $M_L = U(\mathfrak{sl}_2)/I_L$, $\dim_{\mathbb{K}} M_L = 15$ and $\operatorname{Supp}_Z M_L = \{z, z-8, z-24\}$.

Decomposition of M_L:

$$M_L = M_L^{(z)} \oplus M_L^{(z-8)} \oplus M_L^{(z-24)} = U(\mathfrak{sl}_2)/\mathfrak{m} \ \oplus \ U(\mathfrak{sl}_2)/I_9 \ \oplus \ U(\mathfrak{sl}_2)/I_5$$

We denote $\mathfrak{m} = \langle e, f, h\rangle$, $I_5 = \langle e^3, f^3, ef-6, h\rangle$, $I_9 = \langle 4ef + h^2 - 2h - 8, h^3 - 4h, e^3, f^3, fh^2 - 2fh, eh^2 + 2eh, f^2h - 2f^2, e^2h + 2e^2\rangle$. The $\mathbb{K}$–dimensions of corresponding modules are $1, 5, 9$ respectively.

NC Cohen–Macaulay Program: Foundations

Definition

Let A be an associative $\mathbb{K}$–algebra and M be a left A–module.

1. The **grade** of M is defined to be
   $j(M) = \min\{i \geq 0 \mid \operatorname{Ext}^i_A(M, A) \neq 0\}$,
   or $j(M) = \infty$, if no such i exists or $M = \{0\}$.
2. Given a dimension function γ on A, then A is called a **Cohen–Macaulay** algebra **w.r.t.** γ, if for every fin. gen. nonzero A–module M, $j(M) + \gamma(M) = \gamma(A) < \infty$.

Theorem (Gomez–Torrecillaz, Lobillo)

G–algebra is Cohen–Macaulay and Auslander regular.

NC CM: Exact values of global dimensions

Theorem

Let A be a G–algebra in n variables over $\mathbb{K}$.
If A has finite–dimensional representations in $\mathbb{K}$, then $\operatorname{gl.dim} A = n$.

Conjecture

$\operatorname{gl.dim} A = n$ if and only if A has fin.–dim. representations in $\mathbb{K}$.

Open Question

Given a GR–algebra $\mathcal{A}$, determine $\operatorname{gl.dim} \mathcal{A}$ algorithmically.

Exact values of global dimensions: Example

Example

Consider the algebra $X_{\mathbb{K}} = \mathbb{K}\langle x, y \mid yx = xy + y^2 + 1\rangle$.
We know, that $\operatorname{gl.dim} X_{\mathbb{K}} \leq 2$. At the same time, $\operatorname{gl.dim} X_{\mathbb{K}} \geq 1$, since the ideal $I = {}_{X_{\mathbb{K}}}\langle x, y^2 + 1\rangle$ is proper and
$\operatorname{syz}(I) = {}_{X_{\mathbb{K}}}\langle(-(y^2 + 1), x + 2y)^t\rangle$.
Since $X_{\mathbb{C}}$ has one–dim. representations $\{(0, \pm i)\}$, $\operatorname{gl.dim} X_{\mathbb{C}} = 2$.
However, $X_{\mathbb{R}}, X_{\mathbb{Q}}, X_{\mathbb{F}_3}$ have no one–dim. representations.
But for any $\mathbb{K}$ there is a family of representations of $X_{\mathbb{K}}$, parametrized by $a \in \mathbb{K}^*$, given by

$$\rho_a : X_{\mathbb{F}} \to M_2(\mathbb{F}), \qquad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -a \\ 1/a & 0 \end{pmatrix}.$$

Hence, $\operatorname{gl.dim} X_{\mathbb{K}} = 2$.

NC Cohen–Macaulay Program: Details

Various Dimensions

CM property is defined with respect to the dimension function

- Krull dimension (various generalizations)
- e. g. Krull–Rentschler–Gabriel dimension
- relative or absolute GK–dimension
- combined dimension ?

Study different dimensions w.r.t. CM property!

NC Cohen–Macaulay Program: Details

More General Algebras

- Factor–algebras
  - e. g. factor–algebras of CM algebras (*G*–algebras)
  - commutative pre–history and lots of results
  - at least 3 different methods for showing CM property
- Ore localizations
  - local commutative rings are classically CM
  - NC extensions of rings like $\mathbb{K}[[x]], \mathbb{K}[x]_{\langle x \rangle}$?
  - NC extensions of skew fields like $\mathbb{K}(\underline{x})$?

Perspectives

Gröbner bases for more non–commutative algebras

• tensor product of commutative local algebras with certain non–commutative algebras
• different localizations of G–algebras

- localization at some "coordinate" ideal of commutative variables (producing e.g. local Weyl algebras $\mathbb{K}[x]_{\langle x \rangle}\langle D \mid Dx = xD + 1 \rangle$)

$\Rightarrow$ local orderings and the generalization of **standard basis** algorithm, Gröbner basics and homological algebra

- localization as field of fractions of commutative variables (producing e.g. rational Weyl algebras $\mathbb{K}(x)\langle D \mid Dx = xD + 1 \rangle$), including **Ore Algebras** (F. Chyzak, B. Salvy)

$\Rightarrow$ global orderings and a generalization **Gröbner basis** algorithm.

Thank you !

SINGULAR PLURAL

Please visit the SINGULAR homepage

- `http://www.singular.uni-kl.de/`